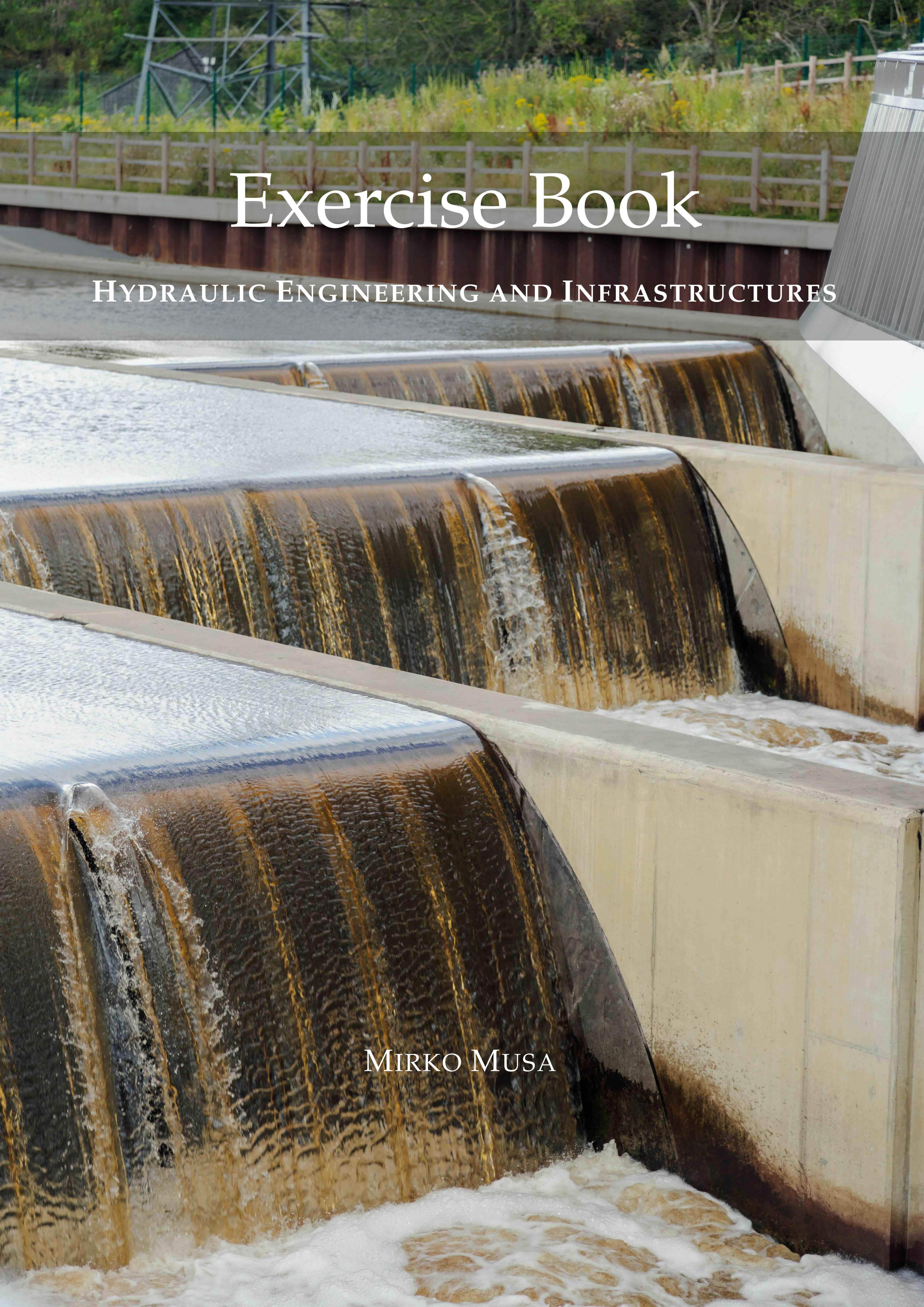
Exercise Book

Hydraulic Engineering and Infrastructures

Mirko Musa



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PREFACE

This collection of exercises has been prepared for the course CIVIL 312 – Hydraulic Engineering and Infrastructures at EPFL, intended for students in Civil Engineering and Environmental Engineering. Its purpose is to provide a progressive and structured set of problems to strengthen understanding of the key concepts covered in the lectures.

The exercises are arranged in order of increasing difficulty. To help guide your practice, each exercise is marked with one to three † symbols, indicating the level of challenge:

- † – introductory exercises that focus on applying basic principles.
- †† – intermediate problems that require combining several concepts.
- † † † – advanced exercises designed to test deeper analytical skills and problem-solving creativity.

Only a few exercises will include fully detailed solutions. This is intentional: while worked-out examples can be very useful, leaving some problems without complete solutions promotes independent reasoning and problem-solving autonomy.

Students are encouraged to start with the lower-difficulty problems, then gradually tackle more challenging ones as their confidence and understanding grow. The aim is to develop both your technical knowledge and your ability to approach real-world engineering challenges with rigor and creativity.

NOMENCLATURE

Greek Symbols

Symbol	Description	Dimensions	Value
γ	Specific weight	$MT^{-2}L^{-3}$	
$\psi = \rho g H$	Bernoulli energy	$MT^{-2}L^{-1}$	
ρ	Specific mass	M/L^3	

Other Symbols

Symbol	Description	Dimensions	Value
$\hat{n}$	Normal of a Control Surface		
$\vec{V}$	Flow velocity vector	L/T	
B	Intensive Property		
b	Extensive Property		
CS	Control Volume		
CV	Control Surface		
E	Energy		
e	Specific energy	L	
H	Hydraulic head	L	
m	Mass	M	
p	Pressure	$MT^{-2}L^{-2}$	
p_0	Atmospheric pressure	$MT^{-2}L^{-2}$	
V	Flow velocity	L/T	

Physics constants

Symbol	Description	Dimensions	Value
g	Standard acceleration of gravity	L/T^2	$9.81\,\mathrm{m/s^2}$

Chapter 1

Fluid Mechanics Review

Brief Summary

1. Reynolds Transport Theorem (RTT)

The general form is:

$$\frac{DB}{Dt} = \frac{\partial}{\partial t}\int_{CV} b\,\rho\,dV \;+\; \int_{CS} b\,\rho\,\vec{u}\cdot d\vec{A} \tag{1.1}$$

where:

- B: extensive property of the system (mass, momentum, energy, etc.)
- $b = \dfrac{B}{m}$: intensive property [per unit mass]
- CV: control volume
- CS: control surface
- ρ: density $[kg/m^3]$
- $\vec{u}$: velocity vector $[m/s]$
- $d\vec{A}$: outward-oriented surface vector $[m^2]$

2. RTT Applications

(a) Continuity Equation ($B = m$, $b = 1$)

$$\frac{\partial}{\partial t}\int_{CV} \rho\,dV + \int_{CS} \rho\,\vec{u}\cdot d\vec{A} = 0 \tag{1.2}$$

Meaning: Conservation of mass. The rate of mass change inside the CV plus the net mass flux through the CS is zero.

(b) Linear Momentum Equation ($B = m\vec{u}$, $b = \vec{u}$)

$$\frac{\partial}{\partial t}\int_{CV} \rho\,\vec{u}\,dV + \int_{CS} \rho\,\vec{u}(\vec{u}\cdot d\vec{A}) = \sum \vec{\mathrm{F}}_{\text{ext}} \tag{1.3}$$

Meaning: The rate of change of momentum inside the CV plus the net flux of momentum through the CS equals the sum of external forces applied.

(c) Energy Equation ($B = E$, $b = e$)

$$\frac{\partial}{\partial t}\int_{CV} \rho e\,dV + \int_{CS} \rho e\,\vec{u}\cdot d\vec{A} = \dot{Q} - \dot{W} \tag{1.4}$$

Meaning: Conservation of total energy. The rate of change of total energy inside the CV plus the net energy flux through the CS equals the balance between the heat added to the system $\dot{Q}$ and the mechanical work done by the system $\dot{W}$.

In hydraulic engineering, the total specific energy e is usually expressed as:

$$e = p + \frac{\rho\, u^2}{2} + \rho g z$$

Dividing by ρg, we define the **total head**:

$$H = \frac{p}{\rho g} + \frac{u^2}{2g} + z$$

where:

- H: total head $[m]$
- $\frac{p}{\rho g}$: pressure head $[m]$
- $\frac{u^2}{2g}$: velocity head $[m]$
- z: elevation head $[m]$

Including power and heat terms: When pumps, turbines, or heat transfer are involved, $\dot{Q}$ and $\dot{W}$ are commonly expressed as equivalent heads:

$$h_p = \frac{\dot{W}}{\rho g Q} \quad ; \quad h_t = \frac{\dot{W}}{\rho g Q} \quad ; \quad h_q = \frac{\dot{Q}}{\rho g Q}$$

where:

- h_p: head added by a pump $[m]$
- h_t: head subtracted by a turbine $[m]$
- h_q: head added due to heating $[m]$ (positive if heat increases energy)
- $\dot{W}$: mechanical power input/output $[W]$
- $\dot{Q}$: heat transfer rate into the CV $[W]$
- Q: volumetric flow rate $[m^3/s]$

Using these definitions, the steady-flow energy equation for two sections becomes:

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 + h_p + h_q = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2 + h_t + h_L$$

where:

- h_L: head losses due to friction and local effects $[m]$

3. HYDROSTATICS

For a static fluid under a (possibly conservative) body-force field $\vec{b}$ [m/s^2]:

$$\nabla p \;=\; \rho\,\vec{b} \qquad \Longrightarrow \qquad \text{if } \vec{b} = -\nabla\Phi \;\Rightarrow\; \nabla p = -\,\rho\,\nabla\Phi.$$

Here Φ is the **body–force potential** (specific potential energy) with units [m^2/s^2]. For conservative fields, the body force per unit mass is $\vec{b} = -\nabla\Phi$; thus the hydrostatic pressure gradient balances the term $-\rho\nabla\Phi$. In a uniform gravitational field with vertical coordinate z [m] positive upward, $\Phi = g\,z$ and $\vec{b} = -g\,\hat{\vec{k}}$, so

$$\frac{\partial p}{\partial z} = -\rho g.$$

Hydrostatic Law

Integrating between z and a reference level z_0:

$$p(z) = p_0 + \rho g (z_0 - z)$$

where p is pressure [Pa], ρ density [kg/m^3], g gravitational acceleration [m/s^2], and z vertical coordinate [m].

Resultant on Plane Surfaces

Resultant force on a fully wetted plane area A [m^2] whose centroid lies at depth h_C [m], using the specific weight $\gamma = \rho g$ [N/m^3]:

$$F_R = \gamma\, h_C\, A$$

Center of pressure (vertical location from the free surface):

$$h_R = h_C + \frac{I_G}{h_C\, A}$$

where h_R is the depth of the line of action [m] and I_G is the second moment of the submerged area about a centroidal axis parallel to the free surface [m^4].

Forces on Curved Surfaces (Projection Method)

$$F_H = \gamma\, h_{C_v}\, A_v, \qquad F_V = W_{\text{(imaginary fluid above)}}, \qquad F_R = \sqrt{F_H^2 + F_V^2}$$

Here F_H and F_V are the horizontal and vertical components of the hydrostatic force [N]; A_v is the area of the vertical projection [m^2], h_{C_v} the depth of its centroid [m]; W is the weight of the (imaginary) fluid above the curved surface up to the free surface [N]; and F_R the resultant magnitude [N]. The line of action follows from moment balance.

Buoyancy (Archimedes)

$$\mathbf{F}_B = \rho_f g\, V_{\text{disp}} \quad \text{(upward through the center of buoyancy)}$$

where ρ_f is the surrounding fluid density [kg/m^3] and V_{disp} the displaced volume [m^3]. Floating equilibrium reads $W = \rho_f g\, V_{\text{disp}}$, with W the body weight [N].

EXERCISES

1. EMPTYING TANK

A vertical storage tank is open to the atmosphere and drains through a small sharp-edged orifice at the bottom. The orifice has a diameter of $3.0\,\mathrm{cm}$. Inside the tank, the cross-sectional area of the water surface at a height h (measured from the bottom outlet) is

$$A(h) = 0.50\sqrt{h}$$

where A is in m^2 and h is in m. Initially, the water level is $h_0 = 1.20\,\mathrm{m}$.

Assumptions:

- Incompressible, inviscid flow.
- No energy losses anywhere in the system.
- Velocity of the free surface is negligible.
- Free surface and outlet jet both at atmospheric pressure.

Take $g = 9.81\,\mathrm{m/s^2}$.

Tasks:

(a) Show that the exit velocity is:

$$v_{\text{out}} = \sqrt{2gh}.$$

(b) Using conservation of mass, derive the differential equation for $h(t)$.

(c) Solve this differential equation to obtain an explicit expression for $h(t)$.

(d) Compute the total time required for the tank to empty.

(e) Compute the volumetric flow rate after $60\,\mathrm{s}$ of draining.

SOLUTION

Given Data

$$d = 0.030\,\text{m}, \quad a = \frac{\pi d^2}{4} = \pi(0.015)^2 = 7.07 \times 10^{-4}\,\text{m}^2$$
$$A(h) = 0.50\sqrt{h}, \qquad h_0 = 1.20\,\text{m}$$

(a) Exit velocity via Bernoulli

Between the free surface (1) and the orifice (2):

$$\frac{v_1^2}{2g} + z_1 = \frac{v_2^2}{2g} + z_2$$

With $v_1 \approx 0$, $z_1 = h$, $z_2 = 0$:

$$h = \frac{v_{\text{out}}^2}{2g} \quad \Rightarrow \quad v_{\text{out}} = \sqrt{2gh}.$$

(b) Differential equation for $h(t)$

Outflow:

$$Q = a\sqrt{2gh}.$$

Volume balance:

$$Q = -\frac{dV}{dt}, \qquad \frac{dV}{dt} = A(h)\frac{dh}{dt}.$$

Thus:

$$a\sqrt{2gh} = -A(h)\frac{dh}{dt}.$$

Substitute $A(h) = 0.50\sqrt{h}$:

$$a\sqrt{2g}\sqrt{h} = -0.50\sqrt{h}\frac{dh}{dt}.$$

Cancel $\sqrt{h}$:

$$a\sqrt{2g} = -0.50\,\frac{dh}{dt}.$$

Hence

$$\frac{dh}{dt} = -\frac{a\sqrt{2g}}{0.50} = -C, \qquad C = \frac{a\sqrt{2g}}{0.50}.$$

(c) Solution for $h(t)$

$$\frac{dh}{dt} = -C \quad \Rightarrow \quad h(t) = h_0 - Ct.$$

(d) Time required to empty

Set $h(t_{\text{empty}}) = 0$:

$$t_{\text{empty}} = \frac{h_0}{C} = \frac{h_0 \cdot 0.50}{a\sqrt{2g}}.$$

Compute:

$$C = \frac{7.07 \times 10^{-4}\sqrt{2(9.81)}}{0.50} = 6.26 \times 10^{-3}\,\text{m/s}.$$

Thus:

$$t_{\text{empty}} = \frac{1.20}{6.26 \times 10^{-3}} = 1.92 \times 10^{2}\,\text{s} \approx 192\,\text{s}.$$

$$\boxed{t_{\text{empty}} \approx 192 \text{ s } (\approx 3.2 \text{ min})}$$

(e) Flow rate at $t = 60$ s

Height:

$$h(60) = 1.20 - (6.26 \times 10^{-3})(60) = 0.82\text{ m}.$$

Exit velocity:

$$v_{\text{out}}(60) = \sqrt{2gh(60)} = \sqrt{2(9.81)(0.82)} = 4.01\text{ m/s}.$$

Flow rate:

$$Q(60) = av_{\text{out}} = (7.07 \times 10^{-4})(4.01) = 2.84 \times 10^{-3}\text{ m}^3/\text{s}.$$

$$\boxed{Q(60) \approx 2.8 \times 10^{-3}\text{ m}^3/s\ (\approx 2.8\ L/s)}$$

2. MASS CONSERVATION IN A JET ENGINE †

An airplane moves forward at a speed of 971 km/h as shown in Figure 1.2. The frontal intake area of the jet engine is 0.8 m^2 and the entering air density is 0.736 kg/m^3. A stationary observer determines that relative to the Earth, the jet engine exhaust gases move away from the engine with a speed of 1050 km/h. The engine exhaust area is 0.558 m^2, and the exhaust gas density is 0.515 kg/m^3.

Estimate the mass flow rate of fuel into the engine in kg/h.

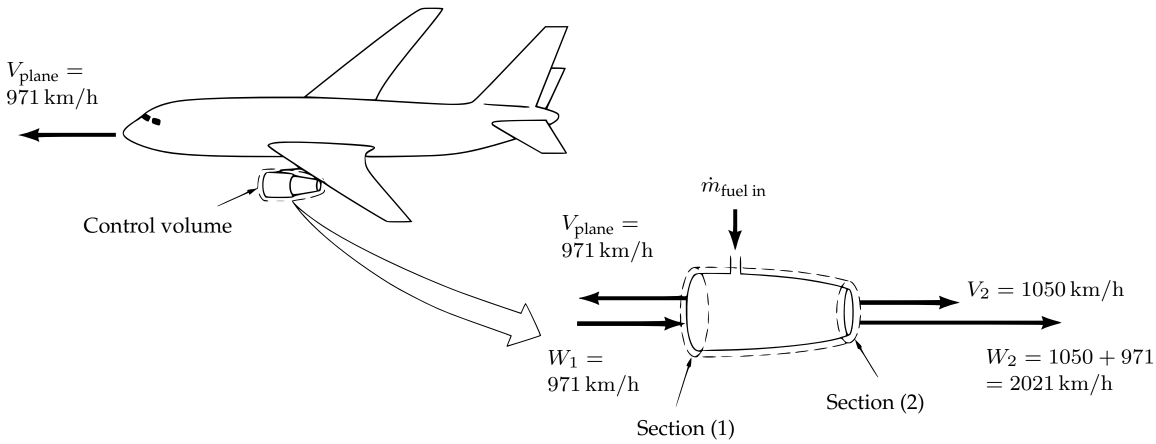


Figure 1.1: Airplane jet engine intake and exhaust rates.

SOLUTION

Choosing a control volume surrounding the engine, the following mass conservation equation is obtained, and the mass flow rate can be directly computed.

$$\dot{m}_{\text{fuel in}} = \frac{d}{dt} m_{\text{fuel in}} = \rho_2 A_2 W_2 - \rho_1 A_1 W_1 = 9100\,\mathrm{kg/h}$$

3. MOMENTUM CONSERVATION AROUND A WEDGE SPLITTING A FREE JET ††

A free jet of fluid strikes a wedge as shown in Figure 1.2. Of the total flow, a portion is deflected 30°; the remainder is not deflected. The horizontal and vertical components of force needed to hold the wedge stationary are F_H and F_V, respectively. Gravity is negligible, and the fluid speed remains constant. Determine the force ratio, F_H/F_V.

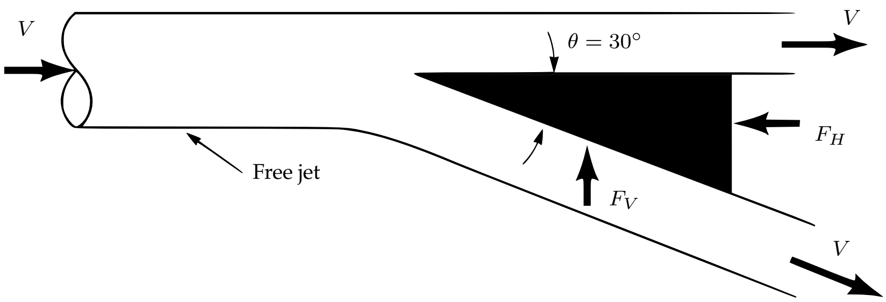


Figure 1.2: Wedge splitting a free jet.

SOLUTION

Since there are no body forces and the flow is stationary, the control volume's momentum conservation equation simplifies to a direct expression for the force applied on the fluid :

$$\vec{F} = \begin{pmatrix} F_H \\ F_V \end{pmatrix} = \int_S \rho \vec{u}(\vec{u} \cdot \vec{n})\, dS = -A_1 \rho V \begin{pmatrix} 1 \\ 0 \end{pmatrix} + A_2 \rho V \begin{pmatrix} 1 \\ 0 \end{pmatrix} + A_3 V \begin{pmatrix} +\cos\theta \\ -\sin\theta \end{pmatrix}.$$

Using the continuity equation, the relation between the areas is $A_1 V = A_2 V + A_3 V$. The force is then $\vec{F} = (\cos\theta - 1, -\sin\theta)^\intercal \rho A_3 V$ and the ratio between the horizontal and vertical forces is :

$$\frac{F_H}{F_V} = \frac{1 - \cos\theta}{\sin\theta} = 0.27.$$

4. ENERGY CONSERVATION FOR A VENTURI METER ††

Kerosene ($\rho_k = 0.85\rho_w$) flows through the Venturi meter shown in Figure 1.3 with flow rates between 0.005 and 0.050 m^3/s. Determine the range in pressure difference, $p_1 - p_2$, needed to measure these flow rates.

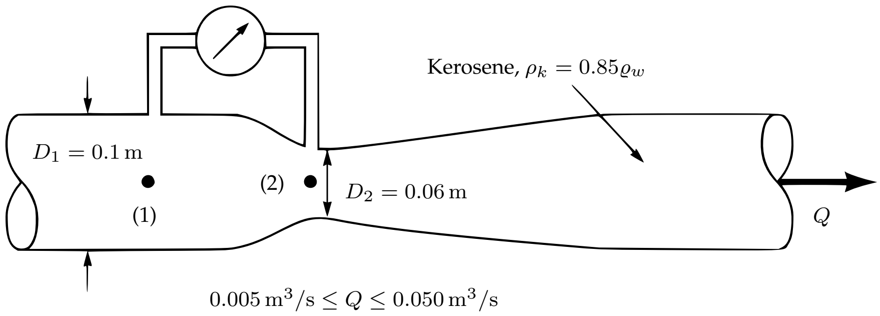


Figure 1.3: Venturi meter.

SOLUTION

Assuming the flow is horizontal ($z_1 = z_2$), steady, inviscid, and incompressible between points (1) and (2), the Bernoulli equation $\psi_1 = \psi_2$ becomes:

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

And thus $p_1 - p_2 \in [1.16; 116]$ kPa.

5. ENERGY CONSERVATION, FLOW FROM A TANK †

Air flows steadily from a tank, through a hose of diameter $D = 0.03\,\mathrm{m}$, and exits to the atmosphere from a nozzle of diameter $d = 0.01\,\mathrm{m}$ as shown in Figure 1.4. The pressure in the tank remains constant at 3.0 kPa (gage) and the atmospheric conditions are standard temperature and pressure.

Determine

(a) the flowrate

(b) the pressure in the hose.

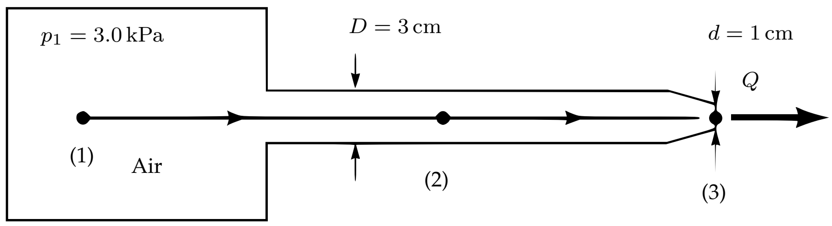


Figure 1.4: A gas tank. The flow is pressure driven.

SOLUTION

1. Assuming the gas is perfect, its unit mass is $\rho = 1.26\,\mathrm{kg/m^3}$.

   If the flow is assumed steady, inviscid, and incompressible, we can apply the Bernoulli equation along the streamline from (1) to (2) to (3) and obtain the flow rate $Q = \frac{1}{4}\pi d^2 \sqrt{2p_1/\rho} = 0.005\,42\,\mathrm{m^3/s}$.

2. The pressure in the point (2) can be obtained from the continuity equation (to get the velocity v_2) and the energy equation as in the previous question. $p_2 = 2963\,\mathrm{Pa}$.

6. MOMENTUM CONSERVATION FOR A JET ON A FLAT SURFACE †

A circular jet whose section has a surface S_0 and a speed u meets a wall as shown in Figure 1.5. The fluid has a unit mass ρ. Compute the force applied on the wall.

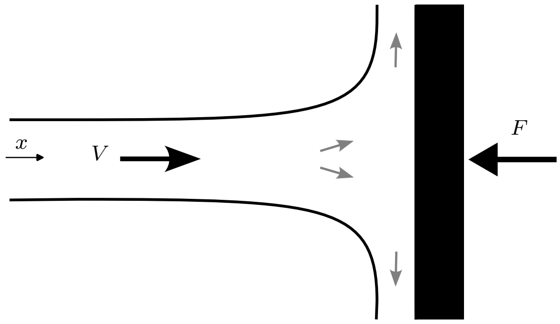


Figure 1.5: Jet hitting a flat surface.

SOLUTION †

No body forces are considered, the atmospheric pressure is neglected and a steady-state is assumed. The control volume is shown in Figure 1.6.

$$\vec{F}_{\text{Exterior}\to\text{Control Volume}} = \begin{pmatrix} F_H \\ F_V \end{pmatrix} = \int_S \rho\vec{V}(\vec{V}\cdot\vec{n})\,dS = -S_0\rho\begin{pmatrix} V^2 \\ 0 \end{pmatrix} + \int_{\text{Exiting}} \rho\vec{V}(\vec{V}\cdot\vec{n})\,dS.$$

Since the flow is symmetric around the x-axis, all forces parallel to the wall are balanced. The force applied by the flow on the wall is then

$$F_{\text{Fluid}\to\text{Wall}} = -F_{\text{Exterior}\to\text{Control Volume}} = S_0\rho V^2$$

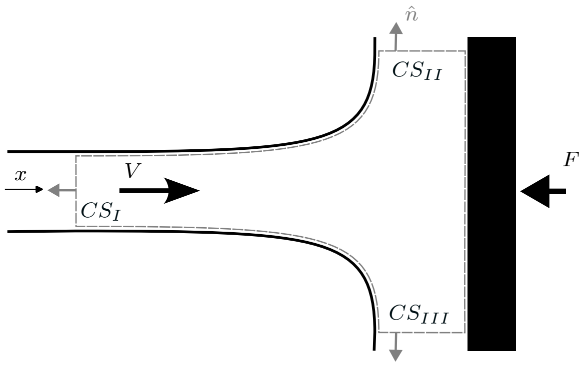


Figure 1.6: Control volume for the jet on a flat surface.

7. MOMENTUM CONSERVATION FOR A BENT PIPE ††

A circular pipe of section S_1 transports a fluid of unit mass ϱ at a flow rate Q as shown in Figure 1.7. The pipe has a bend with an angle α and ends in a narrower section S_2. Knowing the pressure at both ends p_1 and p_2, compute the force applied on the bend.

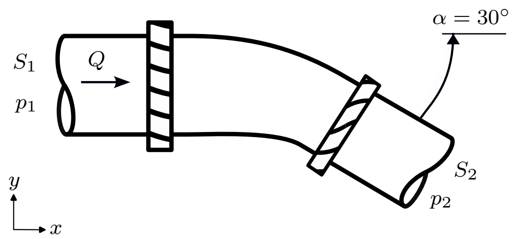


Figure 1.7: Bent pipe.

SOLUTION ††

No body forces are considered, the forces applied on the fixed control volume (shown in Figure 1.8) are the bend's force and the pressures. Let the average flow speeds be $V_1 = Q/S_1$ and $V_2 = Q/S_2$. The control volume's surface in contact with the bend is named $\mathcal{B} = CS \setminus (CS_I \cup CS_{II})$. Let's first define the force of the bend on the Control Volume:

$$\vec{F}_{\text{Ext}} = \int_C S - p\vec{n}\, dS = \underbrace{\int_{\mathcal{B}} -p\vec{n}\, dS}_{\vec{F}_{\text{Bend}\to\text{Control Volume}}} + \int_{\mathcal{S}_1\cup\mathcal{S}_2} -p\vec{n}\, dS.$$

The momentum balance then makes the link with flow properties :

$$F_{\text{Bend}\to\text{Control Volume}} + \int_{\mathcal{S}_1\cup\mathcal{S}_2} -p\vec{n}\, dS = \int_C S\rho\vec{V}(\vec{V}\cdot\vec{n})\, dS.$$

And the force applied on the bend can be extracted :

$$\vec{F}_{\text{Control Volume}\to\text{Bend}} = p_1 S_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + p_2 S_2 \begin{pmatrix} -\cos\alpha \\ +\sin\alpha \end{pmatrix} + \rho Q(\vec{V}_1 - \vec{V}_2).$$

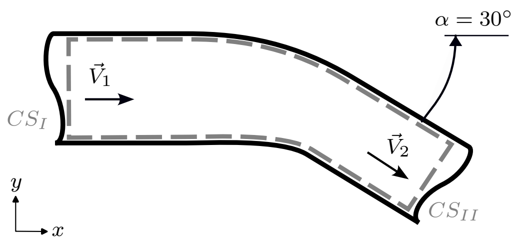


Figure 1.8: Control volume for the bent pipe.

8. HYDROSTATICS OF A VERTICAL FLAT SURFACE †

A tank is filled with three liquids with different specific weights, namely oil, water, and glycerin. These fluids are at rest and have the following specific weights: $\gamma_{\text{oil}} = 8000\ \text{N/m}^3\text{m}$ $\gamma_{\text{w}} = 9806\ \text{N/m}^3\text{m}$ and $\gamma_{\text{gly}} = 11500\ \text{N/m}^3$.

Calculate:

(a) the pressure at point A.

(b) the pressure at the oil-water interface (point B).

(c) the pressure at point C.

(d) the pressure at the water-glycerin interface (point D).

(e) the pressure at the bottom of the tank (point E).

(f) draw the diagram of the pressures acting along the wall of the tank.

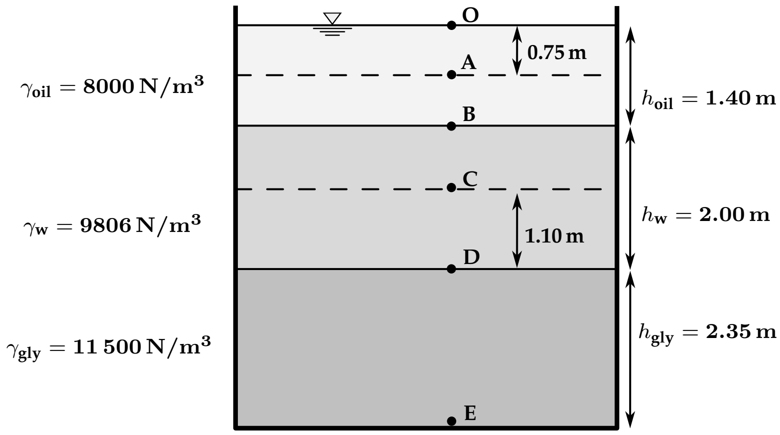


Figure 1.9: Vertical tank with oil, water, and glycerin layers showing heights, specific weights, and points A–E for pressure calculation.

SOLUTION

The solution to the exercise requires calculating the pressure inside the tank, which contains multiple stratified incompressible fluids at rest, at a generic depth. For simplicity, since atmospheric pressure can be assumed constant, in fluid mechanics exercises we refer to the gauge pressure, unless otherwise specified.

If a single fluid is in direct contact with the atmosphere, its pressure is proportional to the depth of the point measured from the free surface, generally indicated by the letter z, multiplied by the specific weight of the fluid. This occurs, in the proposed exercise, within the oil layer, where i denotes a generic point inside the layer.

Following **hydrostatic law**: $P + \gamma \cdot h =$ constant, hence $P_i = P_0 + \gamma_{\text{oil}} \cdot z_i = 0 + \gamma_{\text{oil}} \cdot z$

This relation indicates that, in an incompressible fluid at rest, the pressure varies linearly with depth z.

For the solution of points C, D, and E, located in the underlying layers, within each individual layer the overlying layers contribute a *constant load* equal to the sum of the specific weights of the fluids above multiplied by their respective heights. To this pressure, the contribution of the fluid in the current layer must be added, which equals the specific weight of the fluid multiplied by the depth within the layer, measured from the interface with the overlying fluid.

In the proposed exercise, with three stratified fluids, the depths within the respective layers z_S are introduced, as indicated in the figure.

Point A: there is only one fluid above point A. The gauge pressure is therefore equal to the product of the oil's specific weight and the oil height above that point: $P_A = P_0 + \gamma_{\text{oil}} \cdot z_{\text{oil},A} = 0 + \gamma_{\text{oil}} \cdot 0.75$

Point B: located at the interface between oil and water, also has only one overlying fluid. The expression for the pressure calculation therefore remains unchanged: $P_B = \gamma_{\text{oil}} \cdot z_{\text{oil},B} = \gamma_{\text{oil}} \cdot h_{\text{oil}} = \gamma_{\text{oil}} \cdot 1.40$

Point C: two fluids overlie point C, the oil layer, which contributes a constant pressure equal to the oil's specific weight multiplied by the height of the entire oil layer, and the water layer, which is the fluid in which point C is located. The contribution of the water equals its specific weight multiplied by the depth of the point within the layer, measured from the interface with the oil: $P_C = \gamma_{\text{oil}} \cdot h_{\text{oil}} + \gamma_W \cdot z_{W,C} = P_B + \gamma_W \cdot 0.90$

Point D: the calculation follows the same procedure, with the only difference being that the height of the overlying water is the entire water layer: $P_D = \gamma_{\text{oil}} \cdot h_{\text{oil}} + \gamma_W \cdot z_{W,D} = P_B + \gamma_W \cdot h_W = P_B + \gamma_W \cdot 2.00$

Point E: is obtained by adding to the pressure at point D the weight of the glycerin layer: $P_E = \gamma_{\text{oil}} \cdot h_{\text{oil}} + \gamma_W \cdot h_W + \gamma_{\text{glycerin}} \cdot z_{\text{glycerin},E} = P_D + \gamma_{\text{glycerin}} \cdot h_{\text{glycerin}} = P_D + \gamma_{\text{glycerin}} \cdot 2.35$

General Case $P = \sum_i (\gamma_i \cdot h_i)$

The pressure–depth diagram is piecewise linear, with slopes equal to the specific weight of the fluid in each layer.

Numerical results: $P_A = 6.000$ kPa, $P_B = 11.20$ kPa, $P_C = 20.03$ kPa, $P_D = 30.81$ kPa, $P_E = 57.84$ kPa

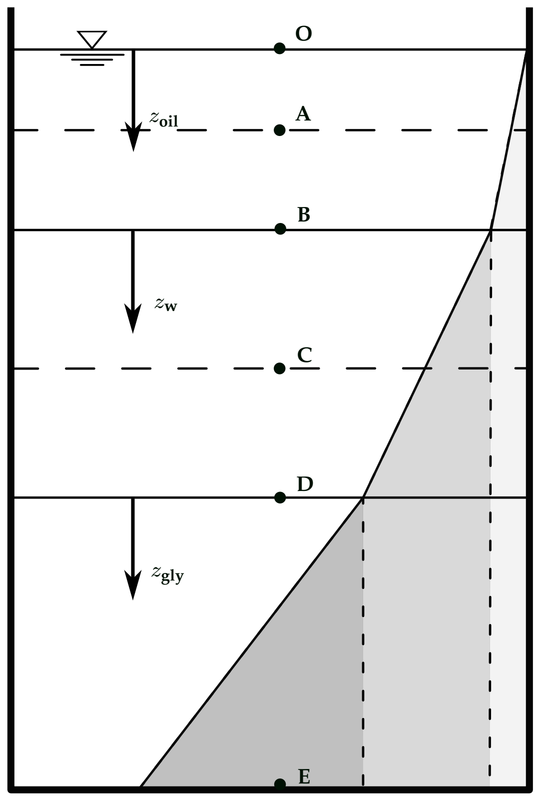


Figure 1.10: Pressure diagram on the tank walls.

9. HYDROSTATICS OF A INCLINED FLAT SURFACE †

The 4-m-diameter circular gate in figure 1.11 is located in the inclined wall of a large reservoir containing water ($\gamma = 9.80\ \mathrm{kN/m^3}$). The gate is mounted on a shaft along its horizontal diameter, and the water depth is 10 m above the shaft.

Determine:

(a) the magnitude and location of the resultant force exerted on the gate by the water and

(b) the moment that would have to be applied to the shaft to open the gate.

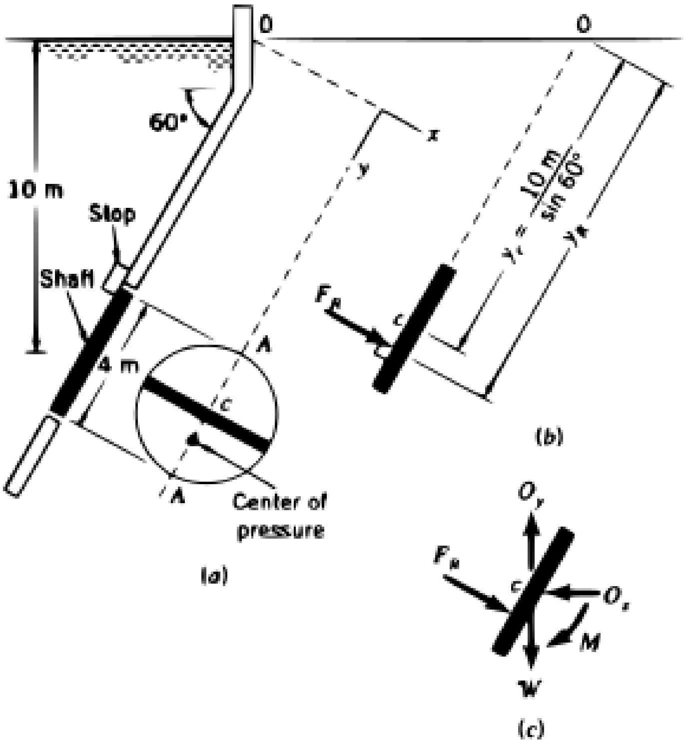


Figure 1.11: Inclined gate.

SOLUTION

(a) To find the **magnitude** of the force exerted on the gate, we can use the hydrostatic force formula: $F_R = \gamma sin(\theta) y_c A = \gamma h_c A$.
With A, the area of the gate, and $h_c = 10$ m, the distance between the water surface and the centroid of the area, it follows:

$$\begin{aligned} F_R &= (9.80 \times 10^3 \text{ N/m}^3)(10 \text{ m})(4\pi \text{ m}^2) \\ &= 1230 \times 10^3 \text{ N} = \boxed{1.23 \text{ MN}} \end{aligned}$$

The point of coordinate (x_R, y_R) through which F_R acts is the center of pressure. As the gate is symmetrical, we know that the center of pressure must lie along the $A - A$ diameter, meaning that $x_R = 0$ m. Then, to find y_R, we can use the fact that, along the x-axis, the moment of the resultant force must equal the moment of the distributed pressure force, or:

$$\begin{aligned} F_R y_R &= \int_A y dF \\ \gamma sin(\theta) y_c A y_R &= \int_A \gamma sin(\theta) y^2 dA \\ y_R &= \frac{\int_A y^2 dA}{y_c A} \end{aligned}$$

The numerator of the fraction is the moment of inertia with respect to the x-axis, I_x. The moment of inertia with respect to the axis parallel to the x-axis and passing through the centroid is determined using the parallel axis theorem:

$$\begin{aligned} I_x &= I_{xc} + y_c^2 A \\ &= \frac{\pi}{4} R^4 + y_c^2 A \end{aligned}$$

Then,

$$
\begin{aligned}
y_R &= \frac{\frac{\pi}{4}R^4}{y_c A} + y_c \\
&= \frac{\frac{\pi}{4}(2\text{ m})^4}{(10\text{ m}/sin60)(4\pi\text{ m}^2)} + \frac{10\text{ m}}{sin60} \\
&= 0.0866\text{ m} + 11.55\text{ m} = \boxed{11.6\text{ m}}
\end{aligned}
$$

(b) The moment required to open the gate can be obtained with the help of the free-body diagram of Fig. 1.11 (c). In this diagram, W is the weight of the gate and O_x and O_y are the horizontal and vertical reactions of the shaft on the gate. We can now sum the moments about the shaft:

$$\sum M_c = 0$$

and, therefore:

$$
\begin{aligned}
M &= F_R(y_R - y_c) \\
&= (1230 \times 10^3\text{ N})(0.0866\text{ m}) \\
&= \boxed{1.07 \times 10^5\text{ Nm}}
\end{aligned}
$$

10. HYDROSTATICS OF A CURVED SURFACE †

A 2-m diameter drainage conduit is half full of water at rest (see figure 1.12).Determine the magnitude and line of action of the resultant force that the water exerts on a 1 m length of the curved section BC of the conduit wall.

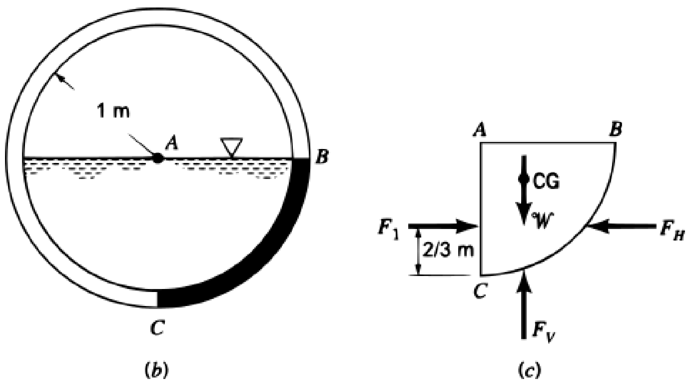


Figure 1.12: 2-m conduit at rest.

SOLUTION

Given: $r = 1$ m, length 1 m, $\gamma = 9.81$ kN/m^3, curved section BC (quarter circle), water at rest.

Horizontal component on BC:

The horizontal force equals the hydrostatic force on the vertical projection (rectangle $1 \times r$):

$$A = 1 \cdot r = 1 \text{ m}^2, \qquad \bar{h} = \frac{r}{2} = 0.5 \text{ m},$$

$$F_H = \gamma A \bar{h} = 9.81\,(1)(0.5) = \boxed{4.905 \text{ kN}}.$$

Center of pressure on the rectangle:

$$h_{CP} = \bar{h} + \frac{I_G}{A\bar{h}} = 0.5 + \frac{(1 \cdot r^3/12)}{(1)(0.5)} = 0.5 + \frac{1/12}{0.5} = \boxed{0.667 \text{ m below the free surface at } A}.$$

Vertical component on BC: Equal to the weight of the imaginary water above the curved surface (quarter cylinder of radius r and length 1):

$$V = \frac{\pi r^2}{4} \cdot 1 = \frac{\pi}{4} = 0.7854 \text{ m}^3, \qquad F_V = \gamma V = 9.81 \cdot \frac{\pi}{4} = \boxed{7.705 \text{ kN}} \text{ (downward)}.$$

Line of action through the centroid of a quarter circle:

$$x_c = y_c = \frac{4r}{3\pi} = \boxed{0.424 \text{ m from } A}.$$

Resultant magnitude and direction:

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{(4.905)^2 + (7.705)^2} = \boxed{9.13 \text{ kN}}.$$

Angle from the horizontal:

$$\alpha = \tan^{-1}\left(\frac{F_V}{F_H}\right) = \tan^{-1}\left(\frac{7.705}{4.905}\right) = \boxed{57.5^\circ} \quad \text{(or } 32.5^\circ \text{ from vertical, into the wall)}.$$

Line of action : The line of action of the resultant force passes through the center of the conduit. In retrospect, this is not a surprising result since at each point on the curved surface of the conduit, the elemental force due to the pressure is normal to the surface, and each line of action must pass through the center of the conduit. It therefore follows that the resultant of this concurrent force system must also pass through the center of concurrence of the elemental forces that make up the system.

$$\boxed{F_H = 4.905 \text{ kN}, \quad F_V = 7.705 \text{ kN}, \quad F_R = 9.13 \text{ kN}, \quad \alpha = 57.5^\circ.}$$

11. PRESSURE OF WATER FLOWING INSIDE A PIPE †

A U-tube piezometer is used to measure the pressure of water flowing inside a pipe. The fluid used in the piezometer is mercury, which has a specific weight $\gamma_{Hg} = 133000\ \mathrm{Nm^{-3}}$. Two configurations are shown in Fig. 1.13, the first on the left and the second on the right.

Calculate:

(a) the gauge pressure P_G of the water measured by the manometer.

(b) the new heights h_1' and h_2' if the pressure inside the pipe is $P_G' = P_G - 30$ kPa.

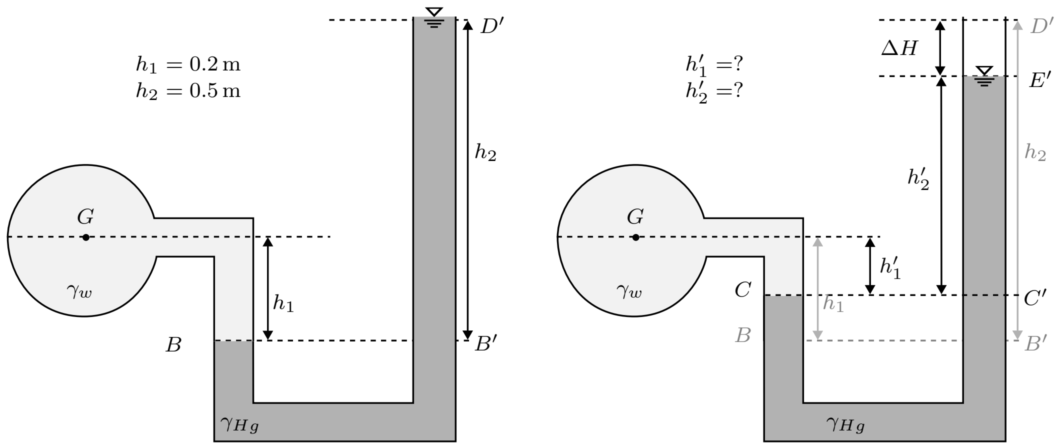


Figure 1.13: Two configurations of the U-tube piezometer.

SOLUTION

The U-pipe piezometer consists of two vertical transparent tubes that communicate with each other. One of the tubes also communicates with the atmosphere, while the other one communicates with the measurement environment.

Recalling the **hydrostatic law** $p + \gamma_h = const$, we can see that if two points belong to the same fluid while being at the same depth, then they are subjected to the same pressure. Then, any horizontal surface contained within the same fluid is an isobaric surface.

The horizontal surface passing through the water-mercury interface (BB' in the first case and CC' in the second) is an isobaric surface. Conveniently, point B (and point C) can be considered as belonging to the water or to the mercury fluids indistinctly.

For the first configuration, using the isobar definition applied to the BB' surface, we have $P_B = P_{B'}$. Using the hydrostatic law for mercury, $P_{B'} = P_{D'} + (\gamma_{Hg} \times h_2)$. As $P_{D'}$ is the reference atmospheric pressure, it is null and we get: $P_{B'} = \gamma_{Hg} \times h_2$.

The pressure at point B can be expressed with respect to the centroid of the conduit, G: $P_B = P_G + (\gamma_W \times h_1)$.

Then,

$$\begin{aligned} P_B &= P_{B'} \\ P_G + (\gamma_W \times h_1) &= \gamma_{Hg} \times h_2 \\ P_G &= \gamma_{Hg} \times h_2 - \gamma_W \times h_1 \end{aligned}$$

Which gives:

$$P_G = 64.54 \text{ kPa}$$

In the second configuration, we apply the same procedure on the isobaric surface CC', noting that the free surface is at point E'.

The new mercury height above the isobaric surface is $h_2' = h_2 - 2\Delta H$. The new water height is $h_1' = h_1 - \Delta H$.

Applying the hydrostatic law on both sides, we get:

$$\begin{aligned} P_{C'} &= \gamma_{Hg} \times h_2' = \gamma_{Hg} \times (h_2 - 2\Delta H) \\ P_C &= P_{G'} + \gamma_W \times h_1' = P_{G'} + \gamma_W \times (h_1 - \Delta H) \end{aligned}$$

Since $P_C = P_{C'}$, we have:

$$P_{G'} + \gamma_W \times (h_1 - \Delta H) = \gamma_{Hg} \times (h_2 - 2\Delta H)$$

After expressing ΔH, we can obtain h_1' and h_2':

$$\begin{aligned} h_1' &= 8.291 \times 10^{-2} \text{ m} \\ h_2' &= 2.658 \times 10^{-1} \text{ m} \end{aligned}$$

12. CONCOMITANT CHAMBERS †

A tank consists of two independent chambers connected by a square opening of side 0.4 m, which is kept closed by a gate of the same shape as the opening and hinged at the point O. The left chamber is sealed and contains pressurized gas, while the right chamber contains water up to a height y above the hinge. The left chamber is connected to a piezometer containing the gas with a column of water of height $h = 1.2$ m above.

Calculate:

(a) the pressure in the left chamber P_{GAS}.

(b) the level y of water in the right tank for which the conditions shown in Fig. 1.14 give equilibrium for the gate.

(c) sketch all the forces acting on the gate, specifying magnitude, direction, and sense.

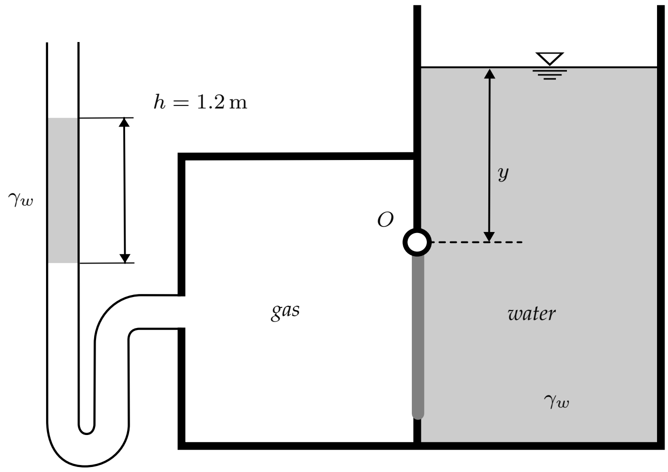


Figure 1.14: Problem configuration.

SOLUTION

The height of the fluid inside the piezometer allows the pressure of the gas in the left chamber to be calculated using the hydrostatic equation: $P_{GAS} = P_{atm} + \gamma_W \times h$. As P_{atm} is the reference atmospheric pressure, we set it to 0 Pa.

$$P_{GAS} = 11.78 \text{ kPa}$$

Knowing this value, the height of the water above the hinge, y, is found by solving the moment-equilibrium equation about point O. in which the only unknown is the water level y.

$$y = 0.93 \text{ m}$$

13. WATER INFLUENCE OF A VERTICAL GATE †

A tank with a width $B = 3$ m contains water up to a height $H = 5$ m. Along the right wall of the tank, there is a gate hinged at point O, of length equal to the height of the container (see Fig. 1.15).

Calculate:

(a) the force exerted by the water on the gate (S_W).

(b) the moment exerted by the water with respect to the hinge point O (M_W).

(c) draw the diagram of the pressures acting along the wall of the tank.

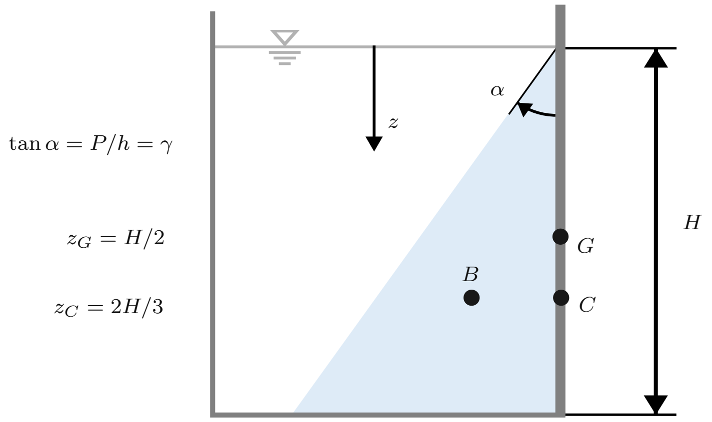


Figure 1.15: Tank, container and gate.

SOLUTION

The force on a flat surface (generally a gate, regardless of its inclination) caused by a single fluid is a force perpendicular to the surface, with a magnitude equal to the product of the pressure acting at its center of pressure (which corresponds to the centroid in this case) multiplied by the area of the surface.

The point of application of the force, called the center of pressure, generally does not coincide with the centroid of the gate.

If the gate is rectangular, the position of the center of pressure can be determined by plotting the pressure distribution along the gate. According to the hydrostatic law, the pressure distribution increases linearly with depth. The projection of the centroid of the pressure diagram on the gate corresponds to the position of the center of pressure (see Fig. 1.16).

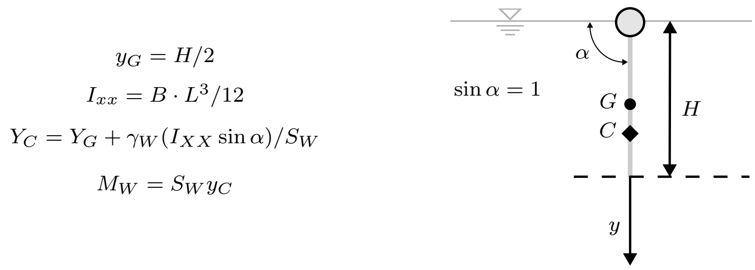


Figure 1.16: G: centroid of the gate, B: centroid of the pressure diagram; C: center of pressure of the gate (projection of B onto the gate)

If the shape of the gate is more complex, the position of the center of pressure (y_C) with respect to the position of the centroid (y_G) can be determined using the general theoretical formula : $y_C = y_G + \gamma_W \frac{I_{xx} sin(\alpha)}{S_W}$ where S_W if the fluid force, α is the angle formed by the gate with respect to the horizontal axis, and I_{xx} is the moment of inertia with respect to the horizontal centroidal axis.

The integral of the pressures that gives the force on the gate, in this example of a single fluid producing a continuous linear distribution of pressures along the height of the gate, is calculated as the product of the pressure at the centroid of the surface and the area of the surface.

Here, the area of the gate is $A = H \times B$ and the centroid is located at $z_G = \frac{H}{2}$.

The average pressure at z_G is calculated using the hydrostatic law: $P_G = \gamma_W z_G = \gamma_W \frac{H}{2}$.

Then, the force of the fluid on the gate is: $S_W = P_G A = \gamma_W \frac{H^2 B}{2}$.

From this expression of S_W, we see that the force can also be considered as the volume of the pressure prism, defined as the product of the area of the pressure diagram and the width of the gate B (see Fig. 1.16).

Regarding the point of application of the force, the diagram has a triangular shape: the center of pressure is then located at the centroid of this triangle, positioned at two-thirds of the height of the gate starting from the hinge. Then, the projection of the centroid of the triangle on the gate, which is the center of pressure of the gate, is $z_C = 2H/3$.

Once the point of application is known, the moment caused by S_W with respect to the hinge is. $M_W = S_W * z_C$.

14. FORCE ON A CYLINDRICAL AQUARIUM † † †

A tank filled with a fluid ($\gamma_f = \gamma_{water}/2 = 4900\,\mathrm{N/m^3}$) and water ($\gamma_{water} = 9800\,\mathrm{N/m^3}$) has a cylindrical aquarium of width $w = 10\,\mathrm{m}$ and radius $r = 3\,\mathrm{m}$ placed right at the interface between the two fluids as shown in Figure 1.17. For a height $H = 10\,\mathrm{m}$, compute the force applied on the aquarium and its application point on the aquarium's surface.

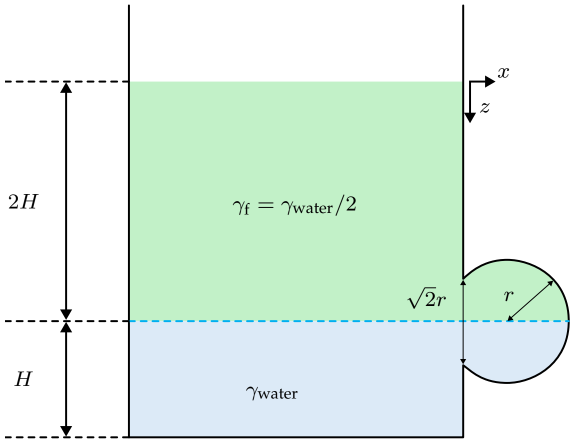


Figure 1.17: A cylindrical aquarium with its bottom half filled with water and the top half with a light fluid.

SOLUTION † † †

To compute the pressure force F within the aquarium, we integrate it on its surface:

$$\mathbf{F} = \int dF = \int -p \cdot \hat{n} dS \text{ with } \hat{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

The pressure inside the aquarium depends on the depth z at which we are. In the aquarium, this means that it depends on which angle θ we are.

$$p(z) = \begin{cases} \gamma_t z & \text{if } z < 2H \\ 2H\gamma_t + \gamma_{\text{water}}(z - 2H) & \text{if } z > 2H \end{cases}$$

$$p(\theta) = \begin{cases} \gamma_t(2H - r\sin\theta) & \text{if } 0 < \theta < 3\pi/4 \\ 2H\gamma_t - \gamma_{\text{water}} r \sin\theta & \text{if } -3\pi/4 < \theta < 0 \end{cases}$$

As $\gamma_t = \gamma_{\text{water}}/2$, we obtain:

$$\mathbf{F} = \begin{pmatrix} F_H \\ F_V \end{pmatrix} = \begin{pmatrix} \gamma_{\text{water}} r(H\sqrt{2} + \frac{1}{8}r)w \\ \gamma_{\text{water}} r^2(\frac{3}{8} + \frac{9\pi}{16})w \end{pmatrix} = \begin{pmatrix} 4.27\,\text{MN} \\ 1.89\,\text{MN} \end{pmatrix}$$

To determine the point of application of the force (x', z'), we can use the following formulas:

$$
\begin{aligned}
F_V \cdot x' = \int x \, dF_V &= \int_{CS} x \cdot p(z) \hat{n}_y \cdot dA \\
&= \int_{-\frac{3\pi}{4}}^{0} r(1+\cos\theta)(2H\gamma_t - \gamma_{\text{water}} r \sin\theta) \sin(\theta) wr \, d\theta \\
&\quad + \int_{0}^{\frac{3\pi}{4}} r(1+\cos\theta)(2H - r\sin\theta)\gamma_t \sin(\theta) wr \, d\theta
\end{aligned}
$$

$$\Rightarrow x' = \left(1 + \frac{2\sqrt{2}}{6+9\pi}\right) r = 3.25\,\text{m}$$

The application point of the horizontal forces can be computed with the projected force on the YZ-plane.

$$
\begin{aligned}
F_H \cdot z' = \int z \, dF_H &= \int_{CS} z \cdot p(z) \hat{n}_x \cdot dA \\
&= \int_{-3\pi/4}^{3\pi/4} z \cdot p(z) \hat{n}_x \cdot wr \, d\theta \\
&= \int_{2H-\sqrt{2}r/2}^{2H+\sqrt{2}r/2} z \cdot p(z) \cdot dA \quad \text{(projected area)}
\end{aligned}
$$

$$\Rightarrow z' = 2H + \frac{\sqrt{2} r^3 w \gamma_{\text{water}}}{8 F_H} = 20.11\,\text{m}.$$

It can be noted that the action line of the force passes through the center of the circle in Figure 1.18 since the application point and the center of the circle are aligned with an angle α with respect to the horizontal.

$$\beta = \arctan \frac{y' - 2H}{x' - r} = \arctan \frac{F_y}{F_h} = \alpha = 23.9^\circ$$

.

This is because all infinitesimal forces $p(z) \cdot \hat{n} \, dA$ are aligned with the radii. Computing the torque around the center of the circle would yield

$$M = \int (r\hat{n}) \wedge p(z)\hat{n} \, dS = \int r \begin{pmatrix} \cos\theta \\ \sin\alpha \end{pmatrix} \wedge \begin{pmatrix} \cos\theta \\ \sin\alpha \end{pmatrix} p(z) \, dS = \int 0 \, dA = 0.$$

The crossing between the force's action line and the circle is finally

$$P_{\text{on circle}} = \begin{pmatrix} r \\ 2H \end{pmatrix} + \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} = \begin{pmatrix} r \\ 2H \end{pmatrix} + \begin{pmatrix} 2.74\,\text{m} \\ 1.21\,\text{m} \end{pmatrix} = \begin{pmatrix} 5.74\,\text{m} \\ 21.2\,\text{m} \end{pmatrix}.$$

Application point

We can make use of the tabulated moments of inertia and the parallel axis theorem to avoid using the integrals. Since the values for rectangular shapes are tabulated (see course notes), we can

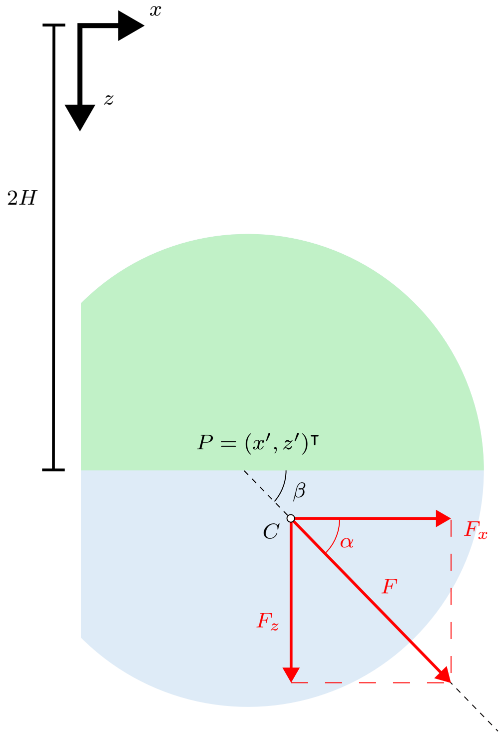


Figure 1.18: Point of application of the force. As long as the force is left along its line of action, its torque is the same.

use them to compute the forces and the application points. Note that we are making use of the projected area. A sketch of the solution is given in Figure 1.21

For the top half, the center of gravity is $z_{C1} = 2H - h/2 = 18.94\,\mathrm{m}$, its area is $A_1 = wh = 21.21\,\mathrm{m}^2$ and the application point is

$$z_{R1} = z_{C1} + \frac{I_{yC1}}{z_{C1}A_1} = 18.96\,\mathrm{m}.$$

And the force is $F_1 = p(z_{C1})A_1 = \gamma_f z_{C1} wh$.

For the bottom half, the change in density pushes us to define a new origin $z' = z - H$ to find a hydrostatic pressure distribution (with slope γ_{water}) matching that of the bottom half for the parallel axis theorem to apply.

Having defined this new origin, the center of mass $z'_{C2} = H + h/2 = z_{C2} - H = 11.06\,\mathrm{m}$, the application point is $z_{R2} = z'_{R2} + H = \frac{I_{yC2}}{z'_{C2}A_2} + H = 21.09\,\mathrm{m}$, and the force is $F_1 = \gamma_{\text{water}} \cdot (H + h/2)A_2$.

The application point of the total force is found with

$$z_R = \frac{F_1 z_{R1} + F_2 z_{R2}}{F_1 + F_2} = 20.11\,\mathrm{m}.$$

Which is in agreement with the previous results.

ALTERNATIVE SOLUTION ††

The force applied on the aquarium can be computed with a geometrical method: instead of computing the integral over the control surface, directly compute the weight of the column of water above the surface. The area of the different columns is drawn on Figure 1.20.

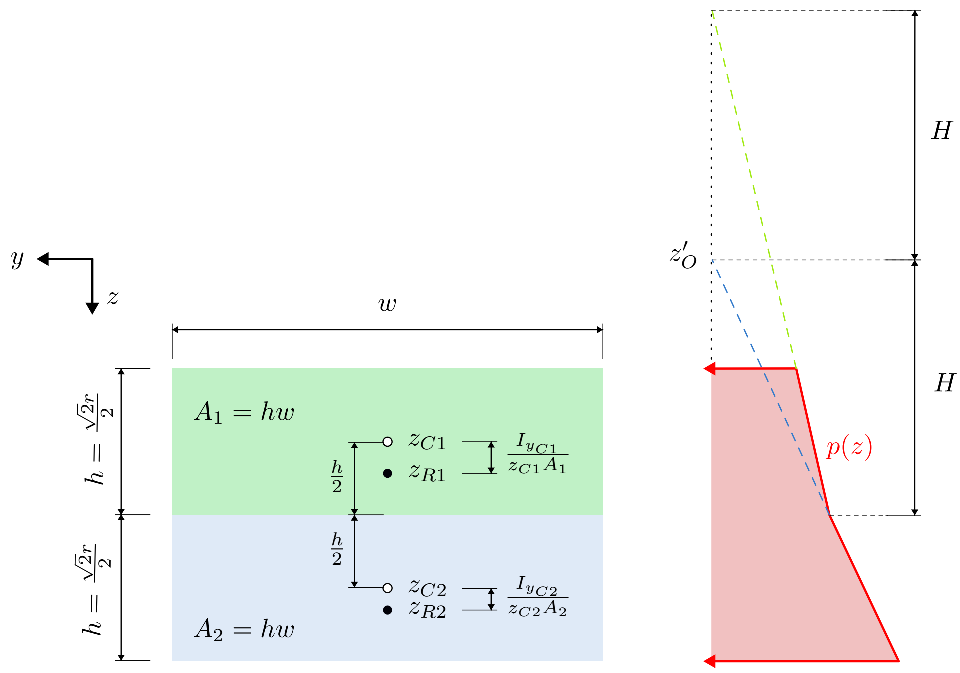


Figure 1.19: Drawing of the opening, its areas and application points.

The downwards force is equal to the weight of the area A_1:

$$F_{y1} = \gamma_1 \cdot A_1 = \gamma_1 \cdot \left(\frac{\sqrt{2}}{2} + 1\right) r \cdot 2H - \gamma_1 \cdot A_2 \tag{1.5}$$

$$= \gamma_1 \cdot Hr\left(\sqrt{2} + 2\right) - \gamma_1 \cdot \left(\frac{3\pi}{8} + \frac{1}{4}\right) r^2. \tag{1.6}$$

Similarly, the weight of the upwards-pushing column is :

$$F_{y2} = \gamma_1 \cdot A_1 + \gamma_1 A_2 + \gamma_2 A_3 = \gamma_1 \cdot \left(\frac{\sqrt{2}}{2} + 1\right) r \cdot 2H + \gamma_2 \cdot A_2 \tag{1.7}$$

$$= \gamma_1 \cdot Hr\left(\sqrt{2} + 2\right) + \gamma_2 \cdot \left(\frac{3\pi}{8} + \frac{1}{4}\right) r^2. \tag{1.8}$$

So the vertical force on the aquarium is

$$F_y = -W_1 + W_2 = (\gamma_1 + \gamma_2) \cdot \left(\frac{3\pi}{8} + \frac{1}{4}\right) r^2$$

Note that for $\gamma_1 = \gamma_2$, the result respects Archimedes' principle. For $\gamma_1 = \gamma_2/2$,

$$F_y = \frac{3}{2}\gamma_2 \cdot \left(\frac{3\pi}{8} + \frac{1}{4}\right) r^2 = 1.89\,\text{MN}.$$

The horizontal force can be computed with the projected area for the top and bottom halves separately:

$$F_{x1} = \frac{\sqrt{2}}{2} r\gamma_1 \cdot \left(2H - \frac{\sqrt{2}/2}{2} r\right) \tag{1.9}$$

$$F_{x2} = \frac{\sqrt{2}}{2} r \cdot \left(\gamma_1 \cdot 2H + \gamma_2 \cdot \frac{\sqrt{2}/2}{2} r\right) \tag{1.10}$$

$$\Rightarrow F_x = \sqrt{2} r\gamma_2 H + (\gamma_2 - \gamma_1)\frac{r^2}{4} \tag{1.11}$$

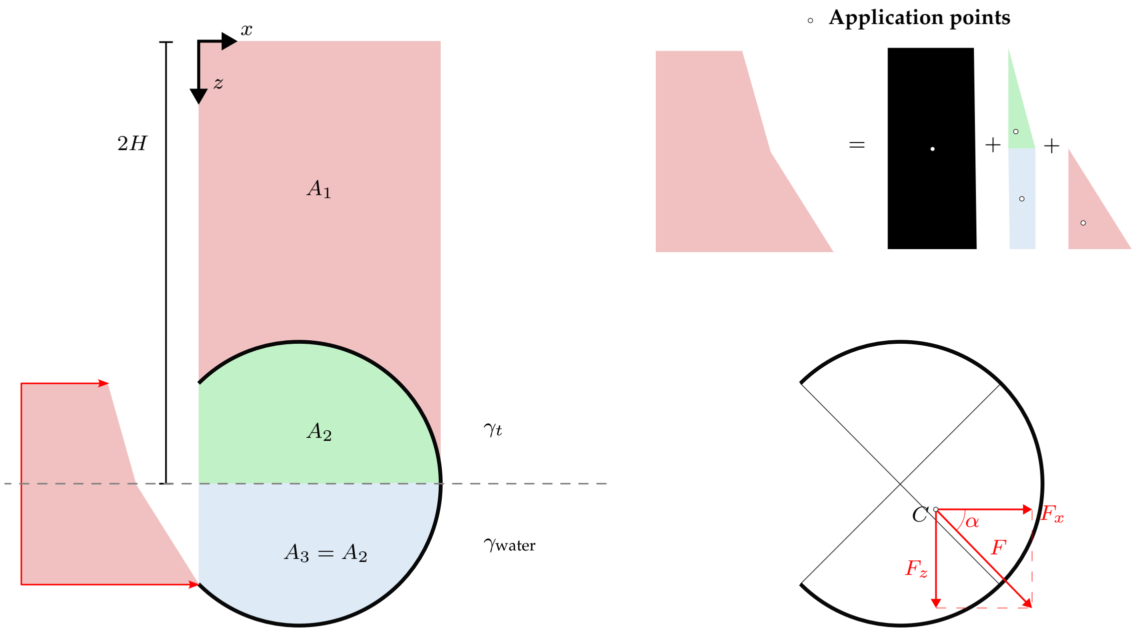


Figure 1.20: Computing the vertical force on the aquarium with a geometric method.

Application point

We can make use of the tabulated moments of inertia and the parallel axis theorem to avoid using the integrals. Since the values for rectangular shapes are tabulated (see course notes), we can use them to compute the forces and the application points. Note that we are making use of the projected area. A sketch of the solution is given in Figure 1.21

For the top half, the center of gravity is $z_{C1} = 2H - h/2 = 18.94\,\mathrm{m}$, its area is $A_1 = wh = 21.21\,\mathrm{m}^2$ and the application point is

$$z_{R1} = z_{C1} + \frac{I_{yC1}}{z_{C1}A_1} = 18.96\,\mathrm{m}.$$

And the force is $F_1 = p(z_{C1})A_1 = \gamma_f z_{C1} wh$.

For the bottom half, the change in density pushes us to define a new origin $z' = z - H$ to find a hydrostatic pressure distribution (with slope γ_{water}) matching that of the bottom half for the parallel axis theorem to apply.

Having defined this new origin, the center of mass $z'_{C2} = H + h/2 = z_{C2} - H = 11.06\,\mathrm{m}$, the application point is $z_{R2} = z'_{R2} + H = \frac{I_{yC2}}{z'_{C2}A_2} + H = 21.09\,\mathrm{m}$, and the force is $F_1 = \gamma_{\text{water}} \cdot (H + h/2)A_2$.

The application point of the total force is found with

$$z_R = \frac{F_1 z_{R1} + F_2 z_{R2}}{F_1 + F_2} = 20.11\,\mathrm{m}.$$

Which is in agreement with the previous results.

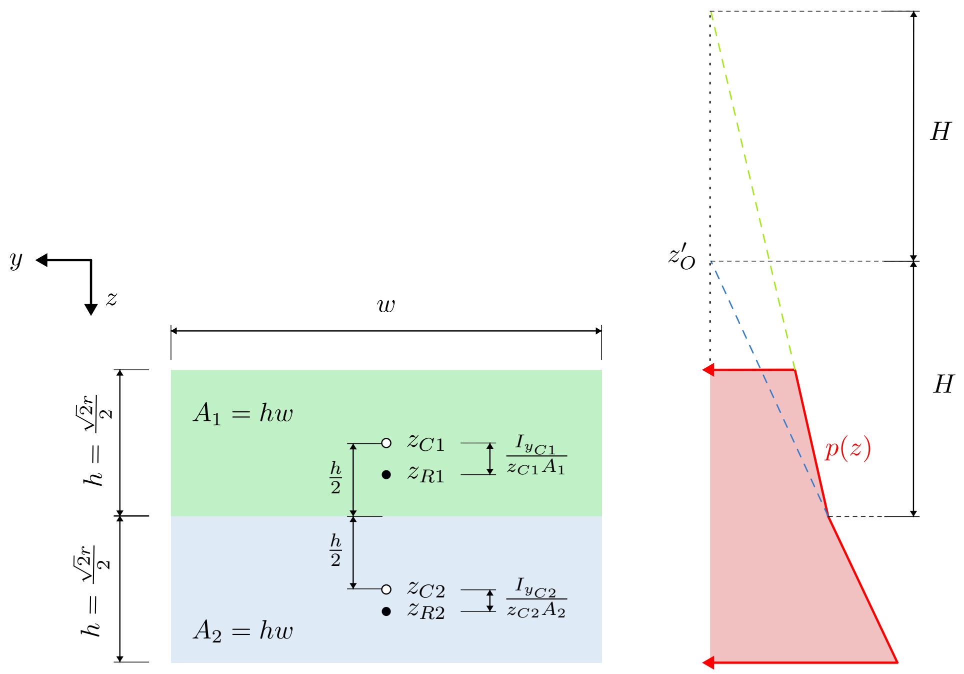


Figure 1.21: Drawing of the opening, its areas and application points.

15. CORK OF A SUBMARINE EMISSARY † † †

A reservoir filled with oil ($\gamma_{\text{oil}} = 8000\,\text{N/m}^3$) has a submarine emissary with an opening $a = \sqrt{2}\,\text{m}$ and is closed with a cylindrical cork of radius $r = 1\,\text{m}$ weighing $10\,\text{kN}$ at a depth $z_a = 5\,\text{m}$. The sea water has a unit weight of $10\,055\,\text{N/m}^3$. Figure 1.17 sketches the situation. Compute the maximum elevation difference z_R between the reservoir and the cork before the cork is pushed away from the emissary.

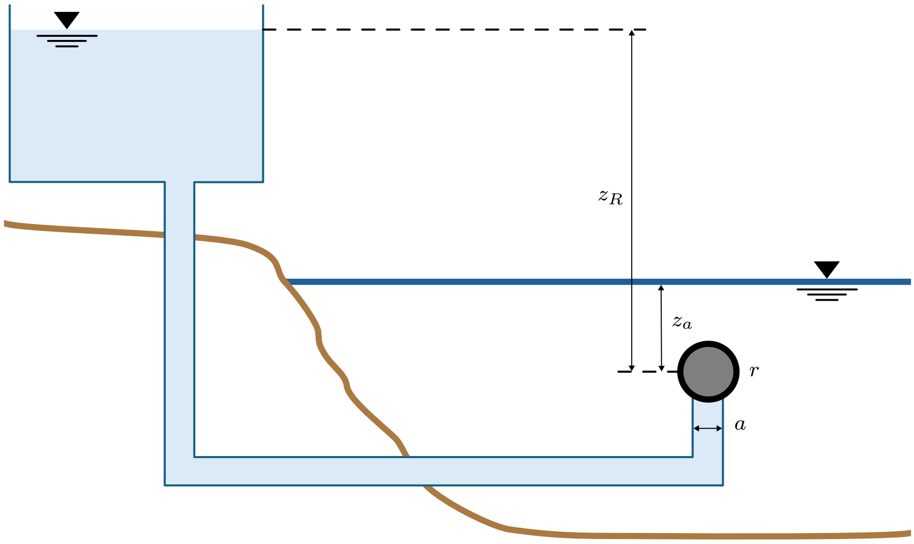


Figure 1.22: Reservoir filled with oil with a submarine emissary. The latter is corked with a cylinder.

SOLUTION † † †

$$z_R = 12.40 \text{ m}$$

Chapter 2

Pressurized Pipe Flow

Brief Summary

1. Energy Equation for Pressurized Pipe Flow

Between sections 1 and 2 (steady flow), including pump head h_P, turbine head h_T, and total headloss h_L:

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} + h_P \ = \ \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_T + h_L$$

Here p is pressure [Pa], $\gamma = \rho g$ specific weight [N/m^3], z elevation [m], V mean velocity [m/s], g gravity [m/s^2], h_P pump head added [m], h_T turbine head extracted [m], and h_L total headloss [m].

Energy & Hydraulic Grade Lines.
Total head (EGL): $H = \frac{p}{\gamma} + z + \frac{V^2}{2g}$.
Hydraulic grade line (HGL): $HGL = \frac{p}{\gamma} + z$.

Along the flow in a pipe, EGL and HGL drop due to losses; a pump causes a jump up while a turbine generates a drop.

2. Reynolds Number and Flow Regimes

$$\mathrm{Re} = \frac{\rho V D}{\mu}$$

with ρ density [kg/m^3], D pipe diameter [m], μ dynamic viscosity [Pa·s]. Regimes: laminar if $\mathrm{Re} \lesssim 2100$, turbulent if $\mathrm{Re} \gtrsim 4000$, transitional in between.

3. Major and Minor Losses

(a) Major Losses Headloss due to wall friction over length L [m] in a pipe of diameter D [m]:

$$h_{L,\text{major}} = f\,\frac{L}{D}\,\frac{V^2}{2g}$$

where f is the Darcy friction factor for fully turbulent flow [–].

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D}{3.71} + \frac{2.51}{\text{Re}\sqrt{f}}\right) \quad \text{(Colebrook–White)}$$
$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D}{3.71}\right) \quad \text{(fully rough asymptote as Re}\to\infty\text{)}$$

Here ε is the equivalent sand roughness height [m], so ε/D is the relative roughness [–].

(a) Minor Losses Each fitting/transition contributes a local loss quantified by a coefficient K_L [–]:

$$h_{L,\text{minor}} = \sum K_L\,\frac{V^2}{2g}$$

Total Headloss in Bernoulli. The loss term in the energy equation is the sum of major and minor losses:

$$h_L = h_{L,\text{major}} + h_{L,\text{minor}} = f\,\frac{L}{D}\,\frac{V^2}{2g} + \sum K_L\,\frac{V^2}{2g}.$$

4. Simple Networks

(a) Pipes in Series

- The discharge is the same in every pipe; total headloss is the sum of the individual losses. (Constant Q, additive losses).
- Governing relations (between endpoints 1–2):

$$Q_1 = Q_2 = \cdots = Q_n = Q$$
$$h_L = \sum_{i=1}^{n} h_{L,i}$$
$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g} - h_L$$

With Darcy–Weisbach: $h_{L,i} = f_i \frac{L_i}{D_i}\frac{V_i^2}{2g} = K_i Q^2$ where $K_i = \dfrac{f_i L_i}{D_i}\dfrac{1}{2gA_i^2}$.

- *Resolution strategy:* use continuity (Q constant), write Bernoulli between end nodes, and replace h_L by the sum of segment losses. If diameters change, remember $V = Q/A$ changes and so does h_L.

(b) Pipes in Parallel

- The headloss between the common nodes is the same in every branch; total flow splits among branches. (Variable Q_i, equal loss).
- Governing relations (between the same pair of junctions):

$$Q_{\text{in}} = \sum_{i=1}^{n} Q_i = Q_{\text{out}}, \qquad h_{L,1}(Q_1) = h_{L,2}(Q_2) = \cdots = h_{L,n}(Q_n) = \Delta H.$$

  Typically $h_{L,i} = K_i Q_i^2$ (fully turbulent assumption or given f_i).

- *Resolution strategy:* write node continuity and enforce equal headloss in each branch to solve for Q_i. If f_i depends on Reynolds number, iterate on f_i and Q_i until consistent.

(c) Branching (one junction connecting multiple branches)

- One interior node couples several pipes; unknowns are branch discharges and junction head. (Node continuity + one Bernoulli per branch).
- Governing relations (at junction D, neglecting velocity heads in reservoirs):

$$\sum Q_{\text{in}} = \sum Q_{\text{out}}, \qquad H_D = H_i - h_{L,i}(Q_i)\ (\forall i), \quad H = \frac{p}{\gamma} + z$$

  With $h_{L,i} = f_i \frac{L_i}{D_i} \dfrac{Q_i^2}{2gA_i^2}$. This yields a square system (e.g., 4 eqs–4 unknowns)

- *Resolution strategy:* choose the junction as reference; apply (i) continuity at the node and (ii) one Bernoulli from each external node/reservoir to the junction; solve for Q_i and junction head.

5. COMPLEX NETWORKS

Principles. Networks satisfy both node continuity and loop energy balance:

$$\sum_{i\in\text{node}} q_i = 0, \qquad \sum_{i\in\text{loop}} h_{f,i} = 0, \quad \text{with } h_{f,i} = K_i\,|q_i|^{n-1} q_i \ \ (n \approx 2 \text{ for D–W}).$$

The algebraic sum around each oriented loop must vanish (choose a sign convention). A network with m loops and n junctions supplies $m + (n-1)$ independent equations.

Formulation and correction. Using $h_{f,i} = K_i q_i^2$ and a small correction $q_i \leftarrow q_i + \delta q$ in one loop,

$$h_{f,i} \approx K_i q_i |q_i| + 2K_i |q_i|\ \delta q,$$

which leads to the loop correction

$$\delta q \;=\; -\frac{\sum s_i\, h_{f,i}}{\sum(2K_i\,|q_i|)} \;=\; -\frac{\sum s_i\, h_{f,i}}{\sum \left|\dfrac{h_{f,i}}{q_i}\right|} \quad (\text{for } n = 2),$$

where $s_i = \pm 1$ depends on the pipe orientation relative to the loop. (Slides compute $h_{f,i}$, $|h_{f,i}/q_i|$ and δq each iteration.)

Algorithm (pseudocode).

1. **Initialization:** Guess flows q_i so that node continuity holds at every junction ($\sum q_i = 0$).
2. **Loop corrections:** For each independent loop L:
   (a) Compute $h_{f,i}$ in every pipe of L ($h_{f,i} = K_i q_i^2$ or other headloss model).
   (b) Form the oriented sum $\sum s_i h_{f,i}$ and denominator $\sum |h_{f,i}/q_i|$.
   (c) Update loop flows with $q_i \leftarrow q_i + \delta q\, s_i$ using the formula above. For pipes shared by two loops, apply both corrections with the proper signs.
3. **Convergence test:** After sweeping all loops, if all $|\delta q|$ are below tolerance (and loop headloss sums are near zero), stop; otherwise repeat Step 2.

EXERCISES

1. PRESSURE DROP ††

Water $\nu = 1.12 \times 10^{-6}\,\mathrm{m^2/s}$ flows from the basement to the second floor through the 1.9 cm diameter copper pipe (a drawn tubing with threaded elbows $K_L = 1.5$) at a rate of $Q = 0.76\,\mathrm{L/s}$, flows through a globe valve (with $K_L = 10$) and exits through a faucet of diameter 1.27 cm as shown in Figure 2.1.

Determine the pressure at point (1) if

1. all losses are neglected,
2. the only losses included are major losses, or
3. all losses are included.

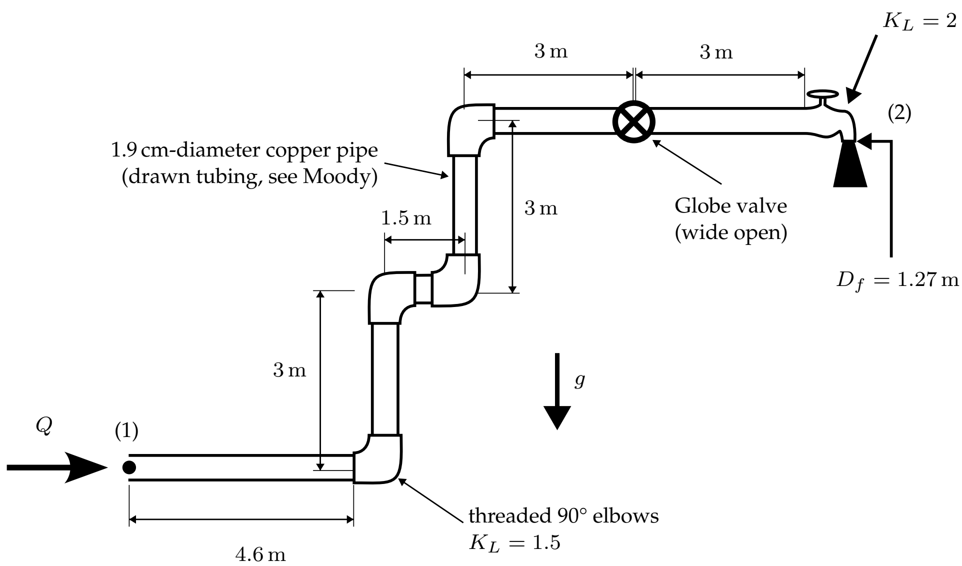


Figure 2.1: Pressure drop through a pipe.

SOLUTION – PRESSURE DROP †

1. The energy equation in the absence of losses is as follows :

$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g}.$$

   Substituting the discharge and the atmospheric pressure (≈ 0),

$$p_1 = \gamma \cdot \left(z_2 - z_1 + \frac{8Q^2}{\pi^2 g} \left[\frac{1}{D_f^4} - \frac{1}{D^4} \right] \right) = 74.2\,\mathrm{kPa}.$$

2. The energy loss is added to the equation :

$$p_1 = \gamma \cdot \left(z_2 - z_1 + \frac{8Q^2}{\pi^2 g} \left[\frac{1}{D_f^4} - \frac{1}{D^4} \right] \right) + h_L \quad \text{with} \quad h_L = f \frac{L}{D} \frac{V_1^2}{2g}.$$

   To compute the friction factor, we can either use the Moody diagram or solve the Colebrook-White equation numerically with $\varepsilon/D = 8 \times 10^{-5}$ (since the rugosity of a drawn tubing is $\varepsilon = 0.0015\,\mathrm{mm}$, see the provided Moody diagram) the and $Re = 45 \times 10^3$). In both cases, $f = 0.0216$. The head loss then has a value of $7.5\,\mathrm{m}$ and the supplied pressure is $p_1 = 148\,\mathrm{kPa}$.

3. Adding minor losses increases the head loss :

$$p_1 = \gamma \cdot \left(z_2 - z_1 + \frac{8Q^2}{\pi^2 g} \left[\frac{1}{D_f^4} - \frac{1}{D^4} \right] \right) + h_L,$$

$$h_L = \left(K_{L,\text{Valve}} + 4K_{L,\text{elbow}} + K_{L,\text{Faucet}} + f \frac{L}{D} \right) \frac{V_1^2}{2g} = 14.1\,\mathrm{m}.$$

   Which needs a pressure $p_1 = 213\,\mathrm{kPa}$.

2. NO MINOR LOSSES †

Air at standard temperature and pressure flows through a horizontal, galvanised iron pipe ($\varepsilon = 0.000152$ m) at a rate of 0.0566 m^3/s. The pressure drop is to be no more than 3.45 kPa per 30.5 m of pipe. Assume the flow is incompressible with $\rho = 1.2266$ kg/m^3 and $\mu = 1.7907 \times 10^{-5}$ Pa·s.

Determine the minimum pipe diameter.

SOLUTION – NO MINOR LOSSES

Note that if the pipe was too long, the pressure drop from one end to the other, $p_1 - p_2$, would not be small relative to the pressure at the beginning, and compressible flow considerations would be required.

For example, if we consider p_1 to be the atmospheric pressure (so, $p_1 = 101$ kPa) a pipe of length 61 m gives

$$\frac{p_1 - p_2}{p_1} = \frac{(3.45\ \text{kPa}/30.5\ \text{m})(61\ \text{m})}{101\ \text{kPa}} = 0.068 = 6.8\%,$$

which is probably small enough to justify the incompressible assumption.

With $z_1 = z_2$ and $V_1 = V_2$, the energy equation becomes:

$$p_1 = p_2 + f\frac{l}{D}\frac{\rho V^2}{2} \tag{1}$$

where $V = Q/A = 4Q/(\pi D^2) = 4(0.0566\ \text{m}^3/\text{s})/\pi D^2 = 0.0721/D^2$, with D in meters. Thus, with $p_1 - p_2 = 3.45$ kPa and $l = 30.5$ m, Eq. (1) becomes:

$$\begin{aligned} p_1 - p_2 &= 3.45\ \text{kPa} \\ &= f\frac{30.5\ \text{m}}{D}(1.23\ \text{kg/m}^3)\frac{1}{2}\left(\frac{0.0721}{D^2}\frac{\text{m}}{\text{s}}\right)^2 \end{aligned}$$

This yields:

$$D = 0.123\, f^{1/5} \tag{2}$$

Also, using the Reynolds number ($\text{Re} = \rho V D/\mu$) and the roughness, we have:

$$\text{Re} = \frac{8.8 \times 10^4}{D} \tag{3}$$

$$\frac{\varepsilon}{D} = \frac{0.000152}{D} \tag{4}$$

Thus, we have four equations (Eqs. (2), (3), (4), and either the Moody chart, the Colebrook equation) and four unknowns (f, D, ε/D and Re) from which the solution can be obtained by trial-and-error methods. If we use the Moody chart, it is probably easiest to assume a value of f, use Eqs. (2), (3), and (4) to calculate D, Re, and ε/D and then compare the assumed f with that from the Moody chart. If they do not agree, try again.

Thus we assume $f = 0.02$, a typical value, and obtain $D = 0.056$ m, which gives $\varepsilon/D = 0.0027$ and $\text{Re} = 8.8 \times 10^4$. From the Moody chart, for these values of ε/D and Re, we get $f = 0.027$, which is different from the value we initially chose. This means we need to redo this process for another value of f.

If we try again for $f = 0.027$, we get $D = 0.060$ m, $\varepsilon/D = 0.0025$, and $\text{Re} = 8.3 \times 10^4$, which in turn give $f = 0.027$, in agreement with the chosen value of f. Thus, the diameter of the pipe should be:

$$D = 0.060\ \text{m}$$

3. MINOR LOSSES †

Water at 15°C ($\nu = 1.21 \times 10^{-6}$ m^2/s) flows from reservoir A to reservoir B through a pipe of length $L = 518$ m and roughness $\varepsilon = 0.000152$ m at a rate of $Q = 0.737$ m^3/s.
The system contains a square-edged entrance, four regular 45° elbows and a submerged exit.

Determine the pipe diameter needed.

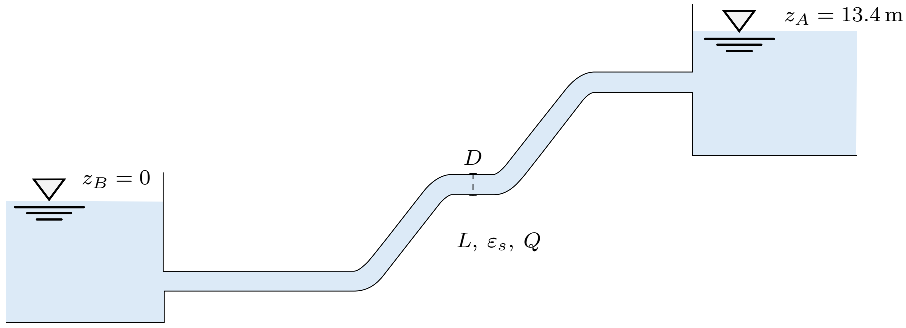


Figure 2.2: Illustration of the problem.

SOLUTION – MINOR LOSSES ††

The energy equation can be applied between two points on the surfaces of the reservoirs ($p_1 = p_2 = V_1 = V_2 = z_2 = 0$) as follows:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$$
$$z_1 = \frac{V^2}{2g}\left(f\frac{\ell}{D} + \sum K_i\right) \tag{1}$$

where we have:

$$V = \frac{Q}{A} = \frac{4Q}{\pi D^2} \tag{2}$$

The loss coefficients are obtained from the table (*Minor loss coefficients for pipe flow (Mays, 2019)*) as $K_{\text{entrance}} = 0.5$, $K_{\text{elbow}} = 0.2$, and $K_{\text{exit}} = 1$. Thus, Eq. (1) can be written as:

$$13.4 = \frac{V^2}{2g}\left(\frac{L}{D}f + \sum_i K_i\right)$$

or, when combined with Eq. (2) to eliminate V,

$$f = f(D) = \frac{\pi^2 g z_1}{8Q^2\ell}D^5 - \frac{D}{\ell}\sum_i K_i \tag{3}$$

Where D and f are the only unknowns.

To determine D we must know f, which is a function of Re and ε/D, such as:

$$\text{Re} = \text{Re}(D) = \frac{VD}{\nu} = \frac{4Q}{\pi D^2}\frac{D}{\nu}$$
$$r = r(D) = \frac{\varepsilon}{D}$$

Where D is the only unknown.

Again, we have four equations (Eqs. (3), (4), (5) and the Moody chart or the Colebrook equation) for the four unknowns D, f, Re, and $r = \varepsilon/D$.

Consider the solution by using the Moody chart. Although it is often easiest to assume a value of f and make calculations to determine if the assumed value is the correct one, with the inclusion of minor losses this may not be the simplest method. For example, if we assume $f = 0.02$ and calculate D from Eq. (3), we would have to solve a fifth-order equation. With only major losses, the term proportional to D in Eq. (3) is absent, and it is easy to solve for D if f is given. With both major and minor losses included, this solution for D (given f) would require a trial-and-error or iterative technique.

Thus, for this type of problem it is perhaps easier to assume a value of D, calculate the corresponding f from Eq. (3), and with the values of Re and ε/D determined from Eqs. (4) and (5), look up the value of f in the Moody chart (or the Colebrook equation).
The solution is obtained when the two values of f are in agreement. A few rounds of calculation will reveal that the solution is given by:

$$D \approx 0.497 \text{ m}$$

4. CAVITATION IN SIPHON †

Siphons are pipes that allow to connect two or more reservoirs winding around obstacles. When they go higher than the highest surface, the siphon will experience low pressures $p = H - z_{\text{max}}$ and might cavitate[1].

Consider a pipe with a diameter $D = 20\,\text{cm}$, a rugosity ε =0.5 mm and the kinematic viscosity of the water ν =1.16 × 10^{-6} m^2/s. What's the maximum head difference ΔH between the reservoirs for which $p/\gamma \geq -10.25\,\text{m}$ throughout the pipe ? What's the corresponding discharge ?

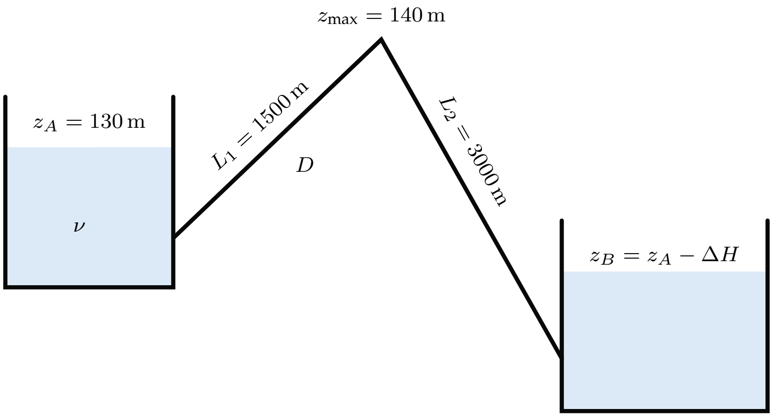


Figure 2.3: Two reservoirs connected by a siphon.

[1] Cavitation is when the the pressure is low enough for the water to vaporize. Here's a cool video introducing the topic: Cavitation - Physics girl.

SOLUTION – CAVITATION IN SIPHON

The energy conservation between A and C gives a relation between V and f:

$$z_A = z_C + \frac{p_C}{\gamma} + \frac{V^2}{2g} + f\frac{L}{D}\frac{V^2}{2g}$$

$$\Rightarrow V = \sqrt{\frac{2g}{1+fL/D}\left(z_A - z_C - \frac{p_C}{\gamma}\right)}$$

Inserting this expression for the speed in the Reynolds number, we have an implicit equation for f.

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon_s}{3.71D} + \frac{2.51\nu}{D\sqrt{\frac{2gf}{1+fL/D}\left(z_A - z_C - \frac{p_C}{\gamma}\right)}}\right).$$

The term p_C/γ being set to −10.25 m, the friction factor is $f = 0.0296$. The head loss has a value of 0.249 m and gives a discharge of 4.66 L/s. Finally, $z_B = 129.25\,\mathrm{m}$.

5. AMMONIA IN A COPPER TUBE

Liquid ammonia at -20°C is flowing through a copper tube at a constant mass flow rate of $Q_m = 0.15\,\mathrm{kg/s}$. The density and dynamic viscosity of liquid ammonia are $\rho = 665.120\,\mathrm{kg/m^3}$ and $\mu = 2.361 \times 10^{-4}\,\mathrm{kg/m \cdot s}$. The tube has a diameter of $D = 5\,\mathrm{mm}$ and length $L = 30\,\mathrm{m}$. The roughness of copper tubing is $\varepsilon = 1.5 \times 10^{-5}$ m. Frictional losses occur along the tube. Determine the pressure drop Δp, the head loss h_L, and the pumping power $\dot{W}_{\text{pump}}$ required to overcome these losses.

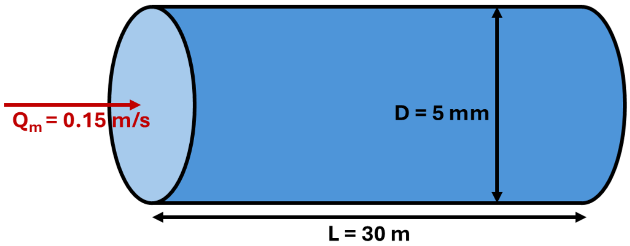


Figure 2.4: Flow of ammonia in a tube.

You can assume a steady, incompressible, and fully developed flow (for which you can neglect the entrance effects). The pipe does not involve any component like bends, valves, and connectors, nor work devices such as pumps and turbines.

SOLUTION – AMMONIA IN A COPPER TUBE

First we need to calculate the average velocity and the Reynolds number to determine the flow regime.

$$V = \frac{Q_m}{\rho A_c} = \frac{Q_m}{\rho(\pi D^2/4)} = \frac{0.15\,\mathrm{kg/s}}{665.1\,\mathrm{kg/m^3}\cdot[\pi\cdot(0.005\,\mathrm{m})^2/4]} = 11.49\,\mathrm{m/s}$$

$$Re = \frac{\rho V D}{\mu} = \frac{665.1\,\mathrm{kg/m^3}\cdot 11.49\,\mathrm{m/s}\cdot 0.005\,\mathrm{m}}{2.361\times10^{-4}\,\mathrm{kg/m\cdot s}} = 1.618\times10^5$$

Since the Reynolds number is greater than 4000, the flow is turbulent.
The relative roughness of the pipe is $\frac{\varepsilon}{D} = \frac{1.5\times10^{-6}\,\mathrm{m}}{0.005\,\mathrm{m}} = 3\times10^{-4}$

The friction factor can be determined from the Moody chart, but to avoid the reading error, we determine it from the Colebrook equation using an equation solver (or an iterative scheme)

$$\frac{1}{\sqrt{f}} = -2.0\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right) \quad\Rightarrow\quad \frac{1}{\sqrt{f}} = -2.0\log\left(\frac{3\times10^{-4}}{3.7} + \frac{2.51}{1.618\times10^5\sqrt{f}}\right)$$

which results in $f = 0.01819$. Then the pressure drop, the head loss, and the useful pumping power required become

$$\Delta P = \Delta P_L = f\frac{L}{D}\frac{\rho V^2}{2}$$

$$= 0.01819\cdot\frac{30\,\mathrm{m}}{0.005\,\mathrm{m}}\cdot\frac{665.1\,\mathrm{kg/m^3}\cdot(11.49\,\mathrm{m/s})^2}{2}\cdot\frac{1\,\mathrm{kN}}{1000\,\mathrm{kg\cdot m/s^2}}\cdot\frac{1\,\mathrm{kPa}}{1\,\mathrm{kN/m^2}} = 4792\,\mathrm{kPa} \approx 4790\,\mathrm{kPa}$$

$$h_L = \frac{\Delta P_L}{\rho g} = f\frac{L}{D}\frac{V^2}{2g} = 0.01819\cdot\frac{30\,\mathrm{m}}{0.005\,\mathrm{m}}\cdot\frac{(11.49\,\mathrm{m/s})^2}{2\cdot 9.81\,\mathrm{m/s^2}} = 734\,\mathrm{m}$$

$$\dot{W}_{\text{pump}} = \dot{V}\Delta P = \frac{Q_m\Delta P}{\rho} = \frac{0.15\,\mathrm{kg/s}\cdot 4792\,\mathrm{kPa}}{665.1\,\mathrm{kg/m^3}}\cdot\frac{1\,\mathrm{kW}}{1\,\mathrm{kPa\cdot m^3/s}} = 1.08\,\mathrm{kW}$$

Therefore, useful power input in the amount of 1.08 kW is needed to overcome the frictional losses in the tube.

6. DETERMINE THE AIR DUCT DIAMETER

Heated air at 1 atm and 35°C is to be transported in a $L = 150\,\mathrm{m}$ circular plastic duct at a rate of $Q = 0.35\,\mathrm{m}^3/\mathrm{s}$. The density, dynamic viscosity, and kinematic viscosity of air at 35°C are $\rho = 1.145\,\mathrm{kg/m}^3$, $\mu = 1.895 \times 10^{-5}\,\mathrm{kg/(m{\cdot}s)}$, and$\nu = 1.655 \times 10^{-5}\,\mathrm{m}^2/\mathrm{s}$.

The head loss in the pipe should not exceed $h_L = 20\,\mathrm{m}$. Determine the minimum diameter of the duct to achieve that. Also verify that the flow is turbulent.
You can assume that the flow is steady and incompressible and the entrance effects are negligible (hence, the flow is also fully developed). The duct involves no components such as bends, valves, and connectors.

SOLUTION

This is a problem of the third type since it involves the determination of diameter for a specified flow rate and head loss. We can solve this problem with three different approaches:

1. An iterative approach by assuming a pipe diameter, calculating the head loss, comparing the result to the specified head loss, and repeating calculations until the calculated head loss matches the specified value.
2. Writing all the relevant equations (leaving the diameter as an unknown) and solving them simultaneously using an equation solver.
3. Using the Swamee–Jain formula.

Following the first approach, the average velocity and the Reynolds number can be expressed as:

$$V = \frac{Q}{A_c} = \frac{0.35\,\mathrm{m^3/s}}{\pi D^2/4}$$

$$Re = \frac{VD}{\nu} = \frac{VD}{1.655 \times 10^{-5}\,\mathrm{m^2/s}}$$

Since the pipe is plastic, the roughness is approximately zero, so (using the Haaland's explicit formula) we get:

$$\frac{1}{\sqrt{f}} = -2.0\,\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right) = -1.8\,\log\left(\frac{6.9}{Re}\right)$$

The head loss is

$$h_L = f\frac{L}{D}\frac{V^2}{2g}$$

and is specified as

$$20 = f\,\frac{150\,\mathrm{m}}{D}\,\frac{V^2}{2(9.81\,\mathrm{m/s^2})}.$$

To start the iteration we choose an initial value for the diameter and then proceeding with the calculation of the velocity, Reynolds number, friction, and head loss. Based on the obtained results, we increase or decrease the value of the diameter. For the friction coefficient, you can
We assign an initial value of of $D = 0.2\,\mathrm{m}$. We obtain $V = 11.15\,\mathrm{m/s}$, $Re = 134743$, $f = 0.0167$, and $h_L = 79.36\,\mathrm{m}$. Since the loss is too high, we need to increase the diameter.

We set $D = 0.25\,\mathrm{m}$. We obtain $V = 7.13\,\mathrm{m/s}$, $Re = 107706$, $f = 0.0175$, and $h_L = 27.71\,\mathrm{m}$. The loss has largely decreased but is still larger than it should be. The next increase should be smaller than the previous one.

We set $D = 0.27\,\mathrm{m}$. We obtain $V = 6.11\,\mathrm{m/s}$, $Re = 99727$, $f = 0.0178$, and $h_L = 18.82\,\mathrm{m}$. Since the loss is smaller than the maximum acceptable value, we can stop here and keep the value of $D = 0.27\,\mathrm{m}$.

If we wanted to slightly reduce the diameter for optimization purposes, we could try setting $D = 0.265\,\mathrm{m}$. In this case we obtain $V = 6.35\,\mathrm{m/s}$, $Re = 101676$, $f = 0.0177$, and $h_L = 20.59\,\mathrm{m}$. Now the loss is slightly larger than the maximum acceptable value, so we are very close to the value for which we get $h_L = 20\,\mathrm{m}$.

Proceeding with the iteration we would find that this is achieved for $D = 0.267\,\mathrm{m}$, which produces $V = 6.25\,\mathrm{m/s}$, $Re = 100848$, $f = 0.01779$.

Given that $Re \gg 4000$, we can state that the flow is turbulent.

7. Pipes fueled by a tank †

From reservoir D, water must be supplied for irrigation to farms F1 and F2. The minimum flow rates to be supplied are 500 L/s and 200 L/s, respectively, with a minimum pressure of 50 m.w.c. (meters of water column) at both points.

By concession, the maximum flow rate that can leave reservoir D is 850 L/s.

Determine:

1. The commercial diameter that the BF2 pipe should have (available commercial diameters: 100, 125, 150, 200, 300, 400, 500, 700, 800, 900, 1000 mm) and the actual flow rate that will circulate through the pipe with the selected commercial diameter. To start, you can assume that Q_1 and Q_2 are at their minimum value.
2. For the selected commercial diameter of BF2, find the maximum and minimum levels that the water in reservoir D must reach so that the maximum and minimum operating flow conditions are satisfied, and calculate their values.
3. Draw the energy gradient lines for each pipe under both operating conditions.

	ϕ (mm)	L (m)	f
AB	800	10000	0.025
BF1	600	5000	0.025
BF2	?	3000	0.025

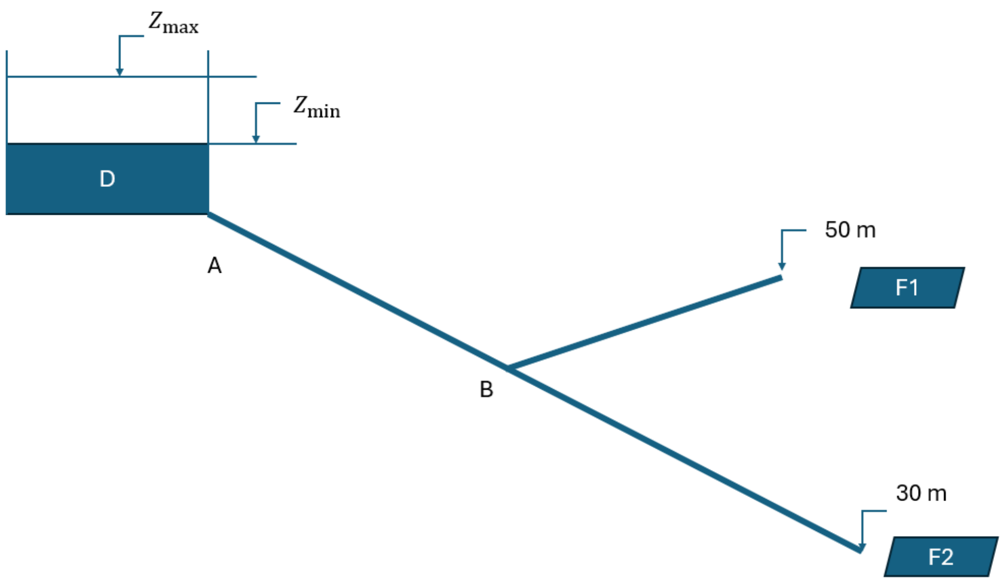


Figure 2.5: Tank fueling the two pipes.

SOLUTION – PIPES FUELED BY A TANK

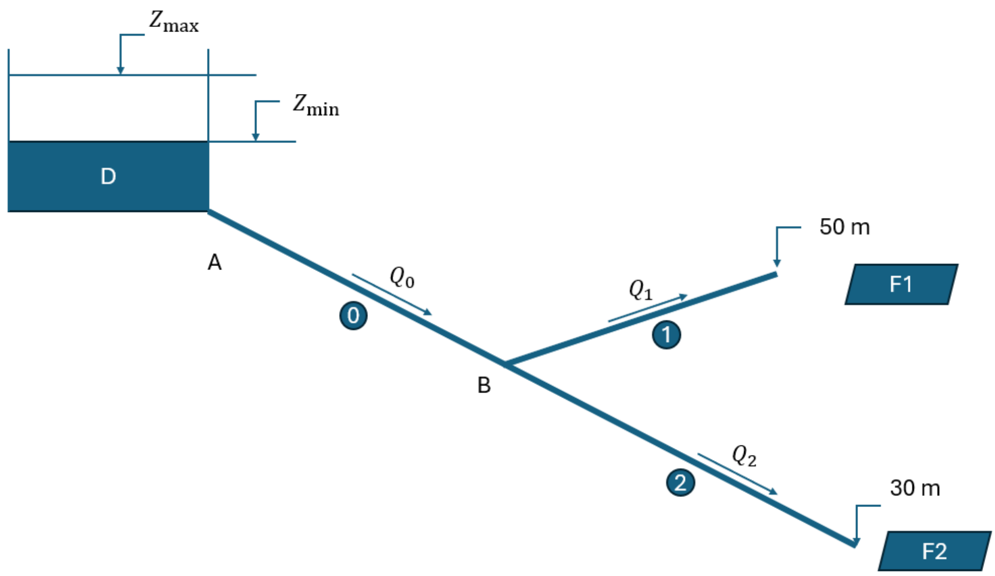


Figure 2.6: Tank fueling the two pipes.

Using the notation as in Fig. 2.6, we know that:

$$Q_0 < 850\ \mathrm{L/s} \qquad \frac{P_1}{\gamma} \geq 50\,\mathrm{m\,a\,s\,l}$$
$$Q_1 > 500\ \mathrm{L/s} \qquad \frac{P_2}{\gamma} \geq 50\,\mathrm{m\,a\,s\,l}$$
$$Q_2 > 200\ \mathrm{L/s}$$

1. Using $\Delta H = \beta \frac{Q_i^2}{D_i^5} L_i$, with $\beta = \frac{8 \cdot 0.025}{\pi^2 \cdot g} = 2.06 \cdot 10^{-3}$, we get:

$$\Delta H_0 = 2.06 \cdot 10^{-3} \cdot \frac{Q_0^2}{0.8^5} \cdot 10000 = 62.86 \cdot Q_0^2$$
$$\Delta H_1 = 2.06 \cdot 10^{-3} \cdot \frac{Q_1^2}{0.6^5} \cdot 5000 = 132.46 \cdot Q_1^2$$
$$\Delta H_2 = 2.06 \cdot 10^{-3} \cdot \frac{Q_2^2}{D_2^5} \cdot 3000 = \frac{6.18}{D_2^5} \cdot Q_2^2$$

- Applying Bernoulli between tank D and F1:

$$z + 0 + 0 = 50 + \frac{P_{F1}}{\gamma} + \frac{V_{F1}^2}{2g} + \Delta H_0 + \Delta H_1$$
$$z = 100 + \frac{V_{F1}^2}{2g} + \Delta H_0 + \Delta H_1 \qquad (1)$$

Then, between tank D and F2:

$$z + 0 + 0 = 30 + \frac{P_{F2}}{\gamma} + \frac{V_{F2}^2}{2g} + \Delta H_0 + \Delta H_2$$
$$z = 80 + \frac{V_{F2}^2}{2g} + \Delta H_0 + \Delta H_2 \qquad (2)$$

By equating (1) and (2), we obtain the following:

$$100 + \frac{V_{F1}^2}{2g} + \Delta H_0 + \Delta H_1 = 80 + \frac{V_{F2}^2}{2g} + \Delta H_0 + \Delta H_1$$
$$20 + \frac{V_{F1}^2}{2g} + \Delta H_1 = \frac{V_{F2}^2}{2g} + \Delta H_2 \qquad (3)$$

For the smallest z, we get the smallest discharges: $Q_1 = 0.500\ \mathrm{m^3/s}$ and $Q_2 = 0.200\ \mathrm{m^3/s}$. Then:

$$V_{F1} = \frac{Q_1}{S_1} = \frac{0.500}{\frac{0.6^2}{4}\pi} = 1.77\ \mathrm{m/s} \Rightarrow \frac{V_{F1}^2}{2g} = 0.16\ \mathrm{m/s}$$
$$V_{F2} = \frac{Q_2}{S_2} = \frac{0.200}{\frac{D_2^2}{4}\pi} = \frac{0.25}{D_2^2} \Rightarrow \frac{V_{F2}^2}{2g} = \frac{0.0033}{D_2^4}$$

Substituting everything into (3), we get $D_2 = 0.341$ m. The commercial diameter to choose is then $D_2 = 400$ mm. We choose a diameter larger than what is necessary so that $Q_0 = Q_1 + Q_2 < 850\ \mathrm{L/s}$.

Using this new value of $D_2 = 400$ mm in equation (3) and $Q_1 = 0.500\ \mathrm{m^3/s}$, we find $Q_2 = 0.297\ \mathrm{m^3/s}$ and $Q_0 = 0.797\ \mathrm{m^3/s}$. These are the new minima values for Q_1, Q_2 and Q_0 as the discharge of Q_2 cannot be lower when $D_2 = 0.400$ mm.

2. To find the minimum value of z, we can use equations (1) and (2) with minimum values of Q_1 and Q_2:

From (1): $z = 173.1\,\mathrm{m}$
From (2): $z = 173.2\,\mathrm{m}$

- The minimal head required in the tank D is thus 173.09 m a s l.

Then, to find the maximum value, we use the maximum value of Q_0, $Q_0 = Q_1 + Q_2$ and equation (3). By finding the values of Q_1 and Q_2 in that case, we can obtain z.

$$Q_1 = 0.539\ \mathrm{m^3/s}$$
$$Q_2 = 0.311\ \mathrm{m^3/s}$$
$$z = 183.9\,\mathrm{m}$$

- The energy gradient lines are represented in red in Fig. 2.7. In order to draw them, the ΔH need to be computed for $z = z_{max}$ and $z = z_{min}$.

$$z_{min} = 173.1\,\mathrm{m};\ \Delta H_0 = 39.9\,\mathrm{m}$$
$$\Delta H_1 = 33.1\,\mathrm{m}$$
$$\Delta H_2 = 53.2\,\mathrm{m}$$

$$z_{max} = 183.9\,\text{m}; \Delta H_0 = 45.4$$
$$\Delta H_1 = 38.5\,\text{m}$$
$$\Delta H_2 = 58.4\,\text{m}$$

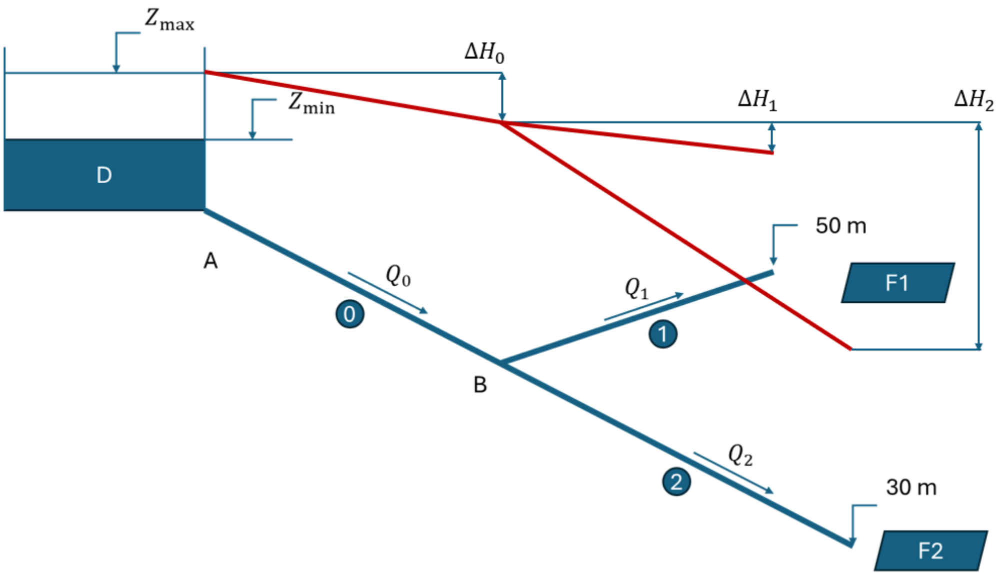


Figure 2.7: Energy gradient lines (red). This figure is when $z = z_{max}$

8. Hardy cross †

Solve the following network using the Hardy Cross method, but instead of the Darcy–Weisbach equation (which gives the head loss as proportional to Q^2), apply the Hazen–Williams equation. This empirical relation expresses the head loss in pressurized pipes as

$$h_f = \frac{10.7L}{C_{HW}^{1.852} D^{4.87}} Q^{1.852}$$

where h_f is the head loss (m), L is the pipe length (m), D is the internal diameter (m), Q is the discharge (L/s), and C_{HW} is the Hazen–Williams roughness coefficient. Accordingly, the discharge correction in the Hardy Cross method changes to

$$\Delta Q = \frac{-\sum h_f}{1.852 \sum \frac{h_f}{Q}}$$

instead of the Darcy–Weisbach formulation with exponent 2. Use a Hazen–Williams factor of $C_{HW} = 100$.

Pipe	L	D
1	305 m	150 mm
2	305 m	150 mm
3	610 m	200 mm
4	457 m	150 mm
5	153 m	200 mm

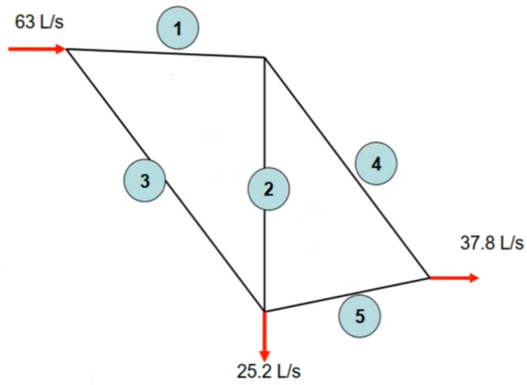


Figure 2.8: Pipeline system.

SOLUTION – HARDY COSS

1. Water is supplied and withdrawn from junction nodes only.
2. Discharge entering the system is equal to discharge leaving the system:

$$\sum Q_{\text{in}} = \sum Q_{\text{out}}$$

3. For long pipe systems, neglect minor losses.
4. Assume flow for each individual pipe network.
5. For any closed loop, the sum of head loss must equal zero:

$$\sum h_f = 0$$

Clockwise flows in a loop are considered positive. Counterclockwise flows are considered negative.

6. To balance head around each loop, flow rate correction, ΔQ, for each loop in the network is computed:

$$\Delta Q = \frac{-\sum h_f}{1.85 \sum \dfrac{h_f}{Q}}$$

7. Discharge is adjusted and another iteration is carried out until summation of head loss approximates to zero:

$$\sum h_f \approx 0$$

The flow in common pipes in two loops is *positive in one loop* and *negative in the other loop*. Pipes in common loops receive both corrections, such that:

For the first loop:	$\Delta Q = \Delta Q_1 - \Delta Q_2$
For the second loop:	$\Delta Q = \Delta Q_2 - \Delta Q_1$

To compute the head losses, we use the Hazen Williams equation:

$$h_f = \frac{10.7L}{C_{HW}^{1.852} D^{4.87}} Q^{1.852}$$
$$h_f = KQ^{1.852}$$

By continuity, we have:

$$\begin{aligned} 63\ \text{L/s} &= Q_1 + Q_3 \\ Q_1 &= Q_2 + Q4 \\ Q_3 + Q_2 &= 25.2\ \text{L/s} + Q_5 \\ Q_4 + Q_5 &= 37.8\ \text{L/s} \end{aligned}$$

FIRST ITERATION

For the first iteration, ensuring that continuity is respected, we assume, in m^3/s:

$$\begin{aligned} Q_1 &= 0.024 \\ Q_2 &= 0.0114 \\ Q_3 &= 0.039 \\ Q_4 &= 0.0126 \\ Q_5 &= 0.0252 \end{aligned}$$

Table 2.1: First iteration results.

Loop	**Pipe**	**Diameter (m)**	**Length (m)**	**K**	**Q (m^3/s)**	h_f	h_f/Q	$\Delta Q \times 10^3$	$Q_{corrected}$ **(m^3/s)**
1	1	0.15	305	6639	0.024	6.64	276	-0.23	0.02377
	2	0.15	305	6639	0.0114	1.67	146	$+0.35$	0.01175
	3	0.20	610	-3271	-0.039	-8.04	206	-0.23	-0.03923
Iteration 1									
2	2	0.15	305	6639	-0.0114	-1.67	146	-0.35	-0.01175
	4	0.15	457	9947	0.0126	3.02	239	-0.58	0.01202
	5	0.20	153	820.5	-0.0252	-0.89	36.0	-0.58	-0.02578

After computing h_f for each pipe, we have:

For the first loop: $h_f = 6.64 + 1.67 - 8.04 = 0.27$

$$\Rightarrow \frac{h_f}{Q} = 276 + 146 + 206 = 628$$

For the second loop: $h_f = -1.67 + 3.02 - 0.89 = 0.46$

$$\Rightarrow \frac{h_f}{Q} = 146 + 239 + 36.0 = 421$$

Then, with these values, we can compute the corrections ΔQ:

$$\begin{aligned} \Delta Q_1 &= \frac{-0.27}{1.85 \times 628} = -0.23 \times 10^{-3} \\ \Delta Q_2 &= \frac{-0.46}{1.85 \times 421} = -0.58 \times 10^{-3} \end{aligned}$$

For the first loop: $\Delta Q = -0.23 \times 10^{-3} + 0.58 \times 10^{-3} = 0.35 \times 10^{-3}$

For the second loop: $\Delta Q = -0.58 \times 10^{-3} + 0.23 \times 10^{-3} = -0.35 \times 10^{-3}$

SECOND ITERATION

For the second iteration, we use the corrected values of Q we obtained with the first iteration.

Table 2.2: Second iteration results.

Loop	**Pipe**	**Diameter (m)**	**Length (m)**	**K**	**Q (m^3/s)**	h_f	h_f/Q	$\Delta Q \times 10^3$	$Q_{corrected}$ **(m^3/s)**
1	1	0.15	305	6639	0.02377	6.52	274	-0.13	0.02363
	2	0.15	305	6639	0.01175	1.77	150	-0.05	0.01169
	3	0.20	610	3271	-0.03923	-8.13	207	-0.13	-0.03937
Iteration 2									
2	2	0.15	305	6639	-0.01175	-1.77	150	$+0.05$	-0.01169
	4	0.15	457	9947	0.01202	2.77	230	-0.08	0.01194
	5	0.20	153	820.5	-0.02578	-0.94	36.3	-0.08	-0.02586

Using the same methodology as for the first iteration, we get the following corrections:

$$\Delta Q_1 = \frac{-0.16}{1.85 \times 631} = -0.13 \times 10^{-3}$$
$$\Delta Q_2 = \frac{-0.06}{1.85 \times 416.3} = -0.08 \times 10^{-3}$$

For the first loop: $\Delta Q = -0.13 \times 10^{-3} + 0.08 \times 10^{-3} = -0.05 \times 10^{-3}$

For the second loop: $\Delta Q = -0.08 \times 10^{-3} + 0.13 \times 10^{-3} = 0.05 \times 10^{-3}$

CHAPTER 3

OPEN CHANNEL FLOW

BRIEF SUMMARY

1. INTRODUCTION TO OPEN CHANNEL FLOW (OCF)

An **open-channel flow** has a free surface exposed to atmospheric pressure and is primarily driven by gravity. The flow depth y can vary in space and time, making the analysis more complex than in pressurized pipe flow.

$$Q = AV, \qquad R_H = \frac{A}{P}, \qquad S_0 = \text{bed slope}, \qquad S_f = \text{friction slope}.$$

Q is flow rate, A is cross-sectional area, V is average velocity, P is wetted perimeter, and R_H is the hydraulic radius.

Types of flow (steady and spatial variation):

- Uniform flow (UF): $\partial y/\partial x = 0$. (Depth is constant)
- Gradually Varied Flow (GVF): $\partial y/\partial x \ll 1$. (Depth changes slowly)
- Rapidly Varied Flow (RVF): $\partial y/\partial x$ is large. (Depth changes abruptly)

2. ENERGY AND MOMENTUM BALANCE

The energy equation between sections 1 and 2 (separated by distance ℓ) is:

$$z_1 + y_1 + \frac{V_1^2}{2g} = z_2 + y_2 + \frac{V_2^2}{2g} + h_L.$$

Assuming bed elevation $z_1 - z_2 = S_0\,\ell$ and head loss $h_L = S_f\,\ell$ (where S_f is the average friction slope):

$$y_1 + \frac{V_1^2}{2g} + S_0\,\ell = y_2 + \frac{V_2^2}{2g} + S_f\,\ell.$$

In **uniform flow**, the flow is steady, and $S_0 = S_f$. The balance between gravity and bed shear stress τ_0 yields:

$$\tau_0 = \gamma R_H S_f = \gamma R_H S_0.$$

Empirical formulas for uniform flow:

$$V = C\sqrt{R_H S_0} \qquad \text{(Chézy equation)}$$

$$V = \frac{1}{n} R_H^{2/3} S_0^{1/2} \qquad \text{(Manning equation)}$$

3. Uniform Flow, Normal and Critical Depth

Normal depth y_n is the depth in steady uniform flow ($S_0 = S_f$). From Manning's equation:

$$Q = \frac{1}{n} A\, R_H^{2/3}\, S_0^{1/2}.$$

For a wide rectangular channel ($A = by$, $R_H \approx y$, $q = Q/b$):

$$y_n = \left(\frac{q\, n}{S_0^{1/2}} \right)^{3/5}.$$

Specific Energy E**:** The energy head relative to the channel bed.

$$E = y + \frac{V^2}{2g} = y + \frac{Q^2}{2g\, A^2}.$$

Froude Number F_r**:** The ratio of inertial to gravitational forces.

$$F_r = \frac{V}{\sqrt{g\, D_m}}, \qquad D_m = \frac{A}{T}.$$

D_m is the *hydraulic depth*, and T is the free-surface width.

$F_r < 1$ (subcritical), $\qquad F_r = 1$ (critical), $\qquad F_r > 1$ (supercritical).

The general form for F_r^2 is:

$$F_r^2 = \frac{Q^2\, T}{g\, A^3}.$$

Critical Flow: Occurs at the minimum specific energy E_c where $F_r = 1$. For a **rectangular channel**:

$$y_c = \left(\frac{q^2}{g} \right)^{1/3}, \qquad E_c = \frac{3}{2} y_c.$$

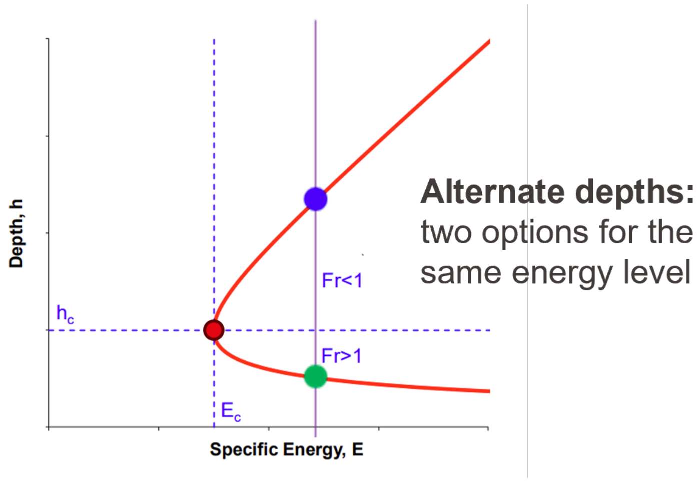


Figure 3.1: Specific Energy curve (E-y) showing subcritical, supercritical, and critical (y_c, E_c) conditions for a constant flow rate q.

4. GRADUALLY VARIED FLOW (GVF)

For a 1D steady flow in a prismatic channel, the differential equation for the water surface profile is:

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F_r^2}.$$

S_f is computed from Manning's equation using the local depth y and velocity V.

For a **wide rectangular channel** (using Manning's $10/3$ exponent):

$$\frac{dy}{dx} = S_0 \frac{1 - (y_n/y)^{10/3}}{1 - (y_c/y)^3}.$$

Steps for hydraulic profile calculations:

1. Compute **normal depth** y_n and **critical depth** y_c for each reach.
2. Identify **control points** where depth is known (e.g., reservoir, gate, weir).
3. Sketch the **qualitative profile**, identifying the profile type (e.g., M1, S2).
4. Follow the direction of information (the "control"):
   - Subcritical flow ($F_r < 1$): Control is downstream; compute profile upstream.
   - Supercritical flow ($F_r > 1$): Control is upstream; compute profile downstream.
5. Compute the **hydraulic profile** numerically (e.g., Direct Step Method).

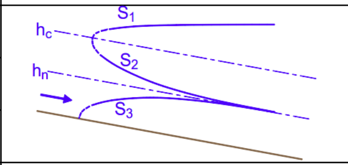
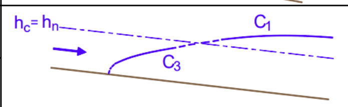
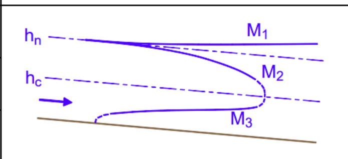
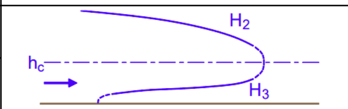
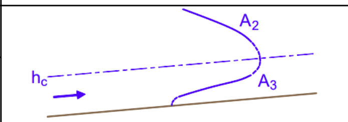
Type	Symbol	Definition	Sketches	Examples
STEEP (normal flow is supercritical)	S1	$h > h_c > h_n$		Hydraulic jump upstream with obstruction or reservoir controlling water level downstream.
	S2	$h_c > h > h_n$		Change to steeper slope.
	S3	$h_c > h_n > h$		Change to less steep slope.
CRITICAL (undesirable; undular unsteady flow)	C1	$h > h_c = h_n$		
	C3	$h_c = h_n > h$		
MILD (normal flow is subcritical)	M1	$h > h_n > h_c$		Obstruction or reservoir controlling water level downstream.
	M2	$h_n > h > h_c$		Approach to free overfall.
	M3	$h_n > h_c > h$		Hydraulic jump downstream; change from steep to mild slope or downstream of sluice gate.
HORIZONTAL (limiting mild slope; $h_n \to \infty$)	H2	$h > h_c$		Approach to free overfall.
	H3	$h_c > h$		Hydraulic jump downstream; change from steep to horizontal or downstream of sluice gate.
ADVERSE (upslope)	A2	$h > h_c$		
	A3	$h_c > h$		

Figure 3.2: Classification of Gradually Varied Flow Profiles (e.g., M1, M2, M3, S1, S2, S3) based on bed slope (Mild, Steep) and depth relative to y_n and y_c.

5. Direct Step Method

This method numerically solves the GVF equation by discretizing it in terms of depth y:

$$\frac{dx}{dy} = \frac{1 - F_r^2}{S_0 - S_f} \quad \Rightarrow \quad \Delta x = \left(\frac{1 - F_r^2}{S_0 - S_f} \right)_{avg} \Delta h.$$

The term in parentheses is evaluated using the average of properties between sections i and $i + 1$.

Algorithm:

1. Define Q, S_0, n, channel geometry, and a known starting depth y_0 at x_0.
2. Choose a step Δy. Set $y_1 = y_0 + \Delta y$.
3. Compute parameters (A, P, R_H, F_r, S_f) at y_0 and y_1.
4. Compute the average value of the slope term and find:

$$\Delta x = \frac{1 - \overline{F_r^2}}{S_0 - \overline{S_f}} \Delta y.$$

5. Update $x_{i+1} = x_i + \Delta x$. Set $y_i = y_{i+1}, x_i = x_{i+1}$ and repeat.
6. Stop when reaching target depth or a transition.

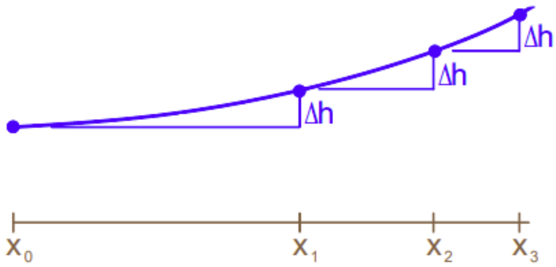


Figure 3.3: Schematic of the Direct Step Method, showing discrete sections i and $i + 1$ along the channel profile.

6. Rapidly Varied Flow: The Hydraulic Jump

A **hydraulic jump** is a rapid transition from supercritical ($F_{r1} > 1$) to subcritical ($F_{r2} < 1$) flow. It conserves momentum but dissipates significant energy.

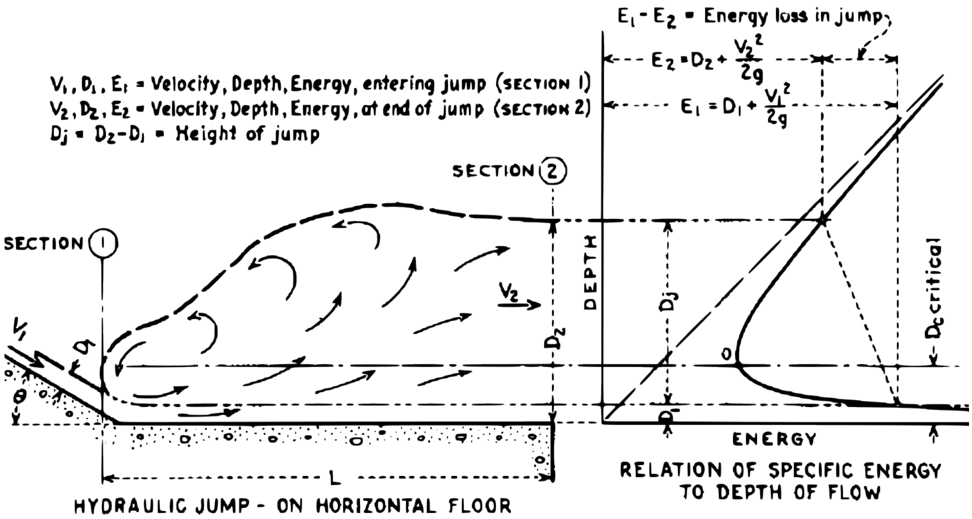


Figure 3.4: Schema of a hydraulic jump, showing conjugate depths (y_1, y_2), the turbulent roller, and the drop in the Energy Grade Line (EGL) due to energy loss ΔE.

Key relations (for a rectangular channel):

$$\frac{y_2}{y_1} = \frac{1}{2}\left(\sqrt{1 + 8\,F_{r1}^2} - 1\right), \qquad F_{r1} = \frac{q}{\sqrt{g}\,y_1^{3/2}}$$

y_1 and y_2 are the **conjugate (or sequent) depths**.

$$\Delta E = E_1 - E_2 = \frac{(y_2 - y_1)^3}{4\,y_1\,y_2} \quad \text{(specific energy loss)}$$

$$L \approx 6y_2 \quad \text{valid for } F_{r1} > 4.5$$

This provides a simple, common estimate of the jump length L.

Algorithm to locate the jump:

1. *Setup:* Define geometry, flow parameters, y_n, and y_c.
2. *Compute GVF branches:* Integrate the GVF equation to get the supercritical profile $y_{\text{super}}(x)$ and the subcritical profile $y_{\text{sub}}(x)$.
3. *Build the conjugate curve:* Transform the supercritical profile into its conjugate depth curve, $y'_{\text{super}}(x) = \text{conjugate}[y_{\text{super}}(x)]$, using the momentum equation above.
4. *Locate the jump:* Find the intersection x_J where the conjugate curve meets the subcritical profile:
$$\text{Find } x_J \text{ such that } y'_{\text{super}}(x_J) = y_{\text{sub}}(x_J).$$
The depths at the jump are: $y_1 = y_{\text{super}}(x_J)$ and $y_2 = y_{\text{sub}}(x_J)$.
5. *Grade Lines:* At x_J, the Energy Grade Line (EGL) drops by ΔE (the energy loss).

Final Summary:

- **Uniform flow** $\rightarrow$ Use Manning's equation to find y_n.
- **Specific energy** $\rightarrow$ Use $F_r = 1$ condition to find y_c.
- **Gradually varied flow** $\rightarrow$ Apply $\dfrac{dy}{dx} = (S_0 - S_f)/(1 - F_r^2)$.
- **Rapid transitions** (jump) $\rightarrow$ Use momentum conservation (conjugate depth formula).

EXERCISES

1. FLOW IN A STRAIGHT CHANNEL †

In a straight channel reach, with width $B = 8\,\mathrm{m}$ and slope $S_0 = 1\,\%$, a discharge $Q = 5\,\mathrm{m}^3/\mathrm{s}$ flows under steady conditions (the discharge per unit width is therefore $q = 0.625\,\mathrm{m}^2/\mathrm{s}$). The roughness coefficient, expressed according to the Gauckler–Strickler[1] formula, is $n = \frac{1}{K_s} = 0.02\,\mathrm{s/m}^{1/3}$.

Determine:

1. the characteristics of the uniform flow y_0, v_0, Fr_0, E_0;
2. the critical depth y_{CR} and the corresponding specific energy E_{CR}.

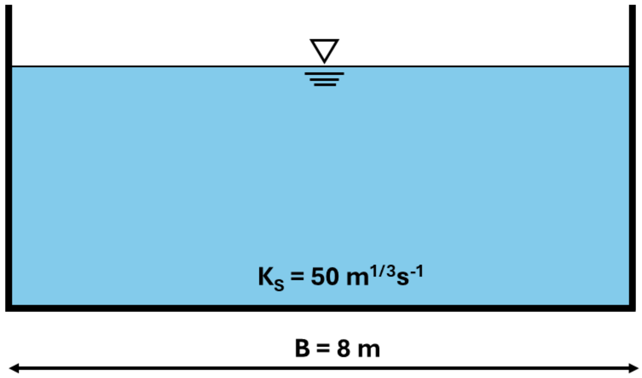


Figure 3.5: Sketch of a straight rectangular channel.

Objectives and guidance

The goal of this exercise is to understand how to determine the main hydraulic characteristics of a steady, uniform open-channel flow. This exercise helps to relate discharge, channel geometry, slope, and roughness to flow depth and velocity, and to distinguish between uniform and critical flow conditions.

Recommended procedure

1. Start by estimating the uniform flow depth using the given discharge, slope, and roughness, typically through an iterative approach.
2. Once the uniform depth is found, compute the corresponding flow velocity and identify the flow regime by evaluating the Froude number.
3. Determine the specific energy associated with the uniform flow conditions.
4. Calculate the critical depth and the corresponding specific energy based solely on the discharge per unit width.
5. Compare the uniform and critical conditions to assess whether the flow is subcritical or supercritical.

[1]The Gauckler-Manning-Strickler (GMS) formula is also known as the Gauckler-Strickler formula, the Manning formula and the Manning-Strickler formula.

SOLUTION

The determination of uniform flow characteristics is fundamental for solving most free-surface flow exercises. The first step consists of calculating the uniform flow depth y_0, which will then allow subsequent determination of other quantities.

This depth is calculated using a reformulation of the Strickler formula, in such a way as to make the depth explicit: an iterative expression is obtained which usually converges in at most 5 iterations:

$$y_0^{n+1} = \frac{nq}{\sqrt{S_0}} \left[\frac{B + 2y_0^n}{By_0^n} \right]^{2/3}$$

Note that the first term of the expression is constant and coincides with the uniform flow depth in the case of an infinitely wide channel ($Rh = y_0$). **The suggested first trial value**, which ensures rapid convergence both in subcritical and supercritical flow, is equal to the value of the uniform flow depth in an infinitely wide channel.

For a very wide channel, the hydraulic radius coincides with the flow depth ($R_h = y$), and the discharge per unit width can be expressed through the Gauckler–Strickler relation:

$$q = \frac{1}{n} y_\infty^{5/3} \sqrt{S_0}.$$

Solving for y gives:

$$y_\infty = \left(\frac{nq}{\sqrt{S_0}} \right)^{3/5}.$$

Substituting the known values $q = 0.625\ \mathrm{m^2/s}$, $K_s = 50\ \mathrm{m^{1/3}/s}$, and $S_0 = 0.01$:

$$y_\infty = \left(\frac{0.625}{50\sqrt{0.01}} \right)^{3/5} = (0.125)^{3/5} \approx 0.287\ \mathrm{m}.$$

Hence, the first trial value for the iterative computation is

$$y_0^{(0)} \approx 0.29\ \mathrm{m}.$$

Once we've iterated enough times to converge to a solution, the second step is to calculate the velocity v_0, expressed as the ratio between discharge per unit width and the uniform depth. The Froude number Fr_0 is then obtained, useful for distinguishing between slow and fast flows depending on whether its value is less than or greater than unity. Finally, the specific energy E_0 is computed:

$$v_0 = \frac{q}{y_0}, \quad Fr_0 = \frac{v_0}{\sqrt{gy_0}}, \quad E_0 = y_0 + \frac{v_0^2}{2g},$$

$$y_0 = 0.30\,\mathrm{m}, \quad v_0 = 2.1\,\mathrm{m/s}, \quad Fr_0 = 1.2, \quad H_0 = 0.52\,\mathrm{m},$$

The depth and specific energy under critical conditions, often very useful in solving more complex exercises, are functions only of the discharge per unit width:

$$y_{CR} = \left(\frac{q^2}{g} \right)^{1/3}, \quad E_{CR} = \tfrac{3}{2} y_{CR}$$

$$y_{CR} = 0.3415\,\mathrm{m}, \quad E_{CR} = 0.5123\,\mathrm{m}$$

2. TRAPEZOIDAL CHANNEL †

A trapezoidal channel has a base width $b = 5\,\mathrm{m}$, side slopes with an angle of 45°, a Manning roughness $n = 0.025\,\mathrm{s/m^{1/3}}$ and a slope $S = 0.17\,\%$. Its cross section is drawn in Figure 3.6.

1. Express $A(y)$, $P(y)$ and $R_H(y)$.
2. Compute the discharge if the normal depth is $y_n = 2\,\mathrm{m}$
3. Compute the normal depth if the discharge is $Q = 100\,\mathrm{m^3/s}$
4. Express the specific energy and find the depth for which it is minimized. What is this water depth, how is it called ? What is its value for $Q = 100\,\mathrm{m^3/s}$?
5. An engineer informs you that the flow speed should be above 2 m/s to ensure sediment transport, what is the minimal required water depth?.

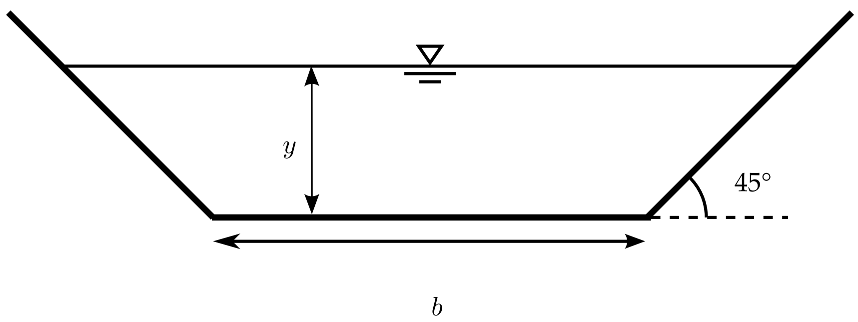


Figure 3.6: Cross-section of a trapezoidal channel

Objectives and guidance

This exercise focuses on expressing the normal depth and deriving the Froude number for a simple cross-section (and can easily be extended to any cross-section). It relies on a prismatic channel and a flow in normal conditions, showing that the computation of the discharge under such conditions is straight-forward while looking for flow depths is more challenging.

Recommended procedure

1. Express the geometric properties of the trapezoidal section: top width, wetted perimeter, cross-sectional area, and hydraulic radius as functions of the water depth.
2. For a known normal depth, compute the discharge using Manning's equation.
3. If the discharge is given, solve Manning's equation numerically (or iteratively) to find the normal depth.
4. Express the specific energy as a function of depth and discharge, then find the depth at which it is minimized. This defines the critical depth.
5. For a specified minimum flow velocity, set up the velocity–depth relation and solve numerically to determine the minimum water depth that satisfies the condition.

SOLUTION

1. The top width $B(y)$ varies linearly with depth. with a slope $2/\tan(45°) = 2$:

$$B(y) = b + 2y.$$

The wetted perimeter follows a similar trend:

$$P(y) = b + 2\sqrt{2}y.$$

And the wetted area is:

$$A(y) = \frac{1}{2}(B(y) + b) \cdot y = (b + y)y.$$

Thus, the hydraulic radius is:

$$R_h(y) = \frac{A(y)}{P(y)} = \frac{y(b + y)}{b + 2\sqrt{2}y}$$

2. Manning's law applies to normal conditions. Therefore, we can use it to estimate the discharge from the normal depth directly:

$$Q = \frac{1}{n}[A(y_n)/P(y_n)]^{2/3} A(y_n)\sqrt{S} \tag{3.1}$$

$$= \frac{1}{n}\left[\frac{(b + y_n)y_n}{b + 2\sqrt{2}y_n}\right]^{2/3} (b + y_n)y_n\sqrt{S} \tag{3.2}$$

$$= 27.7\,\mathrm{m}^3/\mathrm{s} \tag{3.3}$$

3. Solving for y_n more complicated, we have to solve Equation 3.2 numerically (see the tutorial in the additional resources) to get

$$y_n = 3.99\,\mathrm{m}.$$

4. The specific energy is $E = y + \frac{V^2}{2g}$. Rewriting the average speed in terms of discharge and wetted area:

$$E = y + \frac{Q^2}{2gA(y)^2}.$$

Since this function is convex, its minimum is found where its derivative is null. Let's first compute the derivative of the specific energy.

$$\frac{\partial E}{\partial y} = 1 - \frac{Q^2}{gA(y)^3}\frac{\partial A}{\partial y} = 1 - \frac{Q^2 B(y)}{gA(y)^3},$$

Since $\partial A/\partial y$ is simply the top width B. This defines the Froude number.

$$\Rightarrow \frac{Q^2 B(y)}{gA(y)^3} = \left(\frac{V(y_c)}{\sqrt{gA(y_c)/B(y_c)}}\right)^2 = \mathrm{Fr}^2 = 1 - \frac{\partial E}{\partial y} \tag{3.4}$$

The minimum of the specific energy (found at $\partial E/\partial y = 0$) defines the critical depth y_c:

$$\mathrm{Fr} = \frac{V(y_c)}{\sqrt{gA(y_c)/B(y_c)}} = 1 \tag{3.5}$$

Applying it to the trapezoidal section, we get:

$$1 = \frac{Q^2(b + 2y_c)}{g(b + y_c)^3 y_c^3}$$

Whose solution is

$$y_c = 2.83\,\mathrm{m}.$$

5. This is the same problem as question 3 where the speed is imposed instead of the discharge. We seek to solve

$$
\begin{aligned}
V = \frac{Q}{A(y_n)} &= \frac{1}{n} R_H(y_n)^{2/3}\sqrt{S}, \\
&= \frac{1}{n}\left[\frac{(b+y_n)y_n}{b+2\sqrt{2}y_n}\right]^{2/3}\sqrt{S},
\end{aligned}
$$

Which is also an implicit equation. Its solution is

$$y = 2.04\,\mathrm{m}.$$

3. SIMPLE STEP ††

A channel with a width of 1 m has a singularity, as shown in Figure 3.7. Draw an E vs y diagram with the different flow depths corresponding to the singularity in Figure 3.7. Assume that $Q = 5\ \mathrm{m^3/s}$, that the step height is $a = 30\ \mathrm{cm}$, and that there is only one local energy loss of $h_s = 0.15\ \mathrm{m}$ between sections (2) and (3).

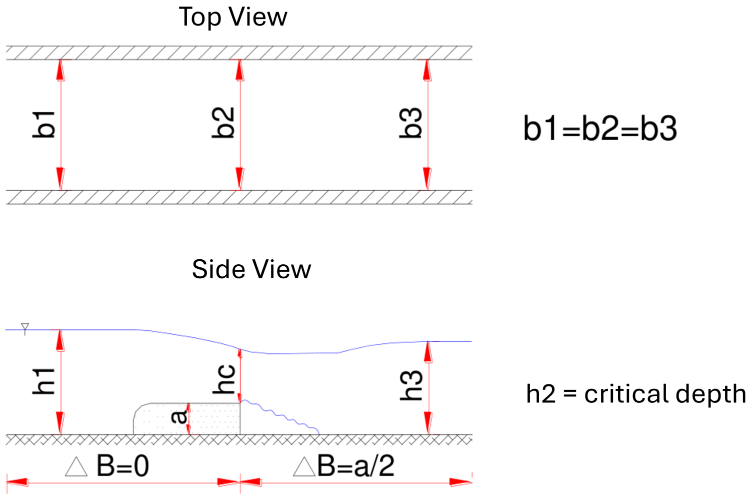


Figure 3.7: Step installed in a rectangular channel of constant width.

Objectives and guidance

This exercise focuses on understanding specific energy in open-channel flow, how a bed step modifies the flow regime, and how the energy equation with bed elevation changes and local energy losses can be used to determine the flow depths at different sections. Students should use the specific energy relationship to calculate the critical, upstream, and downstream flow depths, and represent the results on an E–y diagram.

Recommended procedure

1. Determine the critical depth from the discharge and channel geometry.
2. Apply the specific energy balance between sections (1) and (2), accounting for the step height, to find the upstream flow depth(s).
3. Apply the energy balance between sections (2) and (3), including the specified local loss h_s, to determine the downstream depth(s).
4. Plot the E–y diagram, the line $E = y$, and indicate the relevant points for each section.

SOLUTION

To analyze the effect of a bed step on the flow in a rectangular channel, we use the concept of **specific energy** and apply the **energy equation** between different sections of the channel.

Given data:

- Channel width: $b = 1\ \mathrm{m}$
- Discharge: $Q = 5\ \mathrm{m^3/s}$
- Step height: $a = 0.30\ \mathrm{m}$
- Local energy loss between sections (2) and (3): $h_s = 0.15\ \mathrm{m}$

The analysis involves four main steps:

1. Determine the critical depth.
2. Relate the upstream flow depth (section 1) to the critical section (section 2).
3. Relate the critical section to the downstream section (3), including the local loss.
4. Represent all the results on the E–h diagram.

1. CRITICAL DEPTH

The critical depth corresponds to the minimum specific energy for a given discharge. It is obtained by imposing the critical flow condition:

$$Fr^2 = 1 \quad \Longleftrightarrow \quad \frac{Q^2 b}{g\, A^3} = 1$$

For a rectangular channel, $A = b\, h$ and $b = 1\ \mathrm{m}$, so:

$$\frac{Q^2}{g\, b^2\, y_c^3} = 1 \quad \Longrightarrow \quad y_c = \sqrt[3]{\frac{Q^2}{g\, b^2}}$$

Substituting the numerical values:

$$y_c = \sqrt[3]{\frac{25}{9.8 \times 1^2}} = 1.366\ \mathrm{m}$$

This critical depth will occur just upstream of the steps (section 2), as the abrupt bed elevation change forces the flow to adjust to a critical state.

2. UPSTREAM DEPTH (SECTION 1)

Between sections (1) and (2) there are no local losses. Applying the energy equation:

$$E_1 + z_1 = E_2 + z_2$$

Taking $z_1 = 0$ as reference and $z_2 = a = 0.30\ \mathrm{m}$, and knowing that section (2) is critical:

$$y_1 + \frac{Q^2}{2gb^2y_1^2} = \underbrace{y_c + \frac{Q^2}{2gb^2y_c^2}}_{\frac{3}{2}y_c} + a$$

Substituting:

$$y_1 + \frac{25}{2 \times 9.8\, y_1^2} = 1.366 + \frac{25}{2 \times 9.8 \times 1.366^2} + 0.3 = 2.349\,\mathrm{m}$$

Multiplying through by y_1^2:

$$y_1^3 - 2.349\, y_1^2 + 1.2755 = 0$$

This is a cubic equation, which typically has two positive real roots: one corresponding to supercritical flow (shallow depth, high velocity) and another to subcritical flow (deeper depth, lower velocity). Both solutions can be found iteratively with a first initial value below y_c and a second one above y_c.

The solutions are:

$$y_1 = \begin{cases} 0.957\ \mathrm{m} & \text{(supercritical / torrential)} \\ 2.044\ \mathrm{m} & \text{(subcritical / fluvial)} \\ -0.65\ \mathrm{m} & \text{(non-physical)} \end{cases}$$

Depending on the upstream boundary conditions, the flow may adopt either the supercritical or subcritical depth.

3. DOWNSTREAM DEPTH (SECTION 3)

Between sections (2) and (3), a local energy loss of $h_s = 0.15$ m must be included. The energy equation reads:

$$E_2 + z_2 = E_3 + z_3 + h_s$$

The downstream bed elevation is the same as at section 1, $z_3 = 0$. Therefore:

$$y_c + \frac{Q^2}{2gb^2y_c^2} + a = y_3 + \frac{Q^2}{2gb^2y_3^2} + h_s$$

Substituting numerical values:

$$1.366 + \frac{25}{2 \times 9.8 \times 1.366^2} + 0.3 = y_3 + \frac{25}{2 \times 9.8\, y_3^2} + 0.15$$

Solving this equation gives:

$$y_3 = \begin{cases} 1.056\ \mathrm{m} & \text{(supercritical / torrential)} \\ 1.810\ \mathrm{m} & \text{(subcritical / fluvial)} \\ -0.667\ \mathrm{m} & \text{(non-physical)} \end{cases}$$

Again, two possible flow regimes are possible downstream of the step, depending on the hydraulic control conditions.

4. E–y DIAGRAM

The specific energy values for each section are:

$$E_1 = y_1 + \frac{Q^2}{2gb^2y_1^2} = 2.349 \text{ m}$$
$$E_2 = 2.049 \text{ m}$$
$$E_3 = 2.199 \text{ m}$$

Plotting these points on the E–y diagram allows us to clearly identify the two possible depths (supercritical and subcritical) for each specific energy level, as well as the critical depth.

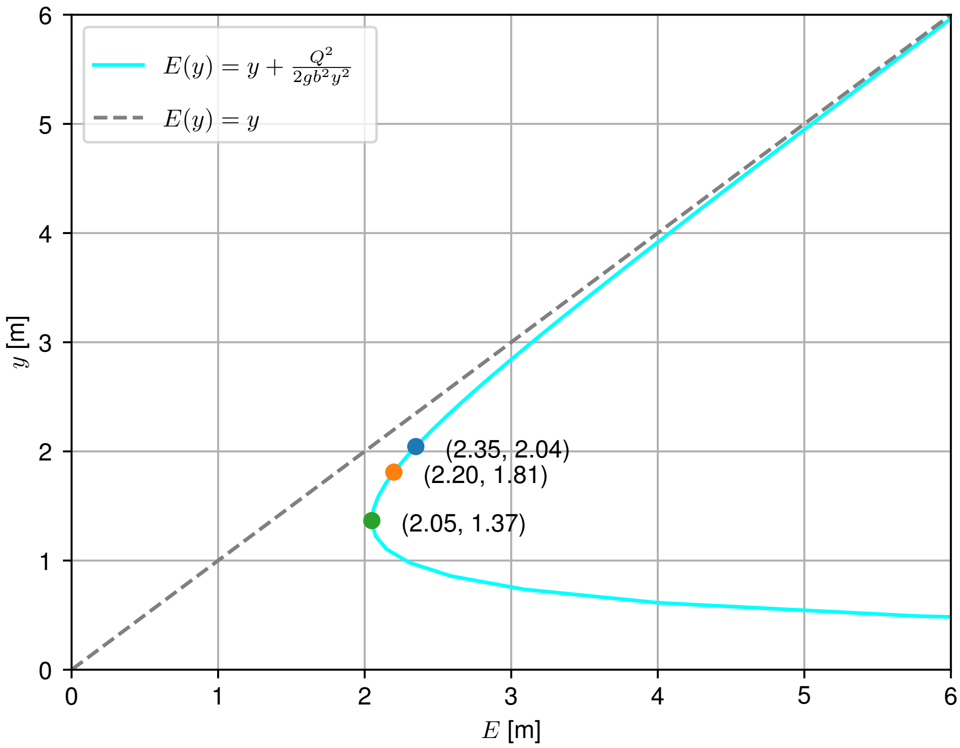


Figure 3.8: Specific energy vs. flow depth diagram. The cyan curve represents the energy–depth relation, the dashed black line corresponds to $E = y$, and the red points indicate the computed solutions at the upstream, critical, and downstream sections.

This diagram is a powerful tool to visualize how the bed step and energy losses affect the flow regime, and why multiple depths are possible for the same discharge.

4. SEA LEVEL VARIATION

A very wide river that discharges its waters into the sea has a discharge per unit width of $q = 5\ \text{m}^2/\text{s}$, a bed slope of $i = 0.0005$, and a roughness coefficient of $n = 0.03$.

As shown in Figure 3.9, after the river mouth it can be assumed that the seabed has an 8% slope. The tidal level may vary from elevation $+0.5$ m up to $+4.5$ m. Determine the normal and critical depths in the river before the mouth, and perform a qualitative analysis of the hydraulic profile in the river under the extreme tidal conditions.

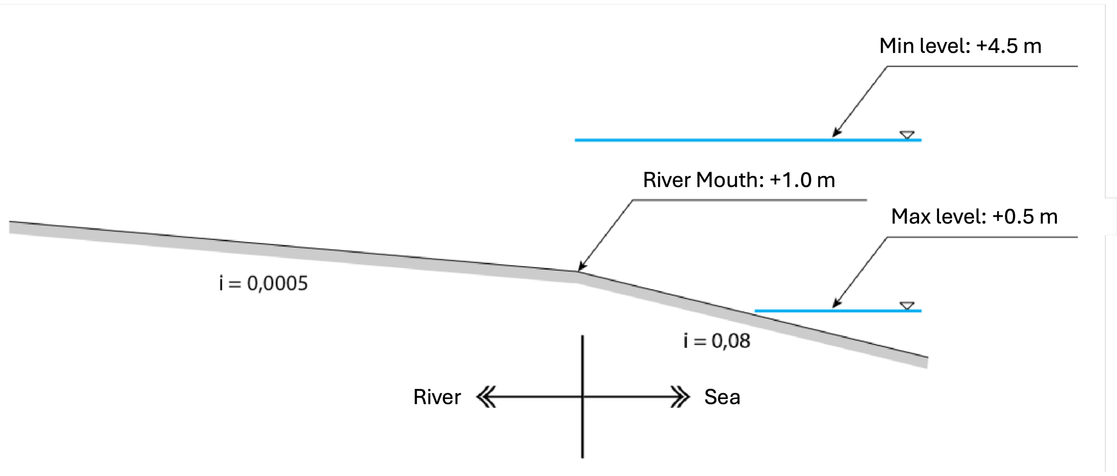


Figure 3.9: Schematic representation of the river–sea system.

Recommended procedure

1. Compute y_c via the Froude number and y_n via the Chézy or Manning equation.

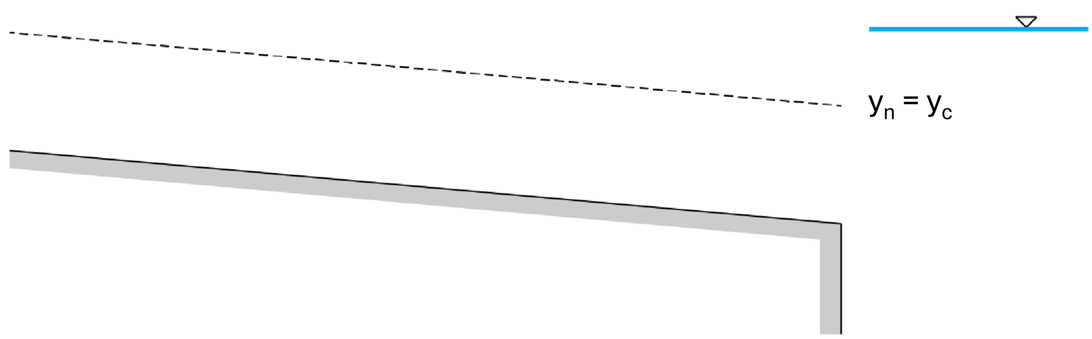
2. Determine the control point(s) (CP) which correspond to where the regime changes (eg. change of slope). When drawing the qualitative profiles, each CP will be the starting point.

3. Determine the direction in which the drawing will be done, starting from the CP.
   If $y > y_c$, the flow is subcritical meaning that you move upstream from the CP.
   If $y < y_c$, the flow is supercritical meaning that you move downstream from the CP.

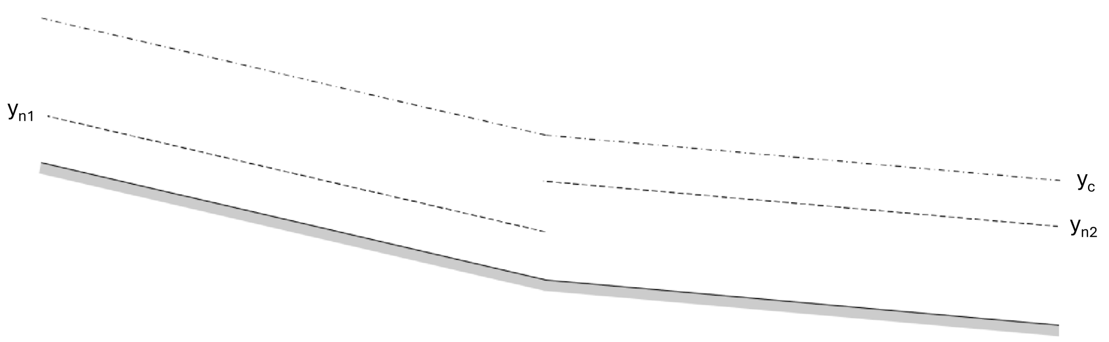
5. QUALITATIVE PROFILES

In the following cases, perform a qualitative analysis of the water-surface profile (hydraulic grade line), identifying the classification (type), control points, and starting points together with their computation direction. Draw your sketches on the diagrams provided below.

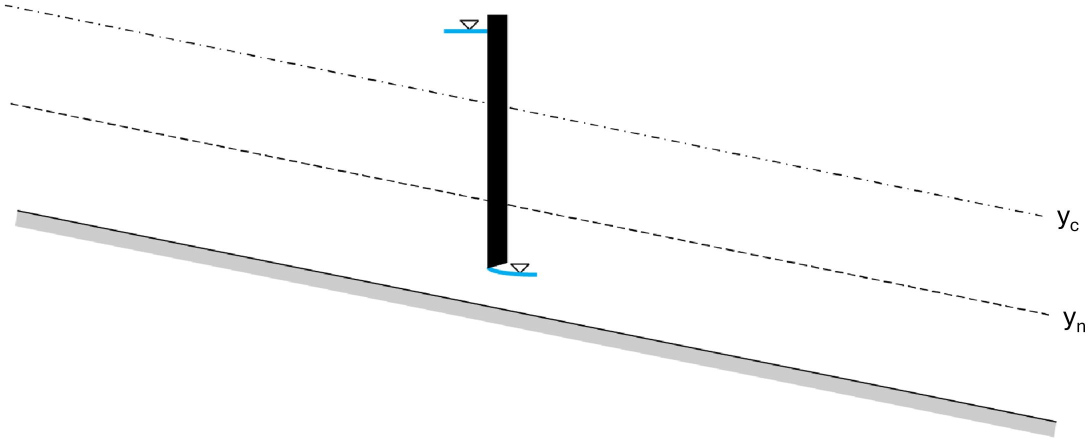
(a)



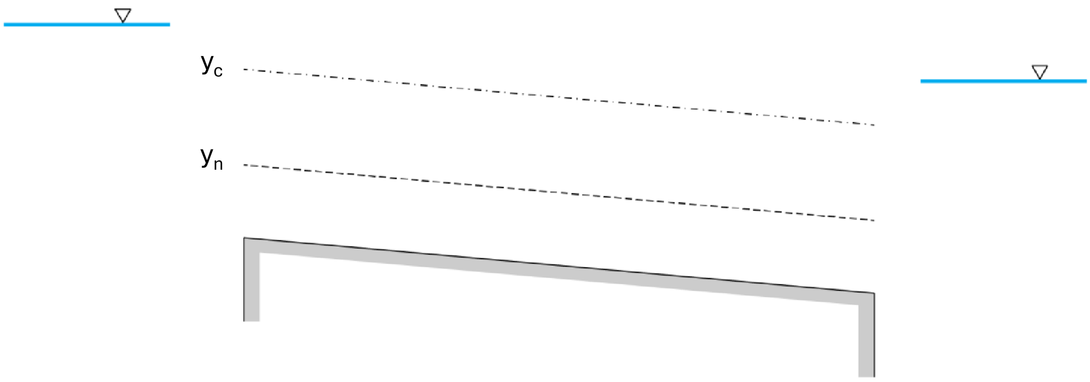
(b)



(c)



(d)



6. CHANNEL WITH A BED SLOPE CHANGE

A channel of width $B = 0.6\,\mathrm{m}$, in which a steady discharge $Q = 1\,\mathrm{m}^3/\mathrm{s}$ flows, presents along its course a change in bed slope. Consider two reaches, upstream and downstream of the slope change, with constant and infinitely long geometry. The geometric characteristics of the two reaches are reported in the table.

	B [m]	n [m$^{1/3}$/s]	S_0 [%]
Reach 1	0.600	0.025	8
Reach 2	0.600	0.025	1

Determine:

1. the position of the possible hydraulic jump,
2. and qualitatively reconstruct the profile.

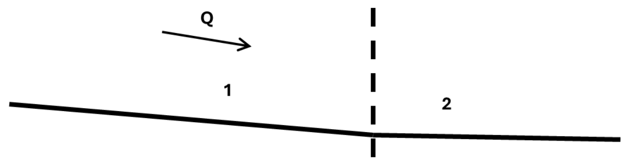


Figure 3.10: Sketch of a channel with a sudden slope change.

7. SPILLWAY INTO STILLING BASIN

The spillway of a reservoir has an approach channel with a rectangular cross-section with a (base of $10\,\text{m}$, side slopes of $1/1$ (H/V)) that ends in a weir of height $a = 2\,\text{m}$.

There is a spillway with a rectangular cross-section after the weir with same width of $10\,\text{m}$, dropping 20 m and ending in a stilling basin where a hydraulic jump is supposed to form.

Downstream of the basin, a step of 4 m discharges the water into a river with a steep slope.

a) Draw the profile qualitatively, noting the supercritical or subcritical states. What happens when the step's height c is small? What happens when it is big? Where are the main head losses?

b) Estimate the water depth y_2 above the weir and the corresponding Froude number.

c) Compute the discharge when the water depth y_1 in the approaching channel is of 5 m.

d) Assuming regular losses along the spillway are small, compute the water depth y_3 at the end of the spillway and the corresponding Froude number.

e) From now on, we assume the step is big. Compute the water depth y_4 after the hydraulic jump.

f) What should be the water depth y_5 over the step? Estimate the value of the necessary step height c to force a hydraulic jump in the stilling basin. Assume the energy losses on the step are negligible.

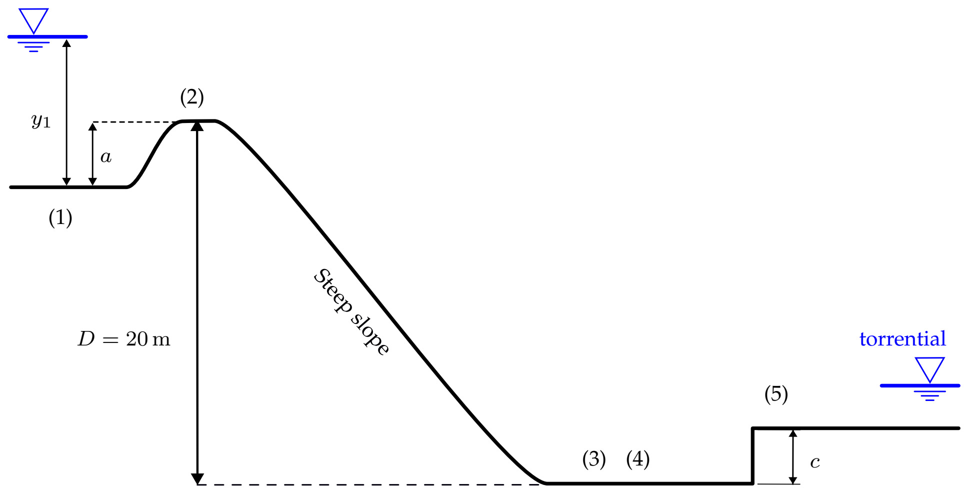


Figure 3.11: Schema of the spillway.

Objectives and guidance

This exercise focuses on using the concepts of specific energy, critical flow, the solving for water depth from a known head (iterative process) and considering head losses around a hydraulic jump. Combining these points, the hydraulic profile can be drawn for this exercise.

8. STEP IN A CHANNEL

A channel of width $B = 2\,\mathrm{m}$, with a bed slope equal to 0.2% and a Strickler roughness coefficient $K_S = 50\,\mathrm{m}^{1/3}\mathrm{s}^{-1}$, has a well-shaped bottom step of variable height a, as reported in the table. A discharge $Q = 3\,\mathrm{m}^3/\mathrm{s}$ flows in the channel under steady conditions. Head losses due to the step can be neglected.

Height	a_1	a_2	a_3
[m]	0.1	0.3	0.8

Determine:

1. the characteristic sections of the hydraulic jump,
2. and qualitatively reconstruct the profile in the three configurations.

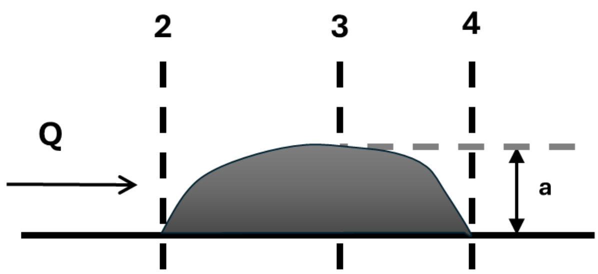


Figure 3.12: A step placed on the bottom of the channel.

9. CHANNEL WITH A BED SLOPE AND BED ROUGHNESS CHANGE

A channel of width $B = 10\,\text{m}$, assumed infinitely long both upstream and downstream, presents at section A–A a change of bed slope and roughness as indicated in the figure. At section A–A, where the channel slope changes from subcritical to supercritical, the specific energy of the flow with respect to the bed is $H_A = 2.4\,\text{m}$.

Determine:

1. the depth y_A that occurs at section A–A;
2. the specific discharge q flowing in the channel;
3. the uniform-flow depth upstream y_{0u} and downstream y_{0d};
4. determine whether the uniform-flow regime in each reach is subcritical or supercritical.

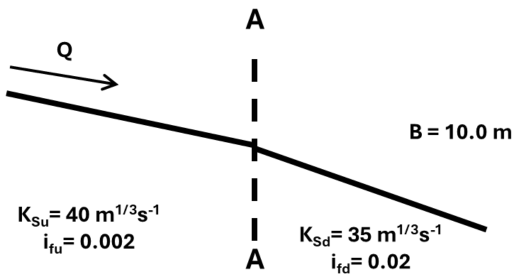


Figure 3.13: Sketch of a channel with a sudden slope and bed roughness change.

10. CHANNEL WITH A SHARP-CRESTED WEIR

In a rectangular channel, infinitely long and with width $B = 4$ m, the slope is $i_f = 0.0003$ and the bed roughness according to Strickler is $K_S = 30\ \mathrm{m}^{1/3}\mathrm{s}^{-1}$. Inside the channel, a sharp-crested weir is placed, characterized by a width $b = 3.5$ m and a crest height $p = 2.8$ m. The head acting on the weir is $h_0 = 0.25$ m. The kinetic head contribution on the weir itself is neglected.

Determine:

1. the discharge Q flowing in the channel,
2. the uniform flow depth y_0 and the flow regime under these conditions,
3. sketch qualitatively the free surface profile, specifying the type of gradually varied flow profiles that occur.

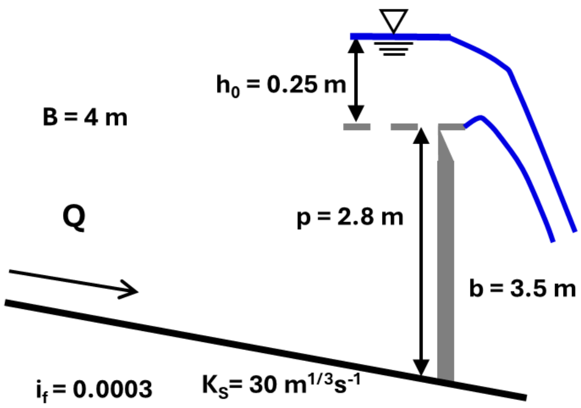


Figure 3.14: Configraation of a sharp-crested weir in a channel.

11. RESERVOIR WITH CONSTANT LEVEL

In a channel of width $B = 8$ m, infinitely long upstream, the flow discharges into a reservoir maintained at a constant level. This level, equal to $y_V = 1.0$ m above the channel bed at its downstream section, coincides with the critical depth of the flow arriving from upstream.

Determine:

1. the discharge Q flowing through the channel,
2. the value of the critical slope i_{CR},
3. the uniform flow depth y_0' for the channel when a discharge $Q' = 0.3Q$ flows with bed slope equal to the previously calculated i_{CR},
4. sketch the free surface profile inside the channel for both configurations.

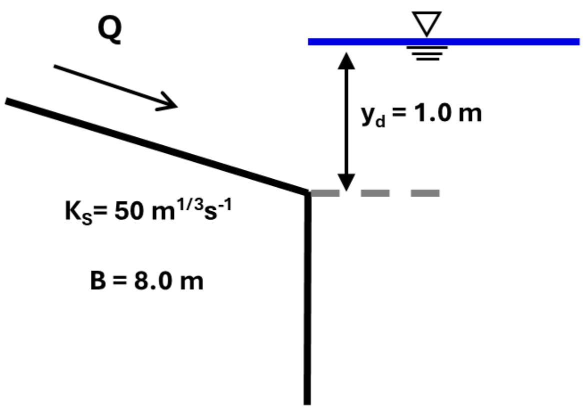


Figure 3.15: Channel discharging in a reservoir with constant level.

12. PRISMATIC CHANNEL

A prismatic channel with the symmetric cross-section shown in Fig. 3.16 carries water at a rate of 1.5 m^3/s. Manning's n may be taken as 0.02 m$^{-1/3}$s and the streamwise slope is 0.1%.

Find:

1. the normal depth (relative to the lowest point),
2. the Froude number at the normal depth,
3. the critical depth at this flow rate.

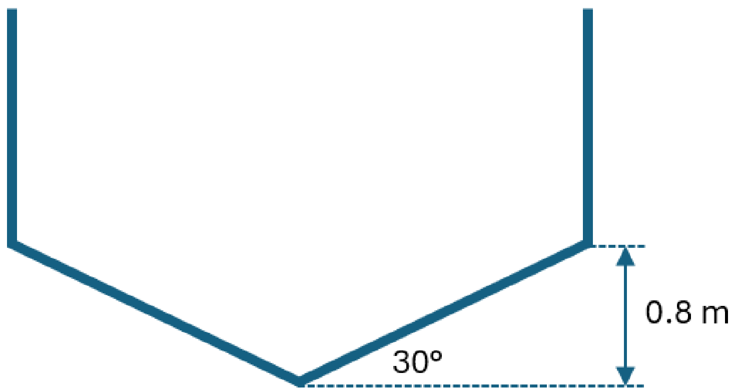


Figure 3.16: Cross-section of the prismatic channel.

Objectives and guidance:

This exercise focuses on expressing the normal and critical depths for a known cross-section.

Recommended procedure

First try to determine if the normal and critical flow depths lie above or below the top of the vee. Note that one can lie above while the other could lie below.
For the normal depth: compute the flow rate for a water depth of 0.8 m.
For the critical depth: compute the Froude number for a depth of 0.8 m and a flow rate of $Q = 1.5$ m^3/s.

13. CHANNEL WITH A GATE

A long rectangular channel of width 2.2 m, streamwise slope 1:100 and Chézy coefficient 80 $m^{1/2}/s$ carries a discharge of 4.5 m^3/s.

1. Find the normal depth and critical depth and show that the slope is steep at this discharge.
2. An undershot sluice gate causes a hydraulic transition in this flow. The depth of parallel flow downstream of the gate is 0.35 m. Find the depth immediately upstream of the gate and sketch the flow.
3. Using 2 steps in the gradually-varied-flow equation, find the distance between the gate and the hydraulic jump.

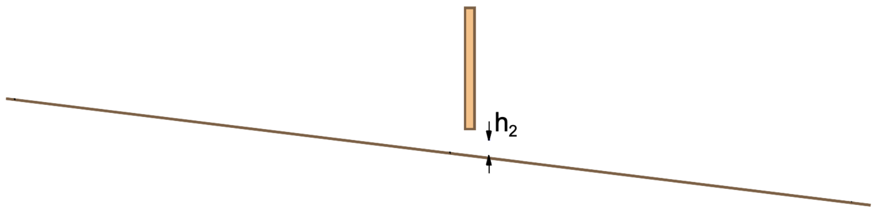


Figure 3.17: Illustration of the problem.

SOLUTION

We know the following parameters:

$$b = 2 \text{ m} \qquad S_0 = 0.01$$
$$C = 80 \text{ m}^{1/2}/\text{s} \qquad Q = 4.5 \text{ m}^3/\text{s}$$

Question 1

For the normal depth, we can use the continuity equation coupled with the Chézy equation,

$$Q = VA$$
$$V = C\sqrt{R_h S}$$

where $A = bh$ and $R_h = \frac{bh}{b+2h}$.

Hence,

$$Q = C\Big(\frac{bh}{b+2h}S\Big)^{1/2} bh \Leftrightarrow Q = C(bh)^{3/2}\Big(\frac{S}{b+2h}\Big)^{1/2}$$
$$\Leftrightarrow h = \Big(\frac{Q}{CbS^{1/2}}\Big)^{2/3}(1+2h/b)^{1/3}$$

To find the normal depth, we must replace S by S_0 as, under normal conditions, they are equal. This gives:

$$h_n = 0.4028(1 + 0.9091 h_n)^{1/3}$$

We start iterating at $h_{n,start} = 0.4028$ to get the final value of the normal depth:

$$h_n = 0.4518 \text{ m}.$$

To compute the critical depth, we can use the critical depth equation for rectangular open-channel flow:

$$h_c = \Big(\frac{q^2}{g}\Big)^{1/3} = 0.7526 \text{ m}$$

where $q = \frac{Q}{b}$.

The normal depth is supercritical since $h_n < h_c$ (i.e. shallower and faster than the case $Fr = 1$, which means its Froude number must be > 1). Hence, by definition, the slope is "steep" at this discharge.

Question 2

We know that $h_2 = 0.35$ m. To find h_1, we impose that the head is equal on both side of the sluice gate.

$$z_{s1} + \frac{V_1^2}{2g} = z_{s2} + \frac{V_2^2}{2g}$$
$$h_1 + \frac{Q^2}{2gb^2h_1^2} = h_2 + \frac{Q^2}{2gb^2h_2^2}$$
$$\Rightarrow h_1 = 2.090 - \frac{0.2132}{h_1^2}$$

Iterating from $h_{1,start} = 2.090$ m, we obtain:

$$h_1 = 2.039 \text{ m}.$$

The depth far upstream (h_n) is supercritical ($h_n < h_c$) and the one right before the gate (h_1) is subcritical ($h_1 > h_c$). To join these two depths, there must be a hydraulic jump done. After the jump, a S1 profile is followed up to the gate.

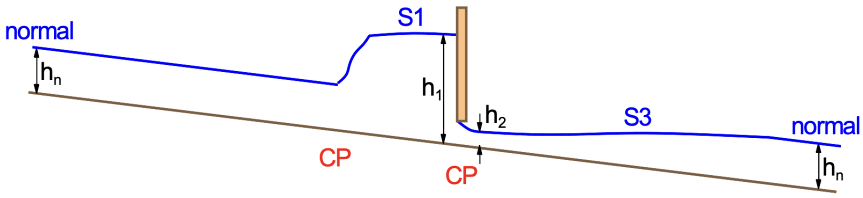


Figure 3.18: Qualitative profile of the flow.

Question 3

Upstream, since the channel is long, the jump goes from normal depth to the corresponding sequent depth in the hydraulic jump. To find the sequent depth (h_J), we use the rapidly varied flow formula:

$$\begin{aligned} h_J &= \frac{h_n}{2}\Big(-1 + \sqrt{1 + 8Fr_n^2}\Big) = 1.166 \text{ m} \\ \text{with: } h_n &= 0.4518 \text{ m} \\ V_n &= \frac{Q}{bh_n} = 4.527 \text{ m/s} \\ Fr_n &= \frac{V_n}{\sqrt{gh_n}} = 2.150 \end{aligned}$$

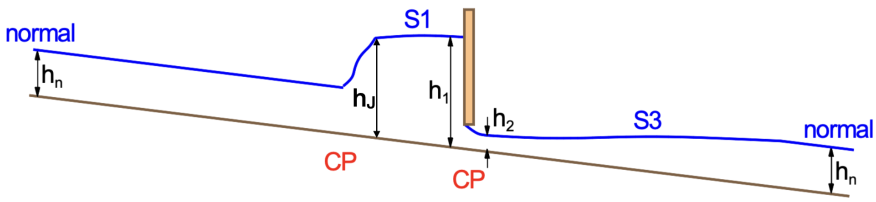


Figure 3.19: Qualitative profile of the flow with the height h_J just downstream of the jump.

As the gradually varied flow (GVF) is subcritical ($h_1 > h_c$ and $h_J > h_c$), we need to work upstream from $h_1 = 2.039$ m (just upstream of the sluice) to $h_J = 1.166$ m (just downstream of the jump).

The GVF equation gives:

$$\frac{dh}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \Leftrightarrow \frac{dx}{dh} = \frac{1 - Fr^2}{S_0 - S_f}$$

with:

$$S_0 = 0.01$$
$$Fr^2 = \frac{V^2}{gh} = \frac{Q^2}{gb^2h^3} = \frac{0.4263}{h^3}$$
$$S_f = \left(\frac{Q}{Cb}\right)^2 \frac{1+2h/b}{h^3} = 6.537 \times 10^{-4} \frac{1+0.9091h}{h^3}$$

Hence:

$$\frac{dx}{dh} = \frac{1 - \frac{0.4263}{h^3}}{0.01 - 6.537 \times 10^{-4} \frac{1+0.9091h}{h^3}}$$

To compute the horizontal distance between the vertical line of h_J and h_1, we need to compute Δx:

$$\Delta x = \left(\frac{dx}{dh}\right)_{mid} \Delta h$$
$$\text{with: } \Delta h = \frac{h_J - h_1}{2} = \frac{1.166 - 2.039}{2} = -0.4365 \text{ m}$$

Starting from $h = h_1$, we get:

Table 3.1: Finding the horizontal distance between the gate and the end of the hydraulic jump.

i	h_i [m]	x_i [m]	h_{mid} [m]	$(dx/dh)_{mid}$	Δx [m]
a	$h_a = h_1 = 2.039$	0			
			1.821	95.69	-41.77
b	$h_b = 1.6025$	-41.77			
			1.384	88.87	-38.79
c	$h_c = h_J = 1.166$	**-80.56**			

14. SECTION INCREASE

In a rectangular channel section, a constant discharge of 6.5 m^3/s flows. The section widens from 3 m to 5 m, without any loss of energy or change in the bed elevation. In the section that is 5 m wide, the water depth is 1.2 m.

Determine the water depth h_1 in the narrow section.

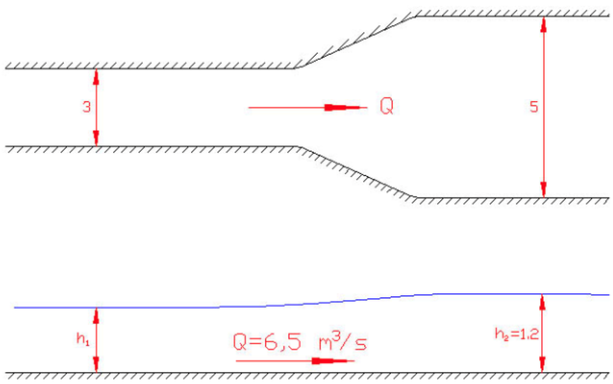


Figure 3.20: Rectangular channel with a section increase.

SOLUTION

Analyze what type of flow occurs in section 2.

$h_{2,c} = 0.467 \cdot q^{2/3} = 0.467 \cdot \left(\frac{6.5}{2}\right)^{2/3} = 0.56\,\text{m}$

$h_2 = 1.2 > 0.56 = h_{2,c}$

Since $h_2 > h_{2,c}$, this section has a non-critical flow regime.

As there are no energy losses at section 1, a fluvial regime must also be verified. (A torrential regime would imply a hydraulic jump.)

By equating $E_1 = E_2$, we get:

$h_1 + \frac{Q^2}{2 \cdot g \cdot b^2 \cdot h_1^2} = h_2 + \frac{Q^2}{2 \cdot g \cdot b^2 \cdot h_2^2}$

$h_1 + \frac{6.5^2}{2 \cdot 9.8 \cdot 2^2 \cdot h_1^2} = 1.2 + \frac{6.5^2}{2 \cdot 9.8 \cdot 2^2 \cdot (1.2)^2}$

$h_1 + \frac{0.2395}{h_1^2} = 1.2598$

This equation has two positive solutions and one negative. By trial and error, we obtain:

$h_1{\prime} = 1.037$ m (fluvial), $h_1{\prime\prime} = 0.598$ m (torrential).

The right solution is $h_1{\prime}$, since there are no energy losses.

The critical height at section 1 is:

$h_{1c} = 0.467 \cdot q^{2/3} = 0.467 \cdot \left(\frac{6.5}{2}\right)^{2/3} = 0.78\,\text{m}$

Which confirms that h_1 is a fluvial height in this section.

15. HYDRAULIC JUMP AT SLOPE CHANGE

A long spillway (slope of 10 % and Manning coefficient 0.025 s/m$^{1/3}$) discharges into a flat stilling basin. There, the water depth is controlled by a step before the water flows into a steep river. A small step height will change the flow lightly, but a larger step will force a subcritical flow further upstream of it and generate a hydraulic jump in the stilling basin. The channel is reactangular with a width of 2 m, has a discharge of 50 m^3/s and the stilling basin is 40 m long. What is the maximum step height before the hydraulic jump affects the spillway?

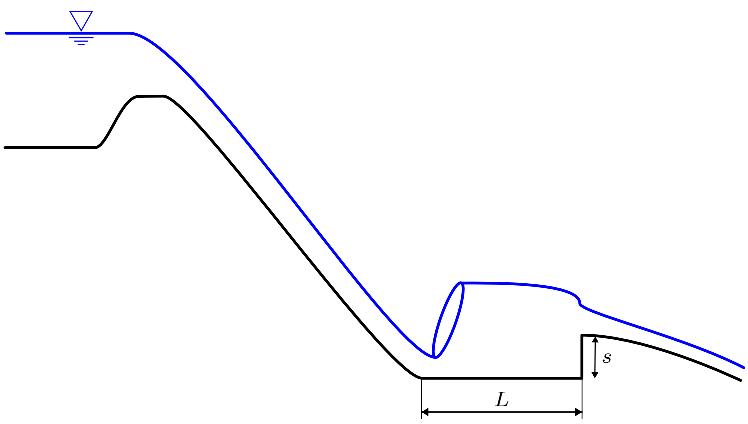


Figure 3.21: Schema of the problem

Objectives and guidance:

The aim of this exercise it to understand the concept of a conjugate height and use it to find the position of a hydraulic jump using the direct-step method.

Hints :

- For the spillway to be free, the hydraulic jump has to happen at the slope change or further downstream. Thus, there is a condition at the slope change: the depth directly downstream is the conjugate of the supercritical depth directly upstream.
- The direct step method can be computed both ways (upstream $\leftrightarrow$ downstream) independently of flow conditions.
- The depth of the step is defined by the jump in energy around the step. Consider that there is no energy loss at the step: $E(x = L - \varepsilon) = s + E(x = L + \varepsilon)$ with ε being a small distance.

SOLUTION

The first step is to compute the critical depth in the stilling basin and the normal depth in the spillway. The assumption of normal flow in the spillway can be made if it is long enough to attain normal conditions.

$$y_{c,\text{basin}} = \left(\frac{Q}{\sqrt{g}b}\right)^{2/3} = 3.99\,\text{m}$$

$$y_{n,\text{spillway}} = \frac{nQ}{R_H^{2/3}b\sqrt{i}} \approx \frac{nQ}{y_{n,\text{spillway}}^{2/3}b\sqrt{i}} \Rightarrow y_{n,\text{spillway}} = 2.48\,\text{m}$$

As the depth after the hydraulic jump should be the conjugate depth of $y_{n,\text{spillway}}$, let it be

$$y_{\text{conjugate}} = \frac{y_{n,\text{spillway}}}{2}\left(\sqrt{1+8F_r^2}-1\right) = 6.04\,\text{m}.$$

The latter is the depth we'll start from at the upstream end of the stilling basin when computing the profile. As the hydraulic jump is set to be at the end of the spillway, we are now looking to find the depth at the position $x = 40\,\text{m}$. It is an M2 profile. First of all, we need to express the Froude number, the hydraulic radius and the friction slope.

The Froude number, the hydraulic radius and the friction slope for a rectangular channel are expressed as follows :

$$F_r = \frac{Q}{\sqrt{gy}by}$$

$$R_H = \frac{by}{b+2y}$$

$$S_f = \frac{n^2V^2}{R_H^{4/3}}$$

The derivative dx/dy comes from the energy balance :

$$\frac{\Delta x}{\Delta y} = \frac{1-F_r^2}{S_0-S_f}$$

The increments Δx are negative since the computation goes from downstream to upstream. The results are shown in Table 3.2 and Figure 3.22.

Once the profile is solved and the depth at $x = 40\,\text{m}$ is interpolated, we can infer the necessary step height by assuming an energy balance between the upstream of the step $E_1 = E(x = 40\,\text{m})$ and the critical flow over the step E_2.

$$E_1 = y_1 + \frac{Q^2}{2gb^2y_1^2} = \frac{3}{2}y_c + s = E_2 + s$$

$$\Rightarrow s = y_1 + \frac{Q^2}{2gb^2y_1^2} - \frac{3}{2}h_c = 0.294\,\text{m}$$

Where y_1 was found at $x = 40\,\text{m}$ by computing the hydraulic grade line from the conjugate depth $y_{\text{conjugate}} = 6.04\,\text{m}$ of the normal depth in the spillway $y_n = 2.48\,\text{m}$ at $x = 0\,\text{m}$.

Table 3.2: Direct-step method to compute the profile in the stilling basin. The step Δy was chosen to attain critical depth in 5 iterations (4 increments): $\Delta y = (y_{\text{conjugate}} - y_{\text{critical}})/4$.

x	dy	y	V	F_r	R_H	S_f	$\Delta x/\Delta y$	$\overline{\Delta x/\Delta y}$	Δx
0.00	-0.51	6.04	4.14	0.54	0.86	0.01	-54.08	-46.55	23.80
23.80	-0.51	5.53	4.52	0.61	0.85	0.02	-39.02	-32.03	16.37
40.17	-0.51	5.02	4.98	0.71	0.83	0.02	-25.04	-18.55	9.48
49.65	-0.51	4.51	5.55	0.83	0.82	0.03	-12.06	-6.03	3.08
52.74		3.99	6.26	1.00	0.80	0.03	0.00		

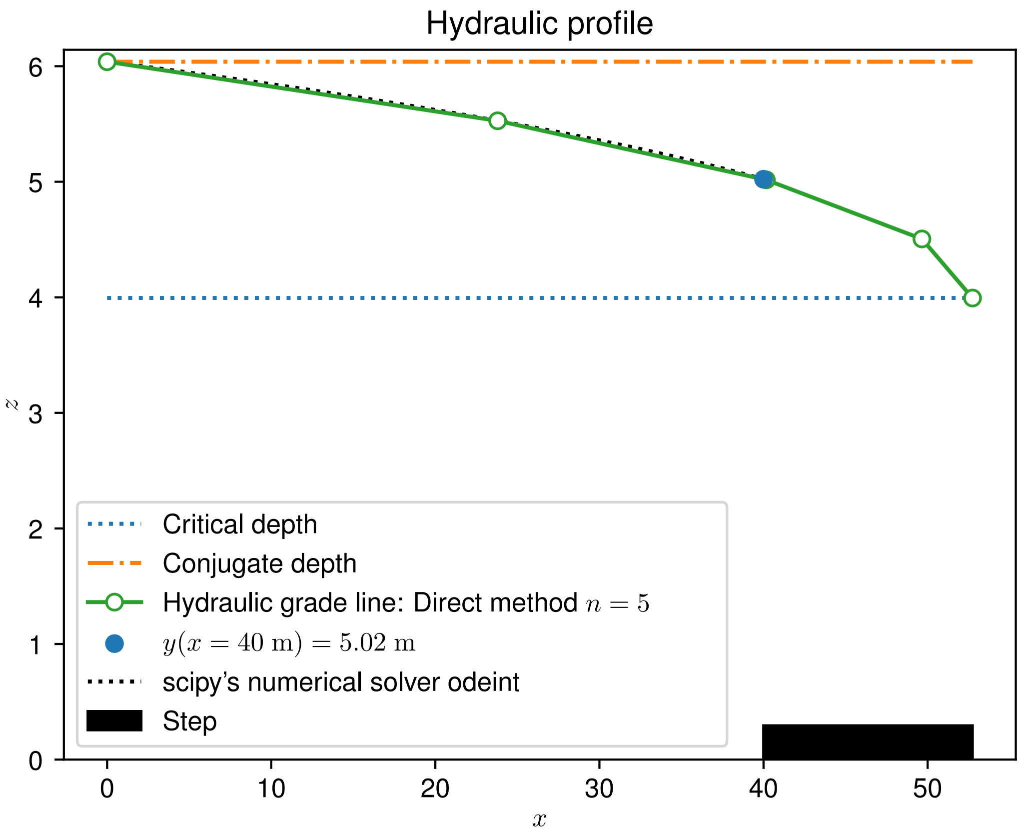


Figure 3.22: Hydraulic profile of the stilling basin.

16. HYDRAULIC JUMP WITH GATE AND WEIR

A group of students is setting up a laboratory experiment to reproduce the formation of a hydraulic jump. They use a flume having a width $b = 0.15$ m, a slope $S_0 = 0.001$, and a Manning coefficient $n = 0.0092$ m$^{-1/3}$s (Plexiglass). In the setup, they place an upstream gate with a constant water level and a constant discharge $Q = 0.011$ m^{3}s^{-1}. The gate vertical opening is $h_{gate} = 0.051$ m, and the minimum water depth reached downstream of the gate (vena contracta) is $y_2 = 0.025$ m. 15 meters downstream of the gate, they also place a $d_{weir} = 0.03$ m long-crested weir where choking occurs. Assume there are no losses at the transition on the weir.

Determine the location of the hydraulic jump x_{jump} and draw the resulting hydraulic profile.

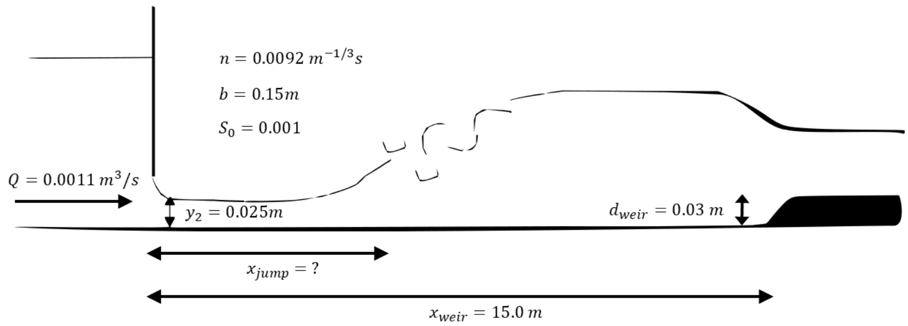


Figure 3.23: Setup of the flume experiment and expected outcome.

SOLUTION

To solve this problem, you should approach it from the two sides. First, compute and draw the profile from the gate opening to downstream, pretending that the long-crested weir is not present. Consequently, compute and draw the conjugate depths that would result in the presence of a hydraulic jump. Similarly, compute and draw the water profile from the long-crested weir to upstream until the gate, assuming that the gate does not affect the flow. To compute these profiles you should use the middle point method for rectangular channels.

Once the three profiles (the one resulting from the gate, its conjugate, and the one resulting from the long-crested weir) are computed and drawn, the location of the hydraulic jump is found on the interpolation between the conjugate and the long-crested weir profiles.

Profile resulting from the gate opening

Before computing the whole profile we should determine the critical and normal flow depths. For that, we first compute the flow velocity at the vena contracta point (minimum depth downstream of the gate y_2, which is the initial point for the computation and drawing of the profile).

Reversing the discharge formula we find

$$v_2 = \frac{Q}{A_2} = \frac{Q}{y_2 b} = \frac{0.011 \text{ m}^3/\text{s}}{0.025 \text{ m} \cdot 0.15 \text{ m}} = 2.87 \text{ m/s}$$

and consequently

$$Fr_2 = \frac{v_2}{\sqrt{g y_2}} = \frac{2.87 \text{ m/s}}{\sqrt{9.81 \text{ m}^2/\text{s} \cdot 0.025 \text{ m}}} = 5.8$$

Therefore, since $Fr > 1$, the flow is supercritical.

We compute the critical depth as

$$y_c = \left(\frac{Q^2}{b^2 g}\right)^{1/3} = \left(\frac{0.011^2 \text{ m}^3/\text{s}}{0.15^2 \text{ m} \cdot 9.81 \text{ m}^2/\text{s}}\right)^{1/3} = 0.082 \text{ m}$$

and the normal depth as

$$y_n = \left(\frac{nQ}{b{S_0}^{1/2}}\right)^{3/5} = \left(\frac{0.0092 \text{ m}^{-1/3}\text{s} \cdot 0.011 \text{ m}^3/\text{s}}{b{S_0}^{1/2}}\right)^{3/5}$$

Consider that for applying this equation we assume that the channel is infinitely large. The result is conservative compared to the equation for real rivers, for which you should solve the equation $nQ - \frac{(b \cdot y_n)^(5/3)}{(b+2y_n)^(2/3)} \cdot \sqrt{S_0} = 0$. Doing in for this exercise returns a value of approximately $y_n = 0.156$ m.

The water depth at the gate outlet is lower than the critical depth, and that confirms that the flow is supercritical. For this reason, when computing the whole profile you should stop at the critical depth. Continuing the computation to the normal depth would be wrong, because when the flow reaches the critical depth, the hydraulic jump occurs.

To compute the profile we use the middle point method. For that, we subdivide the profile in smaller sections. The initial point of the profile is $y_2 = 0.025$ m. The last point of the profile is the critical depth $y_c = 0.082$ m. Hence, the total height variation is $\Delta y = y_c - y_2 = 0.082 \text{ m} - 0.025 \text{ m} = 0.057 \text{ m}$. You are free to choose the step dy, but keep in mind that its value affects the smoothness of the profile: a too large step would make the profile too fragmented; a too small step makes indeed the profile smoother, but would require a longer computation time. For this exercise we suggest to set $dy = 0.01$ m for the first sections. The step of the last section will be slightly different to match the total Δy. In this case, we have five sections with $dy_{1-5} = 0.01$ m, which sum up to $\Delta y_{1-5} = 0.05$ m. Hence, the last step should be $dy_6 = \Delta y - \Delta y_{1-5} = 0.057 \text{ m} - 0.05 \text{ m} = 0.007 \text{ m}$.

Given these steps, we can compute the different water depths as $y_i = y_{i-1} + dy_{i-1}$ with $i = 2, ..., 7$. The various points of the profile are found as $y_i = y_{i-1} + dy_{i-1}$.

Since the profile goes up, for the midpoint method we compute the midpoint water level as $y_{med,i} = y_i + \frac{dy_i}{2}$. As we will see in the following part of the exercise, if the profile is supposed to go down, the midpoint would be computed as $y_{med,i} = y_i - \frac{dy_i}{2}$.

For completing the profile we compute in order:

1. channel section $A_i = b \cdot y_{med,i}$
2. wetted perimeter $P_i = b + 2 \cdot y_{med,i}$
3. hydraulic radius $R_i = \frac{A_i}{P_i}$
4. flow velocity $v_i = \frac{Q}{A_i}$
5. slope of the profile $S_{f,i} = \left(\frac{n \cdot v_i}{{R_i}^{2/3}}\right)^2$
6. Froude number $Fr_i = \frac{v_i}{\sqrt{g \cdot y_{med,i}}}$

Be careful that all these parameters are calculated using $y_{med,i}$ and not y_i. This is relevant for the application of the middle point method.

Once these values are computed, we calculate the longitudinal steps according to the approximation

$$dx_i \approx \left(\frac{1 - Fr_i^2}{S_0 - S_{f,i}}\right) \cdot dy_i$$

from which we derive the longitudinal coordinates as $x_i = x_{i-1} + dx_{i-1}$.

Conjugate profile

The conjugate depths are found with the formula

$$y_{conj,i} = \frac{y_i}{2}\left(-1 + \sqrt{1 + 8Fr_i^2}\right)$$

All values are found in Table 3.3.

Table 3.3: Profile resulting from the gate opening and its conjugate.

Section	y_i **(m)**	dy **(m)**	y_{med} **(m)**	A **(m²)**	P **(m)**	R **(m)**	v **(m/s)**	S_f **(-)**	Fr **(-)**	$S_0 - S_f$ **(-)**	$1 - Fr^2$ **(-)**	dx **(m)**	x **(m)**	h_{conj} **(m)**
1	0.025	0.01	0.030	0.0045	0.210	0.021	2.444	0.0850	4.506	-0.0840	-19.303	2.30	0	0.147
2	0.035	0.01	0.040	0.0060	0.230	0.026	1.833	0.0368	2.927	-0.0358	-7.566	2.12	2.30	0.128
3	0.045	0.01	0.050	0.0075	0.250	0.030	1.467	0.0195	2.094	-0.0185	-3.386	1.83	4.41	0.113
4	0.055	0.01	0.060	0.0090	0.270	0.033	1.222	0.0118	1.593	-0.0108	-1.538	1.43	6.24	0.099
5	0.065	0.01	0.070	0.0105	0.290	0.036	1.048	0.0078	1.264	-0.0068	-0.598	0.89	7.67	0.088
6	0.075	0.007	0.079	0.0118	0.307	0.038	0.934	0.0057	1.065	-0.0047	-0.133	0.20	8.55	0.081
7	0.082	-	-	-	-	-	-	-	-	-	-	-	8.75	0

Profile resulting from the long-crested weir

The water depth on the long-crested weir is not given, but we know that a choking occurs there. In this situation, the water depth is critical. As the channel section does not change, the critical depth also does not change compared to the one previously found. Hence, $y_{weir} = y_c = 0.082$ m.

Since we assume that no losses occur when flowing over the long-crested weir, we know that the specific energy upstream of the long-crested weir (section 1) and along it (section 2) must be equal. Hence, $E_1 = E_2 + d_{weir}$, from which we derive that $y_1 + \frac{v_1^2}{2g} = y_2 + \frac{v_2^2}{2g} + d_{weir}$. The profile is computed starting from y_1 going upstream, until the depth at the gate location. To get y_1, we first rewrite $v = \frac{Q}{bh}$, from which we obtain $y_1 + \frac{Q^2}{2g(by_1)^2} = y_2 + \frac{Q^2}{2g(by_2)^2} + d_{weir}$. Next, we make y_1 explicit. We obtain a 3rd order equation that reads $y_1^3 - y_1^2\left(y_2 + \frac{Q^2}{2g(by_2)^2} + d_{weir}\right) + \frac{Q^2}{2gb^2} = 0$. Since this is a 3rd order equation, it has three solutions for y_1: $y_1\prime = -0.0377$ m, $y_1\prime\prime = 0.0517$ m, and

$y_1''' = 0.139$ m. The negative solution has no physical sense and we already neglect it. The two other solutions are positive and identify a torrential regime (y_1'') and a fluvial regime (y_1'''). As this depth is found downstream of the hydraulic jump, we expect to be in subcritical conditions. Therefore, we pick $y_1 = y_1''' = 0.139$ m.

At this point we apply a similar procedure to the one previously used for computing the profile resulting from the gate opening. We set a constant step $dy = 0.005$ m to compute the various points of the profile. You can choose any value, but the remarks previously mentioned still hold. Since this time we compute the profile from the highest point to the lowest, we invert the sign. For this, we get $y_i = y_{i-1} - dy$, with $i = 2, ..., n$, and consequently $y_{med,i} = y_i - \frac{dy}{2}$. As before, we compute $A_i = b \cdot y_{med,i}$, $P_i = b + 2 \cdot y_{med,i}$, $R_i = \frac{A_i}{P_i}$, $v_i = \frac{Q}{A_i}$, $S_{f,i} = \left(\frac{n \cdot v_i}{R_i{}^{2/3}}\right)^2$, and $Fr_i = \frac{v_i}{\sqrt{g \cdot y_{med,i}}}$. Eventually, $dx_i \approx \left(\frac{1-Fr_i^2}{S_0 - S_{f,i}}\right) \cdot dy_i$. Remember that, as we are computing the profile from downstream to upstream, all dx_i should be negative. The longitudinal coordinates are found similarly as before $x_i = x_{i-1} + dx_{i-1}$. Remember that the first point $x_1 = x_{weir} = 15$ m.

All results are included in Table 3.4.

Table 3.4: Profile resulting from the long-crested weir.

Section	y_i **(m)**	dy **(m)**	y_{med} **(m)**	A **(m^2)**	P **(m)**	R **(m)**	v **(m/s)**	S_f **(-)**	Fr **(-)**	$S_0 - S_f$ **(-)**	$1 - Fr^2$ **(-)**	dx **(m)**	x **(m)**
1	0.139	0.002	0.138	0.0207	0.426	0.049	0.531	0.00135	0.457	-0.00035	0.791	-4.55	15.00
2	0.137	0.002	0.136	0.0204	0.422	0.048	0.539	0.00140	0.467	-0.00040	0.782	-3.94	10.45
3	0.135	0.002	0.134	0.0201	0.418	0.048	0.547	0.00145	0.477	-0.00045	0.772	-3.43	6.52
4	0.133	0.002	0.132	0.0198	0.414	0.048	0.556	0.00150	0.488	-0.00050	0.762	-3.02	3.08
5	0.131	0.002	0.130	0.0195	0.410	0.048	0.564	0.00156	0.500	-0.00056	0.750	-2.67	0.06
6	0.129	0.002	0.128	0.0192	0.406	0.047	0.573	0.00162	0.511	-0.00062	0.739	-2.37	-2.60

Jump location

To localize the hydraulic jump, we look for the longitudinal coordinates where the conjugate profile and the one resulting from the long-crested weir meet. We see that the intersection point is slightly smaller than 0.14 m. To get the exact water depth, we can assume that both profiles are straight lines between the two points closest to the intersection point.

For the conjugate profile, the two points are $y_0 = 0.147$ m and $y_1 = 0.128$ m, and the related coordinates are $x_0 = 0.0$ m and $x_1 = 2.299$ m. Hence, $\Delta y = y_1 - y_0 = 0.128$ m -0.147 m$= -0.019$ m; $\Delta x = x_1 - x_0 = 2.299$ m -0.0 m$= 2.299$ m.

For the long-crested weir profile, the two points are $y_0 = 0.133$ m and $y_1 = 0.135$ m, and the related coordinates are $x_0 = 3.082$ m and $x_1 = 6.516$ m. Hence, $\Delta y = y_1 - y_0 = 0.135$ m -0.133 m$= 0.002$ m; $\Delta x = x_1 - x_0 = 6.516$ m -3.082 m$= 3.434$ m.

We then compute the slope of the lines as $m = \frac{\Delta y}{\Delta x}$, and the intercept as $q = y_0 - x_0 \cdot m$. For the conjugate profile we obtain $m_{conj} = -0.0082$ and $q_{conj} = 0.1473$. For the long-crested weir profile we obtain $m_{weir} = 0.0006$ and $q_{weir} = 0.1312$.

The location of the jump is found at the intersection between the two lines, that is where $x_{jump} \cdot m_{conj} + q_{conj} = x_{jump} \cdot m_{weir} + q_{weir}$, from which we make x explicit and get $x_{jump} = \frac{q_{weir} - q_{conj}}{m_{conj} - m_{weir}} = \frac{0.1312 - 0.1473}{-0.0082 - 0.0006} = 1.8296$ m.

The related depth can be found by using one or the other equations of the lines. Using the equation of the conjugate profile, for instance, we get $y_{jump} = x_{jump} \cdot m_{conj} + q_{conj} = 1.8926$ m $\cdot(-0.0.082) + 0.1473$ m $= 0.1323$ m. The same value can be found using the equation resulting from the long-crested weir profile.

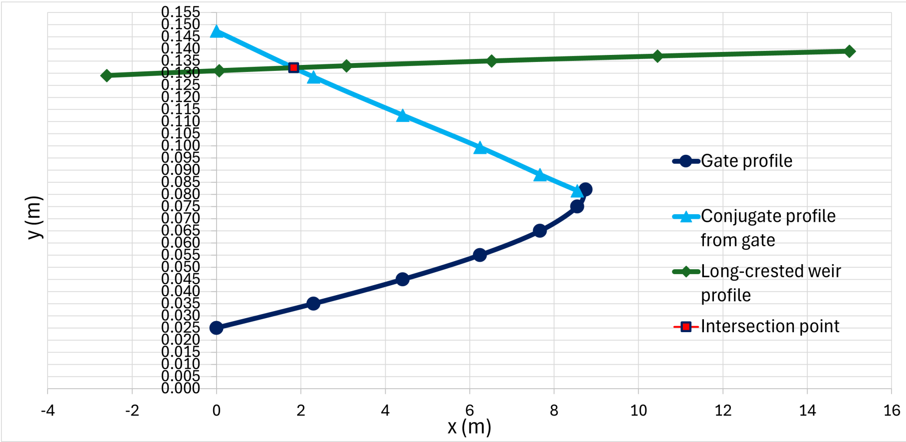


Figure 3.24: Profiles resulting from the gate opening, its conjugate, and from the long-crested weir. The intersection point between the conjugate and the long-crested weir profile results in the jump location.

Chapter 4

Hydraulic Structures

Brief Summary

1. Weirs

1.1. Sharp-Crested Weirs

Basic assumptions

- Hydrostatic pressure upstream.
- Atmospheric pressure in the nappe.
- Negligible head losses.
- Gravity and inertia dominate.

Rectangular sharp-crested weir (free flow)

$$Q = \frac{2}{3} C_d \sqrt{2g}\, b\, H^{3/2}$$

where H is the upstream head above the crest and b the weir width.

Discharge coefficient

$$C_d = f\left(\mathrm{Re},\ \mathrm{We},\ \frac{H}{P_w}\right), \qquad C_d \approx 0.611 + 0.075\,\frac{H}{P_w} \quad (H/P_w < 5)$$

Contracted rectangular weir

$$Q = \frac{2}{3} C_d \sqrt{2g}\,\big(B - 0.1\,n\,H\big)\, H^{3/2}$$

where n is the number of side contractions.

Trapezoidal (Cipolletti) weir

$$Q = \frac{2}{3} C_d \sqrt{2g}\, B\, H^{3/2}$$

The geometry compensates for lateral contraction.

Triangular (V-notch) weir

$$Q = C_{wt}\,\frac{8}{15}\,\tan\!\left(\frac{\theta}{2}\right)\,\sqrt{2g}\, H^{5/2}$$

Typical values $C_{wt} \in [0.58,\ 0.62]$.

Submerged Sharp-Crested Weirs

When the nappe is submerged, free-flow formulas are no longer valid.

Villemonte correction

$$Q^* = Q\left(1 - \left(\frac{h_d}{H}\right)^{3/2}\right)^{0.385}$$

where h_d is the downstream water depth above the crest.

1.2. Broad-Crested Weirs

Broad-crested weirs induce critical flow on the crest.

Critical condition

$$F_r = 1, \qquad y_c = \frac{2}{3}H$$

Discharge

$$Q = C_{db}\, b\, \sqrt{g}\, y_c^{3/2} = C_{db}\, b\, \sqrt{g}\left(\frac{2}{3}H\right)^{3/2}$$

Discharge coefficient

$$C_{db} = 1.125\left(\frac{1 + H/P_w}{2 + H/P_w}\right)^{1/2}.$$

2. Underflow Gates

2.1. Free Outflow Gates

Vena contracta

$$y_2 = C_c\, a$$

with a the gate opening and $C_c \approx 0.61$ for vertical gates.

Energy equation

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g}.$$

Discharge

$$Q = b\, C_c\, a\sqrt{2g\, y_1}$$

Often written as

$$q = \frac{Q}{b} = C_d\, a\sqrt{2g\, y_1}, \qquad C_d = \frac{C_c}{\sqrt{1 + aC_c/y_1}}.$$

Typical values $C_d \approx 0.55$ to 0.60.

2.2. Submerged Underflow Gates

The gate is submerged when the tailwater depth exceeds the conjugate depth of the vena contracta.

Criterion

$$y_3 > y_{2,\text{conj}} \quad \Rightarrow \quad \text{submerged outflow.}$$

Discharge The same equation as free outflow is used, but with a reduced discharge coefficient:

$$Q = C_d(y_3/a)\, b\, a\, \sqrt{2g\, y_1},$$

where C_d decreases with increasing tailwater depth.

3. Spillways

3.1. Definition and Role of Spillways

A **spillway** is a hydraulic structure designed to safely evacuate excess water from a reservoir when the water level exceeds the crest elevation. Spillways act as safety devices for dams, since overtopping is one of the main causes of dam failure.

Typical components

- Control structure (crest or gate)
- Discharge channel (spillway face, chute, or conduit)
- Terminal structure (energy dissipation)

3.2. Discharge over an Overflow Spillway

The discharge over an overflow (ogee) spillway follows the weir-type relation

$$Q = C\, L_e\, H_e^{3/2}$$

where H_e is the **effective head** over the crest, C is the discharge coefficient, and L_e is the effective crest length.

Effective head

$$H_e = H + \frac{V_a^2}{2g}$$

H is the water depth above the crest apex and V_a is the approach velocity computed from the upstream reservoir geometry.

3.3. Effective Crest Length

Due to piers and abutments, not all the geometric crest length contributes to discharge. The effective length is

$$L_e = L - wN - 2NK_p + K_a H_e$$

where L is the total crest length, w the pier width, N the number of piers, K_p the pier contraction coefficient, and K_a the abutment contraction coefficient.

3.4. Discharge Coefficient

The discharge coefficient depends on spillway geometry and head

$$C = C_0 \cdot f\left(\frac{P}{H_d}\right) \cdot f\left(\frac{H_e}{H_d}\right)$$

where P is the crest height above the reservoir bottom and H_d is the **design head** used to define the spillway shape.

For high spillways ($P/H_d \gg 1$), the basic coefficient is approximately

$$C_0 \approx 2.18$$

3.5. Design Head and Under-Design

The spillway crest shape is defined for a single **design head** H_d, while the actual operating head H_e varies with inflow.

- If $H_e < H_d$, pressures on the crest are positive and discharge efficiency decreases
- If $H_e > H_d$, pressures decrease and discharge efficiency increases

To improve efficiency, spillways are often **under-designed**, choosing $H_d < H_e$, while ensuring pressures remain above cavitation limits.

3.6. Cavitation Constraint

Cavitation may occur if pressures drop below a critical threshold

$$p < -7.6 \text{ m of water}$$

Design practice limits the ratio

$$\frac{H_e}{H_d} \lesssim 1.33$$

to avoid cavitation on the crest.

3.7. Ogee Crest Shape

The ogee profile reproduces the lower nappe of a sharp-crested weir at the design head.

Downstream profile

$$\frac{y}{H_d} = \frac{1}{K}\left(\frac{x}{H_d}\right)^n$$

with $n \approx 1.85$ and K depending on P/H_d.

Upstream profile is typically approximated by a quarter-ellipse whose parameters depend on P/H_d and the upstream face slope.

4. Uniform Flow as Design Condition

Engineered channels are designed to operate under **uniform flow**, for which:

$$S_0 = S_f, \qquad Q = Q(y)$$

Key geometric and hydraulic quantities:

$$Q = AV, \qquad R_H = \frac{A}{P}$$

where A is the flow area, P the wetted perimeter, and R_H the hydraulic radius.

4.1. Manning Equation (Design Equation)

Uniform flow discharge is computed using Manning's formula:

$$Q = \frac{1}{n} A \, R_H^{2/3} \, S_0^{1/2}$$

where n is Manning's roughness coefficient.

4.2. Best (Most Efficient) Hydraulic Section

The best hydraulic section minimizes the wetted perimeter P for a given discharge:

$$R_H = \frac{A}{P} \quad \Rightarrow \quad \text{maximize } R_H \quad \Leftrightarrow \quad \text{minimize } P$$

Rectangular channel:

$$A = by, \qquad P = b + 2y$$

Optimal condition:

$$y = \frac{b}{2}$$

Trapezoidal channel (m: side slope):

$$A = by + my^2, \qquad P = b + 2y\sqrt{1+m^2}$$

Optimal side slope:

$$m = \frac{1}{\sqrt{3}} \quad (\theta = 60^\circ)$$

and

$$\frac{b}{y} = 2\left(\sqrt{1+m^2} - m\right)$$

4.3. Velocity and Shear Stress Constraints

Average boundary shear stress in uniform open-channel flow:

$$\tau_0 = \gamma R_H S_0$$

For trapezoidal sections:

$$\tau_b = \gamma y S_0, \qquad \tau_s = K_s \tau_b$$

where K_s depends on the side slope m.

Design requirement:

$$\tau \le \tau_{\text{perm}}$$

with τ_{perm} the permissible (critical) shear stress of the lining material.

4.4. Freeboard

Freeboard F is added above normal depth y_n:

$$F = z_{\text{top}} - y_n$$

Typical empirical recommendations:

$$F = \begin{cases} 0.15\ \text{m}, & y < 0.30\ \text{m} \\ 0.30\ \text{m}, & y \ge 0.30\ \text{m},\ V \le 1.72\ \text{m/s} \\ 0.15 + \dfrac{V^2}{2g}, & y \ge 0.30\ \text{m},\ V > 1.72\ \text{m/s} \end{cases}$$

At bends, additional superelevation:

$$h_s = \frac{V^2 B}{g r_c}$$

5. Culverts: Governing Classification

Culverts are hydraulically short conduits operating under:

$$Q = f(H)$$

where H is the headwater elevation.

Two controlling regimes:

- **Inlet Control (IC)**: discharge controlled by inlet geometry only.
- **Outlet Control (OC)**: discharge controlled by inlet, barrel, and tailwater.

5.1. Inlet Control (IC)

Non-submerged inlet (weir behavior):

$$Q = C_d A_c \sqrt{2g\,(H_W - y_c)}$$

with critical flow at the inlet:

$$F_r = 1$$

Submerged inlet (orifice behavior):

$$Q = C_d A_0 \sqrt{2g\,H_W}$$

5.2. Outlet Control (OC)

Outlet control is governed by the energy equation between headwater (HW) and tailwater (TW):

$$H_W - T_W + S_0 L = \left(1 + K_e + f\frac{L}{4R}\right)\frac{Q^2}{2gA^2}$$

Rearranged discharge form:

$$Q = A\sqrt{\frac{2g\,(H_W - T_W + S_0 L)}{1 + K_e + f\frac{L}{4R}}}$$

The friction term may be rewritten using Manning:

$$f\frac{L}{4R} = \frac{2gn^2 L}{R^{4/3}}$$

5.3. Design Logic for Culverts

1. Select design discharge Q_d from hydrologic analysis.
2. Assume inlet control and compute $H_W(Q)$.
3. Assume outlet control and compute $H_W(Q)$.
4. Governing condition is the one yielding the larger H_W.
5. Verify acceptable headwater and iterate geometry if needed.

EXERCISES

1. FREE AND SUBMERGED FLOW THROUGH A VERTICAL SLUICE GATE

We will use the notation and theoretical background presented in the Gates and Hydraulic Jumps notes available on Moodle.

This exercise concerns flow under a vertical sluice gate in a horizontal, rectangular channel of width b. Upstream of the gate the water depth is uniform and given by y_1. The gate opening is a, and due to contraction the jet thickness at the vena contracta is

$$y_g = C_c\, a,$$

where C_c is the contraction coefficient. The downstream uniform depth is denoted y_d (tailwater depth). We consider two operating regimes: free flow and submerged flow.

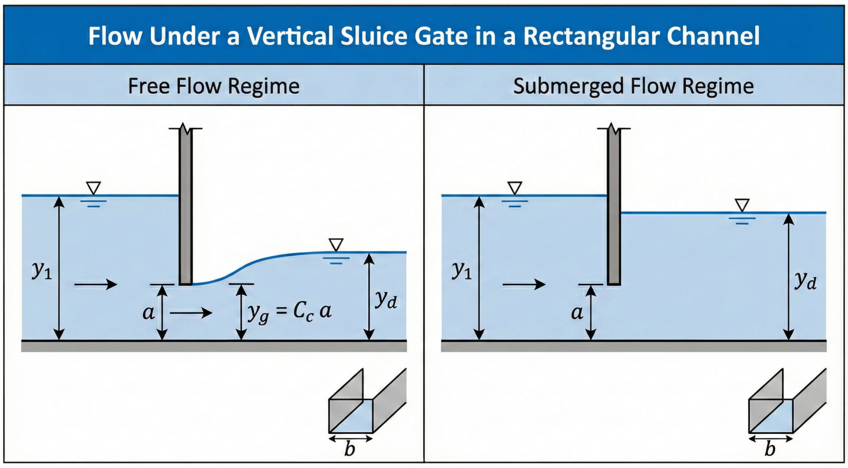


Figure 4.1: Free and submerged flow under a vertical sluice gate, showing upstream depth y_1, gate opening a, contracted jet thickness $y_g = C_c a$, and tailwater depth y_d.

Consider a sluice gate with geometric opening $a = 0.10$ m in a rectangular channel of width $b = 1.5$ m. Upstream depth is $y_1 = 1.0$ m, and the contraction coefficient is $C_c = 0.61$. Take $g = 9.81\ \mathrm{m/s^2}$.

(i) Using conservation of specific energy between the upstream section and the vena contracta, show that the free-flow discharge under the gate may be expressed as

$$Q_{\mathrm{free}} = C_d\, b\, a\, \sqrt{2gy_1}, \qquad \text{with} \qquad C_d = \frac{C_c}{\sqrt{1 + \dfrac{C_c a}{y_1}}}.$$

Start by writing the specific energy at the upstream depth y_1 and at the contracted jet thickness $y_g = C_c a$, then set $E_1 = E_g$ to eliminate q. Show that this leads to the expression above for the discharge coefficient C_d. Finally, evaluate y_g and C_d numerically for the given data.

(ii) Compute the free discharge Q_{free} through the gate.

(iii) Determine the tailwater depth $y_{d,\mathrm{crit}}$ for which the hydraulic jump would attach to the gate and compare it with y_1.

SOLUTION

Given:

$$a = 0.10 \text{ m}, \qquad y_1 = 1.0 \text{ m}, \qquad b = 1.5 \text{ m}, \qquad C_c = 0.61, \qquad g = 9.81 \text{ m/s}^2.$$

The contracted jet thickness at the vena contracta is

$$y_g = C_c\, a = 0.61 \times 0.10 = 0.061 \text{ m}.$$

(i) *Derivation of Q_{free} and C_d*

Let $q = Q/b$ be the discharge per unit width. The specific energy at the upstream section and at the contracted jet section are

$$E_1 = y_1 + \frac{q^2}{2g\, y_1^2}, \qquad E_g = y_g + \frac{q^2}{2g\, y_g^2}.$$

Neglecting losses between section 1 and the vena contracta, conservation of specific energy gives

$$E_1 = E_g.$$

Hence

$$y_1 - y_g = \frac{q^2}{2g}\left(\frac{1}{y_g^2} - \frac{1}{y_1^2}\right) = \frac{q^2}{2g}\frac{y_1^2 - y_g^2}{y_g^2 y_1^2}.$$

Solving for q^2,

$$\frac{q^2}{2g} = \frac{y_g^2 y_1^2}{y_1^2 - y_g^2}(y_1 - y_g) = \frac{y_g^2 y_1^2}{(y_1 - y_g)(y_1 + y_g)}(y_1 - y_g) = \frac{y_g^2 y_1^2}{y_1 + y_g}. \tag{1}$$

For free flow under the gate we define the discharge coefficient C_d by

$$Q_{\text{free}} = C_d\, b\, a\sqrt{2gy_1}, \qquad q_{\text{free}} = \frac{Q_{\text{free}}}{b} = C_d\, a\sqrt{2gy_1}.$$

Thus

$$\frac{q_{\text{free}}^2}{2g} = C_d^2 a^2 y_1. \tag{2}$$

Equating (1) and (2) gives

$$C_d^2 a^2 y_1 = \frac{y_g^2 y_1^2}{y_1 + y_g} \quad \Rightarrow \quad C_d^2 = \frac{y_g^2 y_1}{a^2(y_1 + y_g)}.$$

Using $y_g = C_c a$ so that $y_g^2 = C_c^2 a^2$, we obtain

$$C_d^2 = \frac{C_c^2 a^2 y_1}{a^2(y_1 + y_g)} = \frac{C_c^2 y_1}{y_1 + y_g} = \frac{C_c^2}{1 + \dfrac{y_g}{y_1}} = \frac{C_c^2}{1 + \dfrac{C_c a}{y_1}}.$$

Taking the square root,

$$C_d = \frac{C_c}{\sqrt{1 + \dfrac{y_g}{y_1}}} = \frac{C_c}{\sqrt{1 + \dfrac{C_c a}{y_1}}},$$

which is the required expression.

Numerically,

$$\frac{y_g}{y_1} = \frac{0.061}{1.0} = 0.061, \qquad C_d = \frac{0.61}{\sqrt{1 + 0.061}} \approx \frac{0.61}{\sqrt{1.061}} \approx \frac{0.61}{1.03} \approx 0.59.$$

So

$$\boxed{y_g = 0.061 \text{ m}}, \qquad \boxed{C_d \approx 0.59}.$$

(ii) *Free discharge through the gate*

From the result above,

$$Q_{\text{free}} = C_d\, b\, a\, \sqrt{2gy_1}.$$

Substituting the numerical values

$$Q_{\text{free}} = 0.592 \times 1.5 \times 0.10 \times \sqrt{2 \times 9.81 \times 1.0}.$$

Compute step by step:

$$\sqrt{2 \times 9.81 \times 1.0} = \sqrt{19.62} \approx 4.43,$$
$$0.592 \times 1.5 \times 0.10 = 0.0888,$$

so

$$Q_{\text{free}} \approx 0.0888 \times 4.43 \approx 0.39\ \text{m}^3/\text{s}.$$

Thus

$$\boxed{Q_{\text{free}} \approx 0.39\ \text{m}^3/\text{s}}.$$

(iii) *Tailwater depth for which the jump attaches to the gate*

The limit of free flow corresponds to a hydraulic jump that is just attached to the gate. In that case, the upstream depth of the jump is the jet depth y_g and the downstream depth is the critical tailwater depth $y_{d,\text{crit}}$.

First compute the unit discharge and velocity in the jet:

$$q = \frac{Q_{\text{free}}}{b} \approx \frac{0.39}{1.5} \approx 0.26\ \text{m}^2/\text{s},$$
$$V_g = \frac{q}{y_g} \approx \frac{0.26}{0.061} \approx 4.3\ \text{m/s}.$$

The Froude number at the contracted section is

$$Fr_g = \frac{V_g}{\sqrt{gy_g}} = \frac{4.3}{\sqrt{9.81 \times 0.061}} \approx 5.6.$$

For a hydraulic jump in a rectangular channel, the conjugate depth relation is

$$\frac{y_2}{y_1} = \frac{1}{2}\left[-1 + \sqrt{1 + 8Fr_1^2}\right],$$

where y_1 and Fr_1 are the depth and Froude number on the supercritical side. Here $y_1 = y_g$ and $y_2 = y_{d,\text{crit}}$, so

$$y_{d,\text{crit}} = y_g\, \frac{1}{2}\left[-1 + \sqrt{1 + 8Fr_g^2}\right].$$

With $Fr_g \approx 5.6$ one finds numerically

$$y_{d,\text{crit}} \approx 0.45\ \text{m}.$$

Hence

$$\boxed{y_{d,\text{crit}} \approx 0.45\ \text{m}}.$$

Comparison with the upstream depth,

$$y_1 = 1.0\ \text{m},$$

shows that

$$y_{d,\text{crit}} < y_1.$$

Therefore only relatively shallow tailwater depths allow free flow under the gate. If the tailwater exceeds about 0.45 m, the hydraulic jump is drowned and the gate operates under submerged conditions.

2. OGEE SPILLWAY DESIGN

A high overflow spillway with $P/H_d > 1.5$ has a maximum discharge of $283.2\,\text{m}^3/\text{s}$ with a maximum expected head of $6.10\,\text{m}$. Using the USBR recommendation for the under-design procedure mentioned in class, determine the design head, spillway crest length (neglect contractions), and the minimum pressure (expressed in kPa) on the spillway. Plot the complete spillway crest shape for a compound circular curve in the upstream quadrant of the crest.

SOLUTION

To under design the spillway while ensuring no cavitation, the USBR recommend to adopt an expected-head to design-head ratio no more than 1.33. Thererfore:

$$H_d = H_e/1.33 = 6.10/1.33 = 4.58\,\text{m} \tag{4.1}$$

From the discharge coefficient plots, for high spillways $P/H_d > 1.5$, the basic discharge coefficient $C_0 \approx 2.18$. Using the condition of $H_e/H_d = 1.33$, the basic discharge coefficient must be corrected using the ratio $C/C_o = 1.04$.

Therefore, the final overall design coefficient is:

$$C = 2.18 \times 1.04 = 2.27$$

The spillway crest length is given by the standard discharge equation:

$$L = \frac{Q}{CH_{\text{max}}^{3/2}} = \frac{283.2}{2.28 \times (6.10)^{3/2}} = 8.26\,\text{m} \tag{4.2}$$

The minimum pressure head is obtained from standard design charts (Figure 6.3a) to be -0.43 times the design head with the result:

$$\left(\frac{p}{\gamma}\right)_{\text{min}} = -0.43 \times 4.57 = -1.97\,\text{m} \tag{4.3}$$

The minimum pressure is calculated using $\gamma_{water} \approx 9.81\,\text{kN/m}^3$:

$$p_{\text{min}} = -1.97 \times 9.81 = -19.3\,\text{kPa}$$

For high spillways with $P/H_d \gg 1$, the downstream crest shape recommended by USBR can be expressed in nondimensional form as

$$\left(\frac{y}{H_d}\right) = \frac{1}{K}\left(\frac{x}{H_d}\right)^n,$$

with standard values $n = 1.85$ and $K = 2$. Using the design head $H_d = 4.58\,\text{m}$, the dimensional downstream profile becomes

$$y = \frac{H_d}{K}\left(\frac{x}{H_d}\right)^{1.85} = \frac{H_d^{1-1.85}}{2}\,x^{1.85} = \frac{H_d^{-0.85}}{2}\,x^{1.85}.$$

Numerically,

$$H_d^{-0.85} = 4.58^{-0.85} \approx 0.274, \qquad \frac{0.274}{2} \approx 0.137,$$

so that

$$\boxed{y_{\text{d/s}} = 0.137\,x^{1.85}.}$$

Representative downstream ordinates:

x (m)	0	1	2	3	4	5	6
y (m)	0.000	0.137	0.494	1.047	1.783	2.694	3.774

The upstream profile is described by the USBR recommended elliptical form

$$\frac{x^2}{A^2} = \frac{(B-y)^2}{B^2} = 1,$$

with chart-based coefficients appropriate for high spillways,

$$\frac{A}{H_d} = 0.28, \qquad \frac{B}{H_d} = 0.165.$$

Using $H_d = 4.58$ m:

$$A = 0.28(4.58) \approx 1.28 \text{ m}, \qquad B = 0.165(4.58) \approx 0.76 \text{ m}.$$

Thus,

$$\boxed{\frac{x^2}{1.28^2} + \frac{(0.76 - y)^2}{0.76^2} = 1.}$$

Representative upstream ordinates (for $H_d = 4.58$ m):

x/H_d	0	−0.05	−0.10	−0.15	−0.20	−0.25	−0.28
x (m)	0.000	−0.229	−0.458	−0.687	−0.916	−1.145	−1.282
y (m)	0.000	0.012	0.050	0.118	0.227	0.415	0.756

The upstream and downstream ogee crest profiles for the design head $H_d = 4.58$ m are plotted below. The coordinate system follows hydraulic convention, with positive y downward.

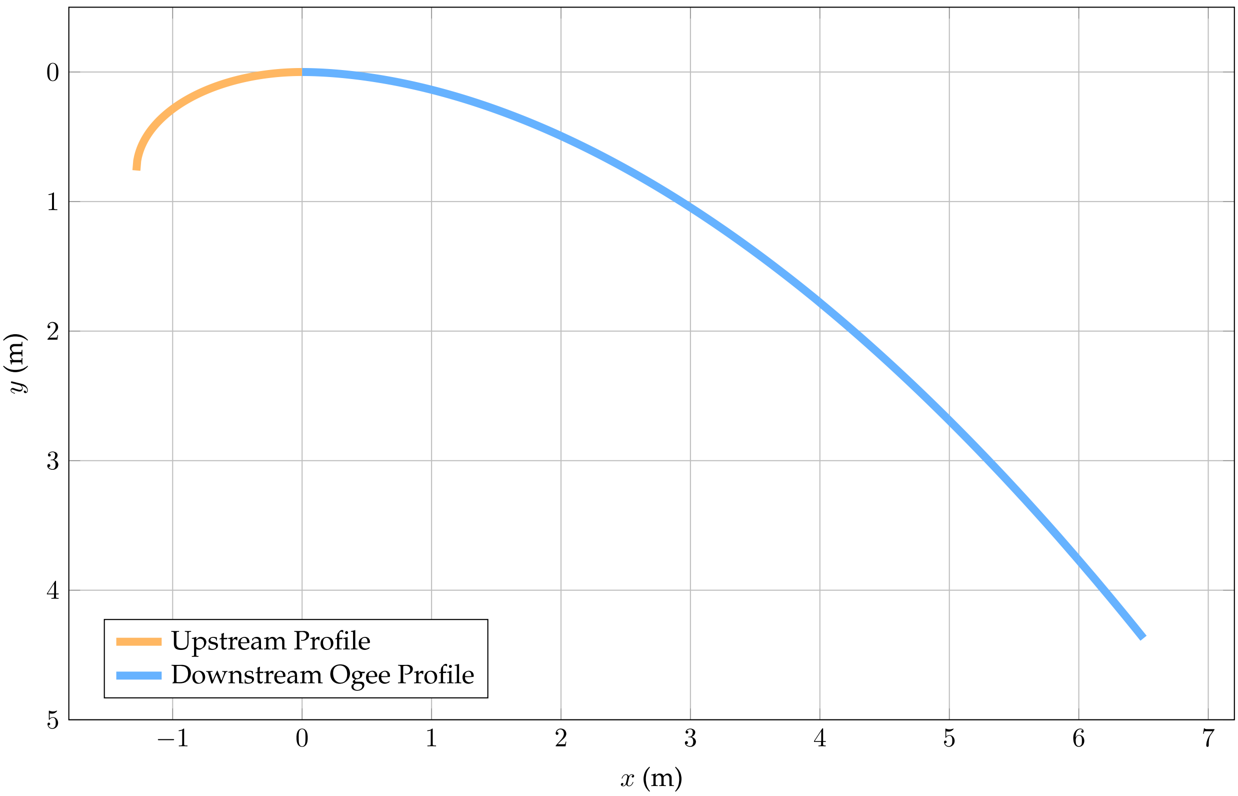


Figure 4.2: Computed upstream and downstream ogee crest profiles for $H_d = 4.58$ m (positive y downward).

3. STILLING BASIN

The maximum design discharge over a spillway is $Q = 280\,\mathrm{m^3/s}$, and both the spillway and stilling basin have width $b = 12\,\mathrm{m}$. The reservoir level upstream of the spillway is El. 60.00 m, and the river water surface immediately downstream of the stilling basin is El. 30.00 m. As the flow descends the spillway, it is estimated to lose 10 % of its hydraulic head.

Determine the elevation of the basin floor so that the hydraulic jump forms in the basin, and design the stilling basin.

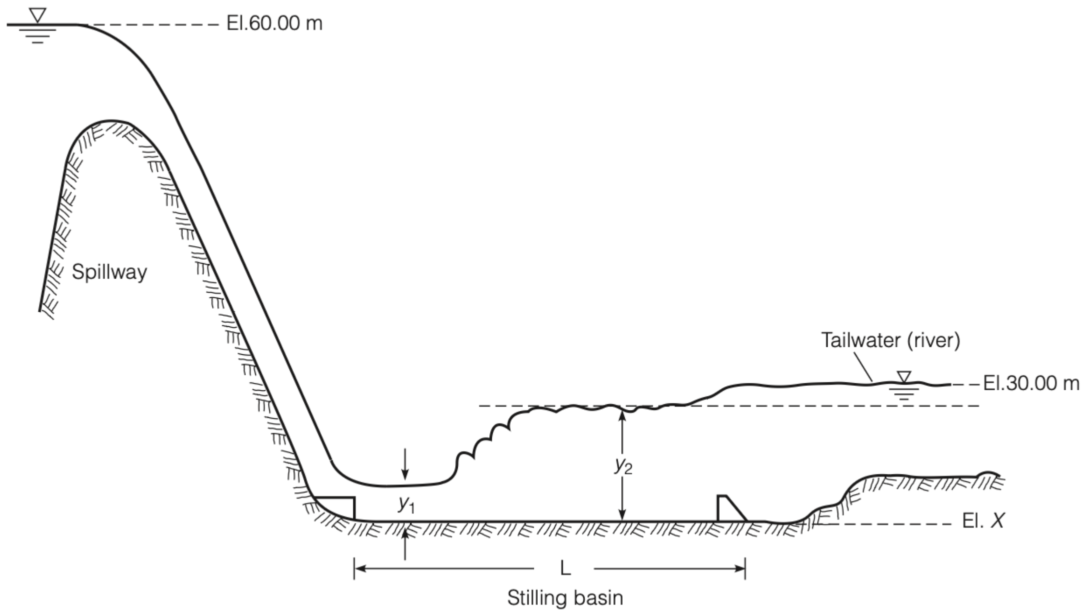


Figure 4.3: Spillway and stilling basin arrangement.