Hydraulic Engineering and Infrastructures

Civil Engineering Department

Fluid Mechanics Review

1 Momentum conservation for a bent pipe ††

A circular pipe of section S_1 transports a fluid of unit mass ϱ at a flowrate Q as shown in Figure 1. The pipe has a bend with an angle α and ends in a narrower section S_2. Knowing the pressure at both ends p_1 and p_2, compute the force applied on the bend.

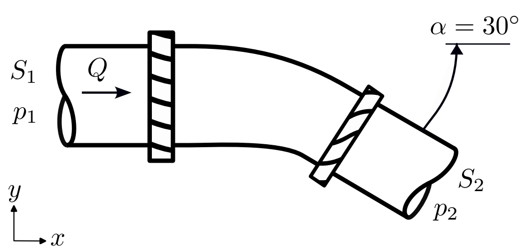

Figure 1: Bent pipe.

Solutions

Ex6 – Momentum conservation for a bent pipe ††

The control volume chosen for this exercise is shown in Figure 2. The conservation of mass states that, in a steady-state regime, we have

$$\underbrace{\frac{\partial}{\partial t}\int_{CV}\rho\, dV}_{=0} + \int_{CS}\rho\vec{V}\cdot\underbrace{\hat{n}\, dA}_{\vec{A}} = 0,$$

Defining the average velocities on both ends $\vec{V}(CS_I) = \vec{V}_1$ and $\vec{V}(CS_{II}) = \vec{V}_2$,

$$\int_{CS_I}\rho\begin{pmatrix}V_1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\0\end{pmatrix}\, dA + \int_{CS_{II}}\rho\begin{pmatrix}V_2\cos\alpha\\-V_2\sin\alpha\end{pmatrix}\cdot\begin{pmatrix}\cos\alpha\\-\sin\alpha\end{pmatrix}\, dA = 0,$$

$$\Rightarrow \int_{CS_I} -\rho V_1\, dA + \int_{CS_{II}}\rho V_2\, dA = 0,$$

The mass conservation equation simplifies to the intuitive relation

$$S_1V_1 = S_2V_2\ (= Q).$$

Since we are looking for a force, let's apply the conservation of linear momentum around the pipe.

$$\underbrace{\frac{\partial}{\partial t}\int_{CV}\rho\vec{V}\, dV}_{=\vec{0}} + \int_{CS}\rho\vec{V}(\vec{V}\cdot\hat{n})\, dS = \int_{CS} -p\hat{n}\, dS$$

The pressure can be decomposed into the pipe's force and the pressures in both ends :

$$
\begin{aligned}
\int_{CS} -p\hat{n}\, dS =& \int_{CS_I} -p\hat{n}\, dS + \int_{CS_{II}} -p\hat{n}\, dS + \int_{CS_{\text{Pipe}}} -p\hat{n}\, dS \\
=& p_1 S_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + p_2 S_2 \begin{pmatrix} -\cos\alpha \\ \sin\alpha \end{pmatrix} + \vec{F}
\end{aligned}
$$

The convection term can be simplified to the sections of the control surface where $\vec{V} \cdot \hat{n} \neq 0$. Assuming the velocity profiles are uniform on both ends of the pipe :

$$
\begin{aligned}
\int_{CS} \rho\vec{V}(\vec{V}\cdot\hat{n})\, dS =& \int_{CS_I} \rho\vec{V}_1(\vec{V}_1\cdot\hat{n})\, dS + \int_{CS_{II}} \rho\vec{V}_2(\vec{V}_2\cdot\hat{n})\, dS \\
=& S_1\rho\vec{V}_1(\vec{V}_1\cdot\hat{n}) + S_2\rho\vec{V}_2(\vec{V}_2\cdot\hat{n}) \\
=& S_1\rho\begin{pmatrix} V_1 \\ 0 \end{pmatrix}\left[\begin{pmatrix} V_1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] + \\
& S_2\rho\begin{pmatrix} V_2\cos\alpha \\ -V_2\sin\alpha \end{pmatrix}\left[\begin{pmatrix} V_2\cos\alpha \\ -V_2\sin\alpha \end{pmatrix}\cdot\begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}\right] \\
=& S_1\rho V_1^2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + S_2\rho V_2^2\begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} \\
=& \rho Q(\vec{V}_2 - \vec{V}_1)
\end{aligned}
$$

Substituting the convective flux and the pressure integrals in the momentum equation, the force applied on the bend is

$$
\vec{F}_{\text{CV}\to\text{Bend}} = -\vec{F} = \rho Q(\vec{V}_1 - \vec{V}_2) + p_1 S_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + p_2 S_2\begin{pmatrix} -\cos\alpha \\ \sin\alpha \end{pmatrix}.
$$

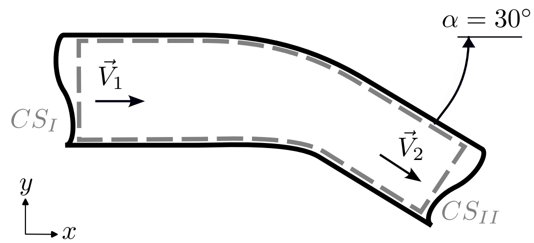

Figure 2: Control volume for the bent pipe.