ChE 330 – Fluid Mechanics and Transport Phenomena

Fall 2021

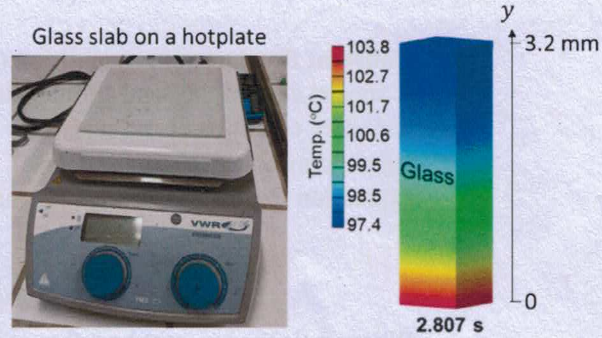
Midterm exam: **40 points**

NAME, First Name KEY

Signature

INSTRUCTIONS: You may use all of the course materials, module notes and exercise corrections for reference. A calculator may be used, but the **internet is not allowed**. Use additional sheets of paper to write your answers. Please show all of your derivations and calculations, but there is no need to re-derive equations already given in the course notes (just mention where the equation comes from). **Write your name on all sheets of paper used and staple them to this sheet when finished**. Write your name and signature above.

1. Heating a glass substrate (6 points)

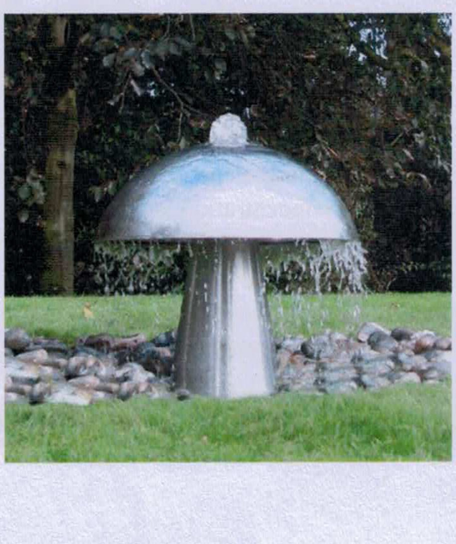


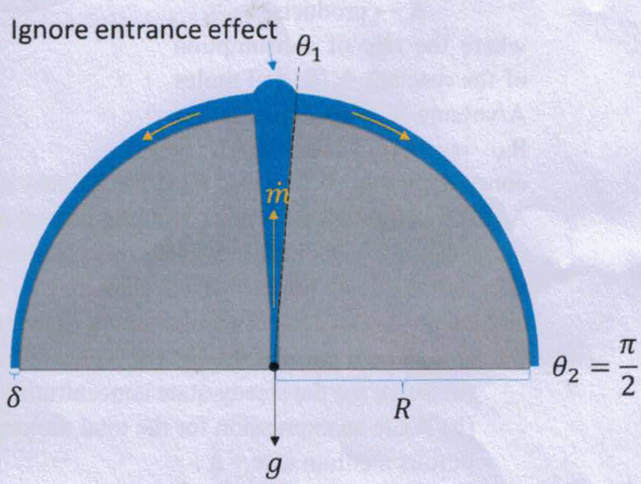
In a recent published paper on the perovskite solar cell [Li, N. *et al. Science* **373**, 561–567 (2021)], the authors were concerned about the heating of the perovskite thin film, which is coated on a large rectangular glass slab (3.2 mm in thickness) as a substrate. The authors performed a simulation to model the temperature of the glass, which was initially at 25 °C. At a time t = 2.807 seconds after the glass was placed on a hot plate with a temperature of 103.8 °C, they found the top surface of the glass to be at a temperature of 97.4°C. Can you estimate the thermal diffusivity of the glass?

2. Mushroom fountain (11 points)

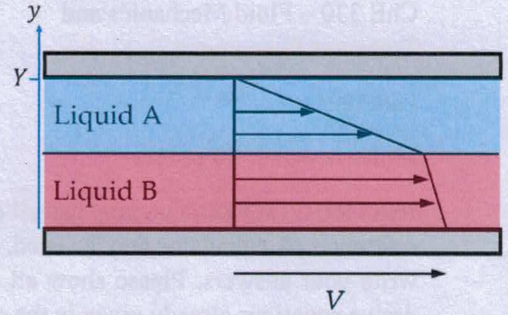
On a walk through a park you see this nice mushroom-shaped fountain where a steady-state flow rate of water, $\dot{m}$, flows up a central cone-shaped tube and then down the hemispherical surface of the fountain as indicated in the diagram. Assume that the flow is only driven by gravity (no pressure driven flow). Further assume that only steady-state Stokes flow occurs (not the case in the picture!) of a Newtonian fluid (constant ρ, μ) and disregard any entrance effects.

Can you propose a set of differential equations and boundary conditions that could be solved to find the fluid velocity as function of position as it flows over the hemispherical surface (i.e. from θ_1 to θ_2), and ultimately define the thickness of the film, δ? Do not solve these equations, but state any further assumptions that you make, and describe how to solve for δ. (Hints: assume a constant value for the velocity at θ_1, and use an integral mass balance at θ_2).





3. **Two phase flow (6 points)**



Two immiscible liquids A and B are flowing in steady laminar shear flow between two solid plates as illustrated in the figure. The bottom plate is moving to the right at a constant velocity V and the top plate is stationary. The velocity profile is indicated in the figure.

a) Which fluid has the larger viscosity? Provide an explanation for your answer with a mathematical statement, if possible.

b) Sketch the shear stress, τ_{yx} as a function of y.

c) Sketch the velocity profile for the two fluids if instead the top plate was moving at a constant velocity V and the bottom plate is stationary.

4. **Constant flux boundary condition (6 points)**

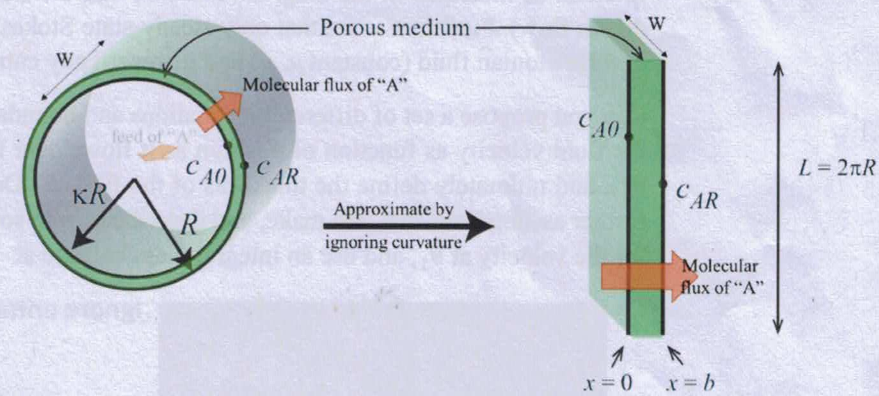
A special case of the semi-infinite slab solution for transient heat transfer can be defined when the boundary condition at the surface ($y = 0$) is a constant heat flux in the y direction (not a constant temperature as we saw in the course). Your colleague solves for this case and arrives at the following expression for the transient temperature profile :

$$T(y,t) = \frac{q_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(\frac{y^2}{4\alpha t}\right) - y\,\mathrm{erfc}\left(\frac{y}{\sqrt{4\alpha t}}\right)\right] + T_0$$

Where q_s is the constant heat flux at the surface, k is the thermal conductivity of the solid, T_0 is the initial temperature of the slab and the other symbols have their usual meanings. Do you think your colleague has arrived at the correct solution ? Justify your answer.

5. **Pharmaceutical production (11 points)**

In the production of a pharmaceutical chemical, an impurity chemical "A" is degraded in a thin, porous, cylindrical catalytic reactor (inner radius κR, outer radius R) as illustrated in the figure below. A dilute solution of reactant A is fed into the inner portion of the reactor ($r \leq$ κR). Species A then diffuses through the porous medium (where the effective diffusivity $\mathcal{D}_A$ is constant). The reactant also undergoes a first-order reaction in the porous medium:



$$A \rightarrow \text{products}$$

where the rate of consumption of the reactant A (R_A [=] moles A/volume·time) is expressed as $R_A = -k_1 c_A$ where k_1 is constant (and $k_1 > 0$ so that $R_A < 0$). We may assume that c_A only depends only on r, and as species A diffuses through and reacts with the porous medium, its concentration is reduced from c_{A0} at $r =$ κR to c_{AR} at $r =$ R. Before solving this problem, we first take advantage of the fact that the shell is thin, and therefore the cylindrical shell may be approximated as a thin slab of thickness b = R − κR, and length L ≈ 2πR, as illustrated on the right side of the figure above (the width into the page W is the same in both parts of the figure). The concentration of A now only depends on x (≡ r − κR).

(a) Solve for the steady-state concentration profile $c_A(x)$

(b) Write an expression for the total molecular flux (moles/time) of the reactant A that exits the porous medium at $x = b$.

1) heating a glass substrate:

use transient solution developed in course

@ $t = 2.807 s$

$$\theta = \frac{T(t,y) - T_0}{T_1 - T_0} = \frac{97.4°C - 25°C}{103.8°C - 25°C} = 0.919$$

2

From graphical solution p.33 (module 4):

$$Fo = \frac{\alpha t}{b^2} \approx 1.05$$

2

thus we can solve for α

$$\alpha = \frac{1.05\,(3.2\,mm)^2}{2.807 s} = 3.83 \frac{mm^2}{s}$$

$$\boxed{= 3.83 \times 10^{-6} \frac{m^2}{s}}$$

2

2) mushroom fountain

- choose spherical coordinate system
- which direction is flow? → θ, $v_r = v_\phi = 0$
- what is driving force? → gravity

⇒ we look to solve for $v_\theta(r,\theta)$ & $p(r,\theta)$ using Stokes flow approx. ($\rho \frac{Dv}{Dt} = 0$)

we can simplify the N-S equations:

Spherical coordinates (r, θ, ϕ):

$$\cancel{\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)} = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\cancel{\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r)} + \cancel{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}}\right] + \rho g_r$$

Note! $g_r = -g\cos\theta$

$g_\theta = g\sin\theta$

$$\cancel{\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2\cot\theta}{r}\right)} = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta)\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}}\right] + \rho g_\theta$$

$$\cancel{\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}\right)} = \cancel{-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}}$$
$$+ \mu\left[\cancel{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)} + \cancel{\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta)\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}} + \cancel{\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}} + \cancel{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}}\right] + \cancel{\rho g_\phi}$$

for incompressible fluids $(\nabla \cdot v) = 0$ we can also apply the continuity equation

$$(\nabla \cdot \mathbf{v}) = \cancel{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta) + \cancel{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}} = 0$$

thus $\frac{\partial}{\partial\theta}(v_\theta \sin\theta) = 0$

Finally the simplified N-S equ's:

$$\boxed{\begin{aligned}\frac{\partial p}{\partial r} &= -g(\cos\theta)\rho \\ \frac{1}{r}\frac{\partial p}{\partial \theta} &= \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)\right] + \rho g\sin\theta\end{aligned}}$$ 3

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}\right]$$

For Boundary conditions @ $r = R + \delta$ we have a "free surface" thus $\tau_{r\theta}\big|_{r=R+\delta} = 0$ from Newton's law of viscosity we then have

$$\boxed{\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\bigg|_{r=R+\delta} = 0}$$ 1

we also have the "no slip" B.C.

$$\boxed{v_\theta\big|_{r=R} = 0}$$ 1

For $p(r,\theta)$ we need 2 B.C.'s

$p\big|_{r=R+\delta} = p_{atm}$

$p\big|_{\theta=\theta_2} = p_{atm}$ 1

atmospheric pressure @ both of the free surfaces

2) mushroom fountain, cont...

as problem states we assume

$v_\theta|_{\theta=\theta_1} = \text{Const.}$ what is this const?

we know:

$\dot{m} = \int_A \rho(v \cdot \hat{n})\, dA$

$\dot{m} = \rho v_\theta \int_0^{2\pi}\int_R^{R+\delta} r \sin\theta_1\, dr\, d\phi$

$\dot{m} = 2\rho v_\theta \pi \sin\theta_1 \int_R^{R+\delta} r\, dr$

$\dot{m} = 2\rho v_\theta \pi \sin\theta_1 \left[\frac{r^2}{2}\right]_R^{R+\delta}$

$\dot{m} = \rho v_\theta \pi \sin\theta_1 (\delta^2 + 2R\delta)$

thus our B.C. Becomes:

$$\boxed{v_\theta|_{\theta=\theta_1} = \frac{\dot{m}}{\pi\rho \sin\theta_1 (\delta^2 + 2R\delta)} \quad \text{where } \delta = \delta(\theta_1)}$$ 2

to solve for $\delta(\theta)$ 1
- Solve Differential equation
- apply B.C.s to obtain expressions for:
  $v_\theta(r,\theta)$
  $p(r,\theta)$

these expressions will depend on δ

Rearrange to solve for $\delta(\theta)$

For the last B.C. we are told to use an integral mass Balance @ $\theta = \theta_2$

in this case

$\dot{m} = \int_A \rho v_\theta|_{\theta=\theta_2}\, dA$

thus:

$\dot{m} = \rho \int_0^{2\pi}\int_R^{R+\delta} v_\theta|_{\theta=\theta_2} \sin\theta_2\, r\, dr\, d\phi$

$\dot{m} = 2\pi\rho \int_R^{R+\delta} v_\theta|_{\theta=\theta_2}\, r\, dr$

OR

$$\boxed{@\ \theta=\theta_2 \quad \int_R^{R+\delta} v_\theta\, r\, dr = \frac{\dot{m}}{2\pi\rho}}$$ 2

3) 2-phase flow

a) the key observation is that @ the interface between the liquids is that the <u>momentum flux</u> must be equal

thus:

$$\tau_{yx}^{A}\Big|_{interface(-)} = \tau_{yx}^{B}\Big|_{interface(+)}$$

Recall Newton's law of viscosity:

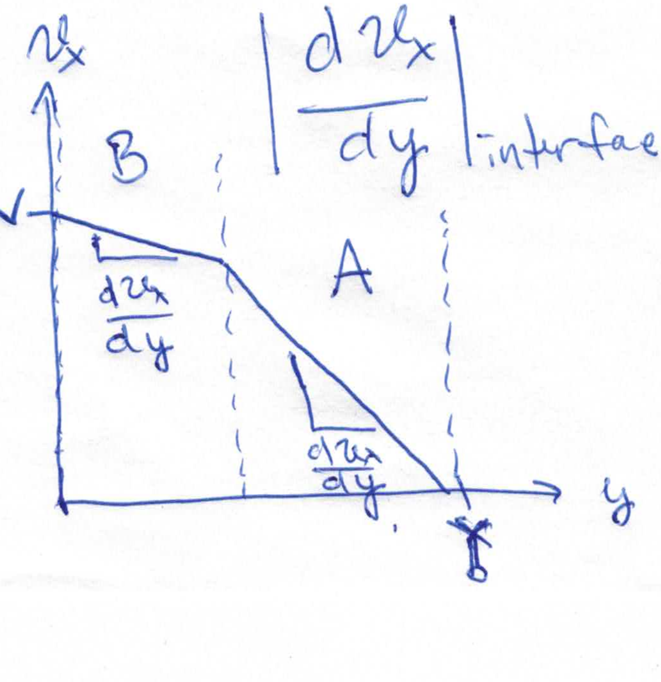
$$\tau_{yx} = -\mu \frac{dv_x}{dy}$$

thus

$$-\mu^{A}\frac{dv_x}{dy}\Big|_{interface(-)} = -\mu^{B}\frac{dv_x}{dy}\Big|_{interface(+)}$$

we see from figure that

$$\left|\frac{dv_x}{dy}\Big|_{interface(-)}\right| > \left|\frac{dv_x}{dy}\Big|_{interface(+)}\right|$$



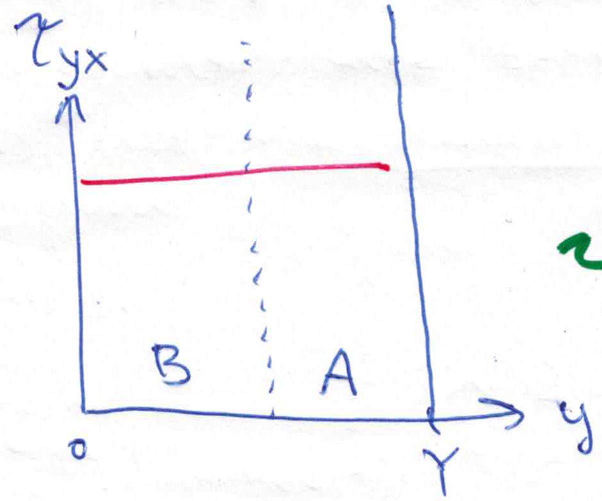
thus to maintain equality

$$\boxed{\mu^{B} > \mu^{A}}$$

2

3) cont...

b)



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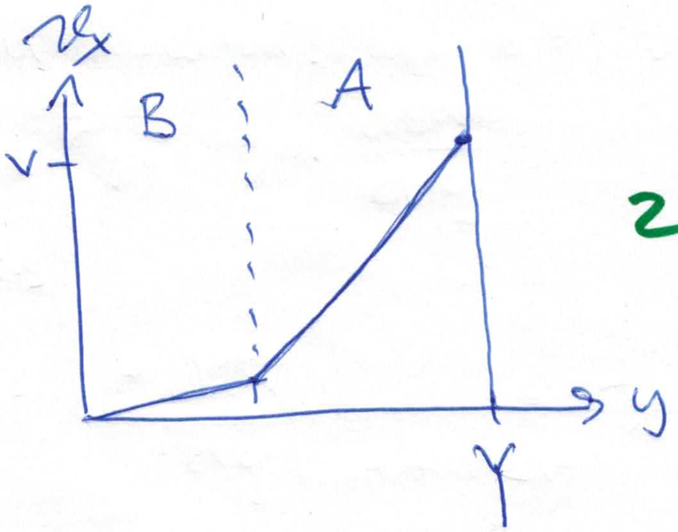
c)



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4) test expected Boundary Conditions!

(1) $T(y=\infty, t) = T_0$

3

(2) $-k \left.\frac{dT}{dy}\right|_{y=0} = q_s$

test (1)

$$T(\infty, t) = \frac{q_s}{k}\left[\underbrace{\sqrt{\frac{4\alpha t}{\pi}} \exp\left(\frac{y^2}{4\alpha t}\right) - y \, \mathrm{erfc}\left(\frac{y}{\sqrt{4\alpha t}}\right)}_{\text{this must be } = 0}\right] + T_0$$

$\downarrow \lim_{y\to\infty}$ ∞ $\qquad \downarrow \lim_{y\to\infty} = ?$

Since $\exp\left(\frac{y^2}{4\alpha t}\right) \to \infty$ as $y \to \infty$

3

then ~~wer~~ we are sure that the proposed solution is <u>incorrect</u>

By the way:

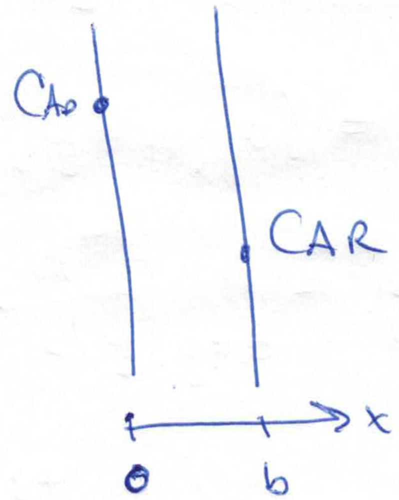
if we tried to verify (2) we would find (with some effort) that $\left.\frac{dT}{dy}\right|_{y=0} = \frac{-q_s}{k}$

this is the correct answer, But the ~~pro~~proposed solution is still incorrect.

the correct solution is

$$T(y,t) = \frac{q_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}} \exp\left(\frac{-y^2}{4\alpha t}\right) - y \, \mathrm{erfc}\left(\frac{y}{\sqrt{4\alpha t}}\right)\right] + T_0$$

5)



Start with eq. 4.14 $\frac{DC_A}{Dt} = \mathcal{D}_{AB} \nabla^2 C_A + R_A$

- steady state
- no fluid (solid continuum)

$$0 = \mathcal{D}_A \nabla^2 C_A + R_A$$

cartesian coordinates and $C_A(x)$

$$\mathcal{D}_A \frac{d^2 C_A}{dx^2} - k_1 C_A = 0$$

⇓ Divide by $\mathcal{D}_A$

$$\frac{d^2 C_A}{dx^2} - \frac{k_1}{\mathcal{D}_A} C_A = 0$$ 3

this matches the form $\frac{d^2 \varphi}{dx^2} - a^2 \varphi = 0$ with $a = \sqrt{\frac{k_1}{\mathcal{D}_A}}$

then the general solution is:

$$C_A(x) = C_1 \exp\left(+\sqrt{\frac{k_1}{\mathcal{D}_A}}\, x\right) + C_2 \exp\left(-\sqrt{\frac{k_1}{\mathcal{D}_A}}\, x\right)$$

-OR- $C_A(x) = C_1' \cosh\left(\sqrt{\frac{k_1}{\mathcal{D}_A}}\, x\right) + C_2' \sinh\left(\sqrt{\frac{k_1}{\mathcal{D}_A}}\, x\right)$ 2

5) cont.

Apply Boundary conditions!

$$C_A(0) = C_{A0}$$
$$C_A(b) = C_{AR}$$

$$C_{A0} = C_1 + 0$$

$$C_{AR} = C_{A0} \cosh\left(\sqrt{\frac{k_1}{D_A}}\, b\right) + C_2 \sinh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)$$

$$C_2 = \frac{C_{AR} - C_{A0} \cosh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)}{\sinh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)}$$

2

thus

$$\boxed{C_A(x) = C_{A0} \cosh\left(\sqrt{\frac{k_1}{D_A}}\, x\right) + \frac{C_{AR} - C_{A0} \cosh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)}{\sinh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)} \sinh\left(\sqrt{\frac{k_1}{D_A}}\, x\right)}$$

6)

$$j_A\Big|_{x=b} = -D_A \frac{dC_A}{dx}\Big|_{x=b}$$

4

$$\frac{dC_A}{dx} = C_{A0} \sqrt{\frac{k_1}{D_A}} \sinh\left(\sqrt{\frac{k_1}{D_A}}\, x\right) + \left(\frac{C_{AR} - C_{A0} \cosh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)}{\sinh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)}\right) \sqrt{\frac{k_1}{D_A}} \cosh\left(\sqrt{\frac{k_1}{D_A}}\, x\right)$$

$$j_A\Big|_{x=b} = -D_A \left[C_{A0} \sqrt{\frac{k_1}{D_A}} \sinh\left(\sqrt{\frac{k_1}{D_A}}\, b\right) + \left(\frac{C_{AR} - C_{A0} \cosh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)}{\sinh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)}\right) \sqrt{\frac{k_1}{D_A}} \cosh\left(\sqrt{\frac{k_1}{D_A}}\, b\right)

5) Alt. if $C_A(x) = C_1 \exp\left(\sqrt{\frac{k_1}{D_A}} x\right) + C_2 \exp\left(-\sqrt{\frac{k_1}{D_A}} x\right)$

then

$$\boxed{C_A(x) = \frac{\exp\left(-\sqrt{\frac{k_1}{D_A}} x\right)\left[C_{A0}\left(\exp\left(\sqrt{\frac{k_1}{D_A}} 2b\right) - \exp\left(\sqrt{\frac{k_1}{D_A}} 2x\right)\right) + C_{AR} \exp\left(\sqrt{\frac{k_1}{D_A}} b\right)\left(\exp\left(\sqrt{\frac{k_1}{D_A}} 2x\right) - 1\right)\right]}{\exp\left(\sqrt{\frac{k_1}{D_A}} 2b\right) - 1}}$$