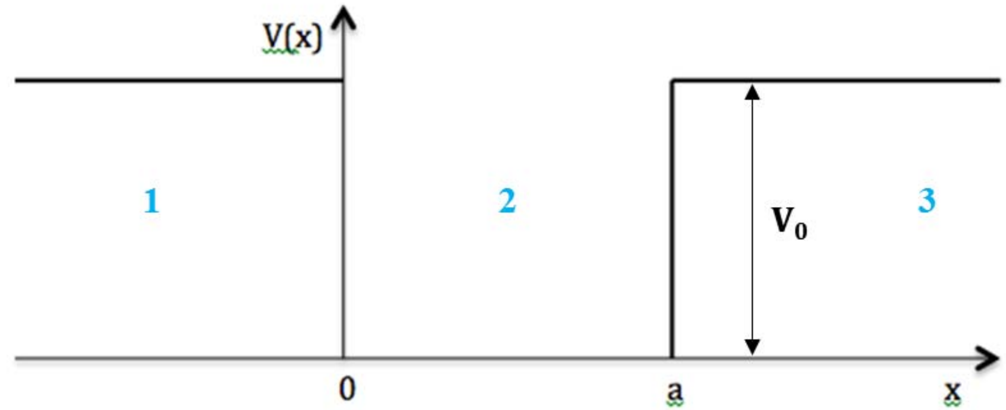
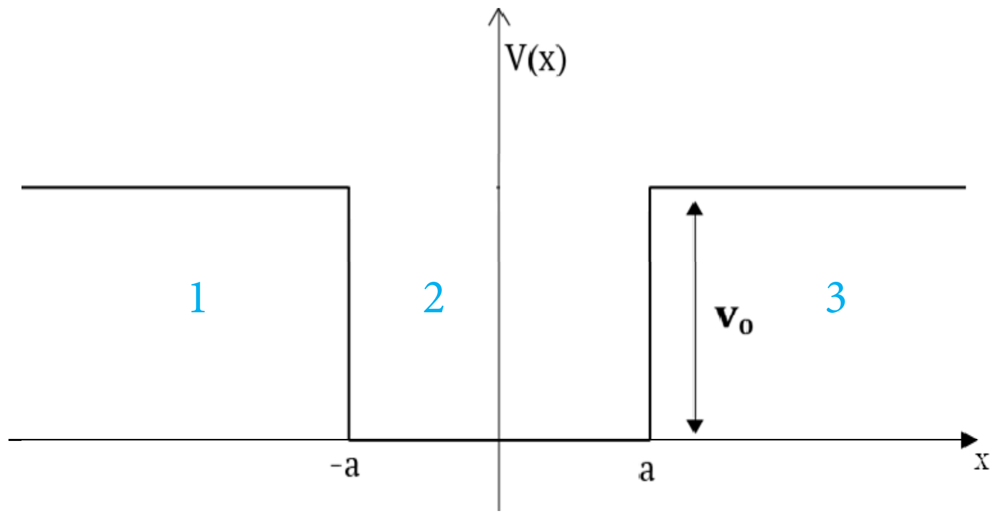
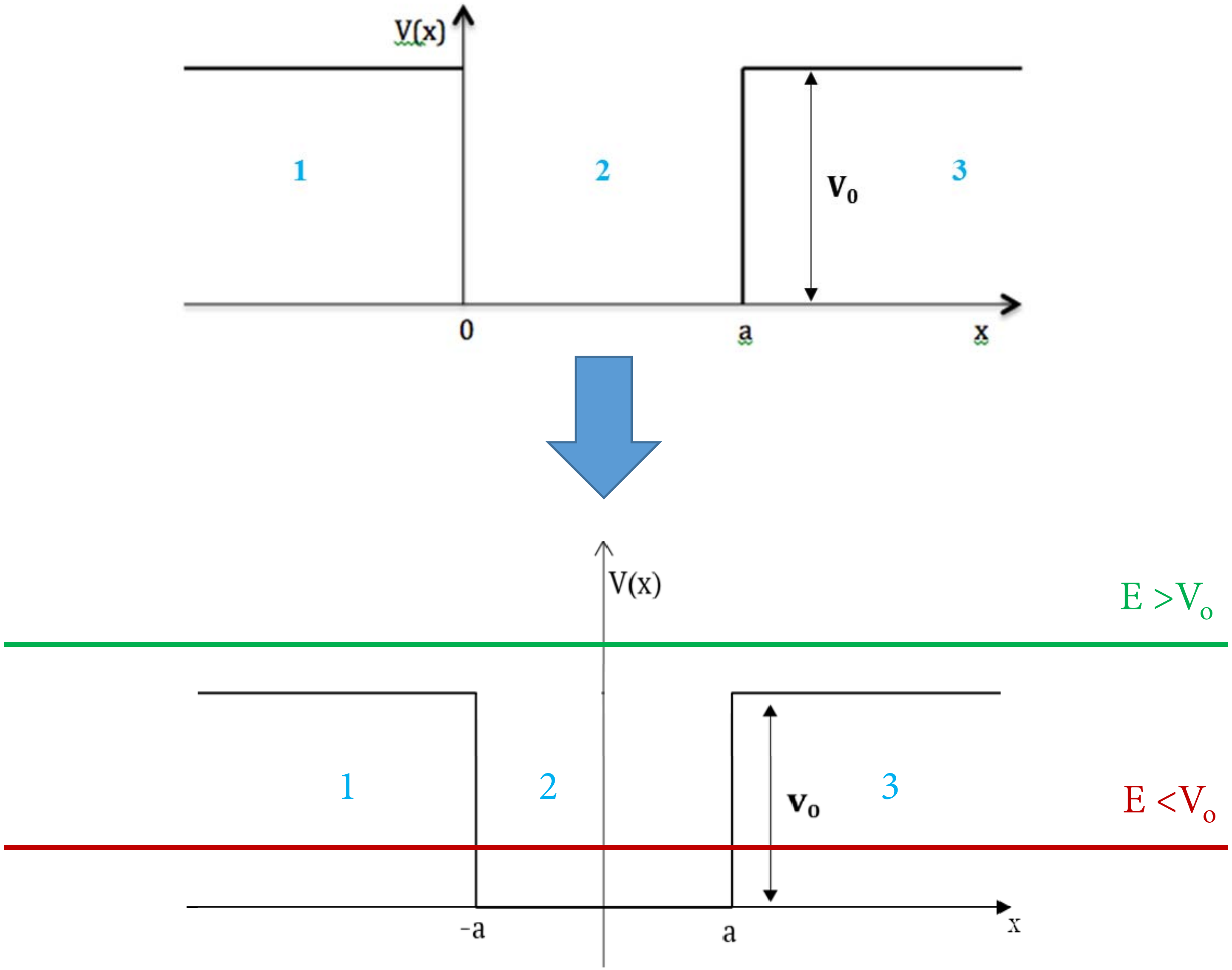
# Finite Potential Well

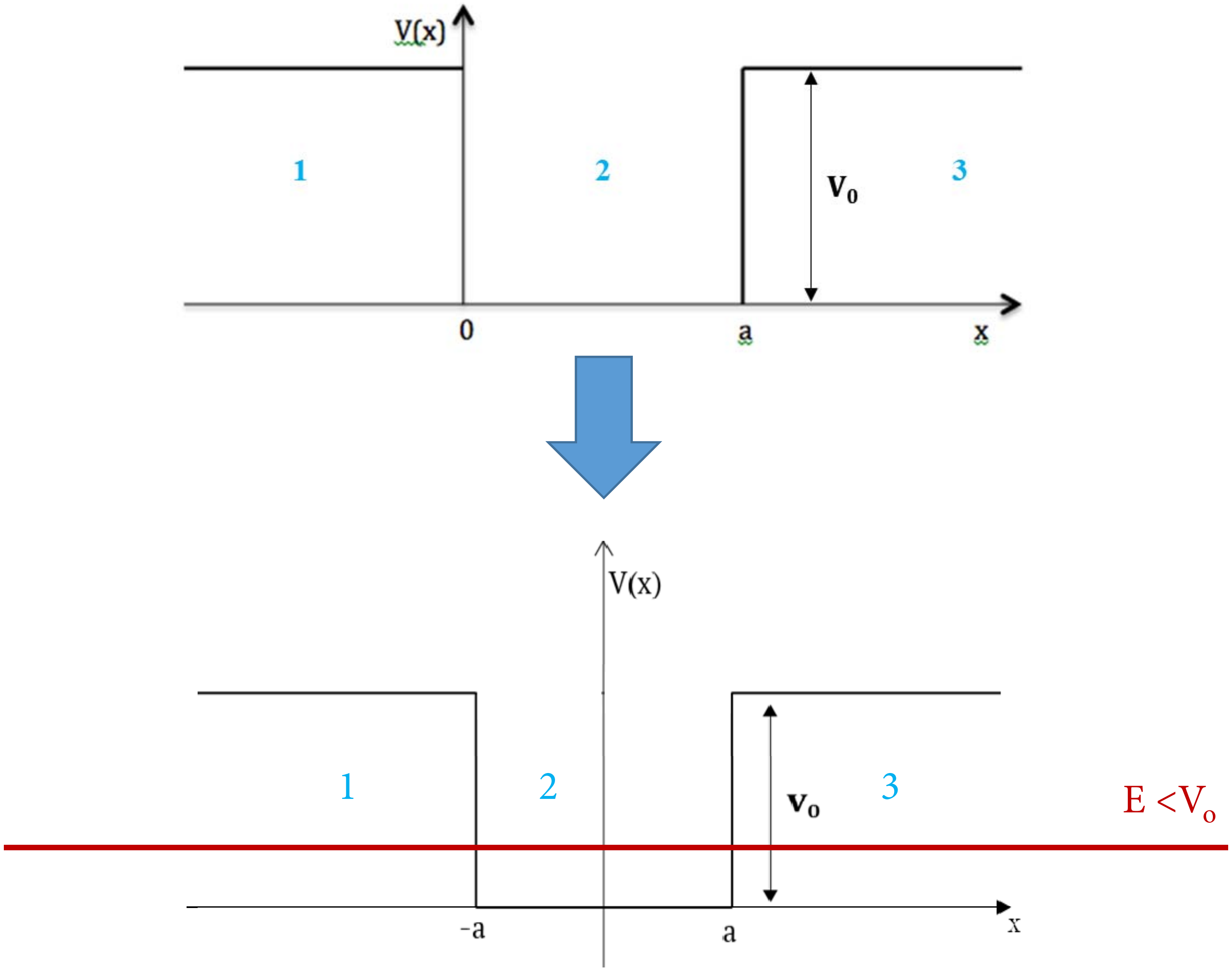



In order to simplify the solution…





V(x)
1
2
3
$V_0$
0
a
x
V(x)
$E > V_o$
1
2
3
$V_0$
$E < V_o$
-a
a
x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1
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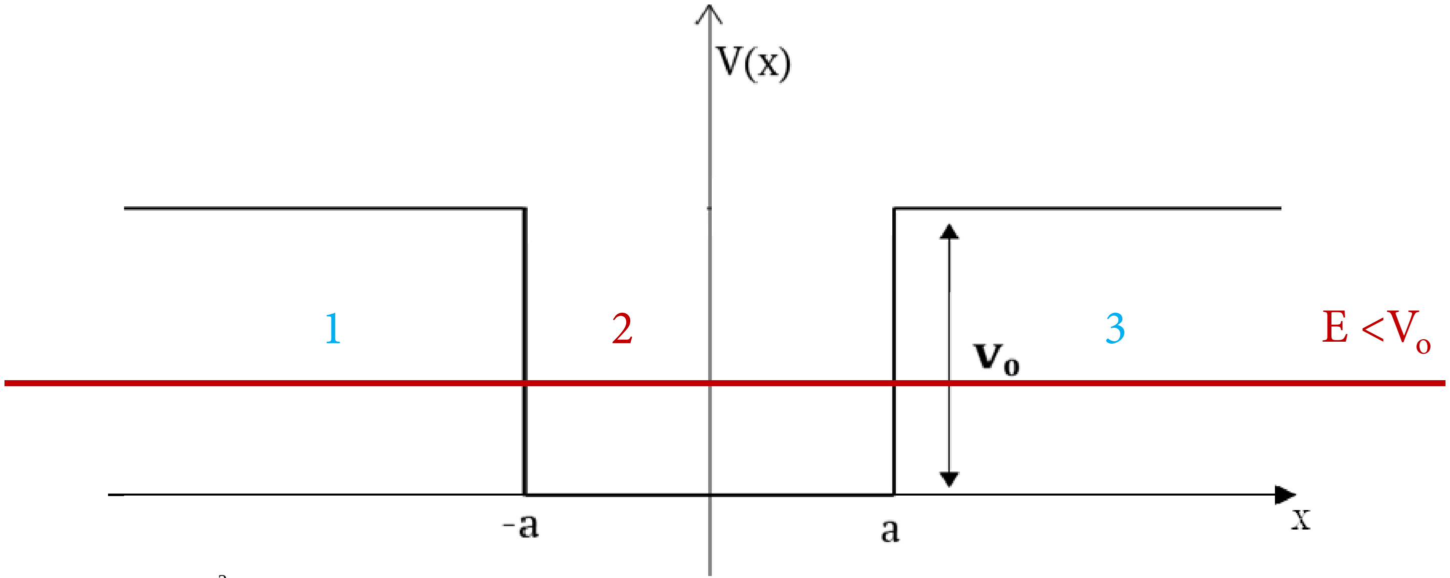



**Region 1**

$$H_{class} = \frac{p^2}{2m} + V_0$$

$$H_{quant} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0$$

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_1 = 0$$

$$\frac{2m}{\hbar^2}(E - V_0) < 0$$

$$\rho^2 = \frac{2m}{\hbar^2}(V_0 - E) > 0$$

$$\frac{\partial^2 \psi_1}{\partial x^2} = \rho^2 \psi_1$$

**Region 2**

$$H_{class} = \frac{p^2}{2m}$$

$$H_{quant} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$\frac{\partial^2 \psi_2}{\partial x^2} + \frac{2m}{\hbar^2}E\psi_2 = 0$$

$$\frac{2m}{\hbar^2}E > 0$$

$$k^2 = \frac{2m}{\hbar^2}E > 0$$

$$\frac{\partial^2 \psi_2}{\partial x^2} = -k^2 \psi_2$$

**Region 3**

$$H_{class} = \frac{p^2}{2m} + V_0$$

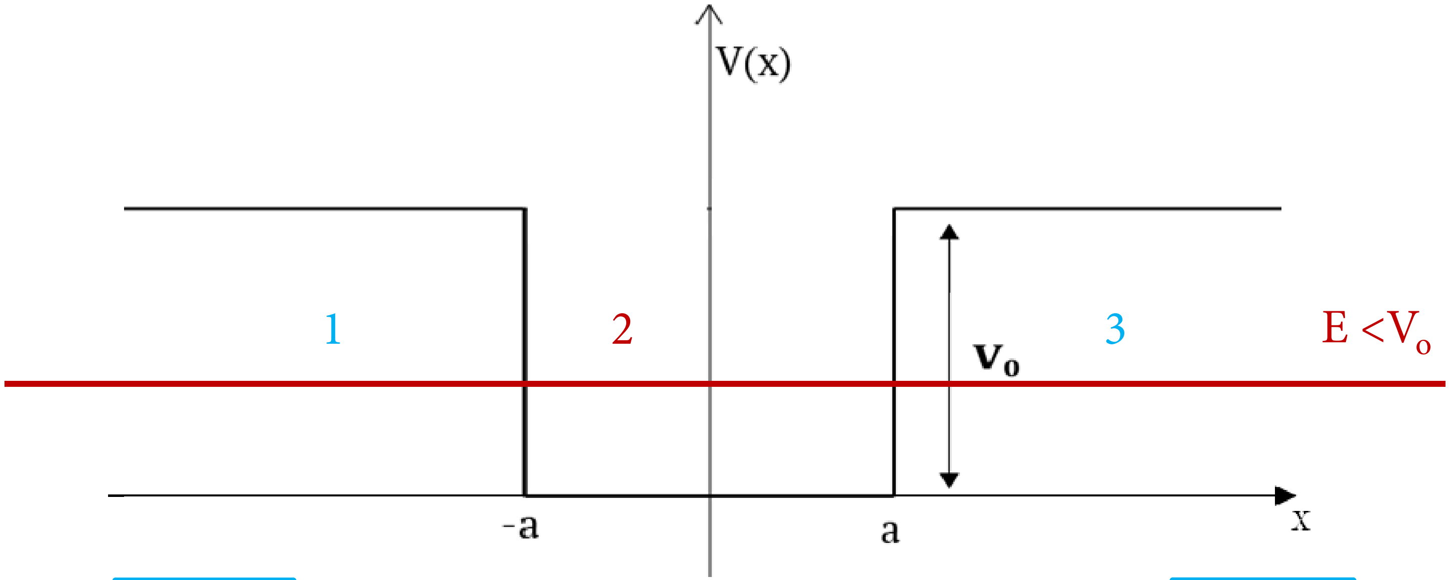
$$H_{quant} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0$$

$$\frac{\partial^2 \psi_3}{\partial x^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_3 = 0$$

$$\frac{2m}{\hbar^2}(E - V_0) < 0$$

$$\rho^2 = \frac{2m}{\hbar^2}(V_0 - E) > 0$$

$$\frac{\partial^2 \psi_3}{\partial x^2} = \rho^2 \psi_3$$




$$\frac{\partial^2\psi_1}{\partial x^2} = \rho^2\psi_1 \qquad \frac{\partial^2\psi_2}{\partial x^2} = -k^2\psi_2 \qquad \frac{\partial^2\psi_3}{\partial x^2} = \rho^2\psi_3$$

$$\psi_1(x) = B_1 e^{\rho x} + B_1' e^{-\rho x} \qquad \psi_2(x) = A_2 e^{ikx} + A_2' e^{-ikx} \qquad \psi_3(x) = B_3 e^{\rho x} + B_3' e^{-\rho x}$$

$x \to -\infty$ $\qquad\qquad$ $x \to \infty$

$$\psi_1(x) = B_1 e^{\rho x} \qquad\qquad \psi_3(x) = B_3' e^{-\rho x}$$

$$+$$

$$A \quad \begin{cases} \psi_1(-a) = \psi_2(-a) \\ \psi_1'(-a) = \psi_2'(-a) \end{cases} \qquad B \quad \begin{cases} \psi_2(a) = \psi_3(a) \\ \psi_2'(a) = \psi_3'(a) \end{cases}$$

$$\psi_1(x) = B_1 e^{\rho x}$$

$$\psi_2(x) = A_2 e^{ikx} + A_2' e^{-ikx}$$

$$\psi_3(x) = B_3' e^{-\rho x}$$

+

A
$$\begin{cases} \psi_1(-a) = \psi_2(-a) \\ \psi_1'(-a) = \psi_2'(-a) \end{cases}$$

B
$$\begin{cases} \psi_2(a) = \psi_3(a) \\ \psi_2'(a) = \psi_3'(a) \end{cases}$$

$$\begin{cases} B_1 e^{-\rho a} = A_2 e^{-ika} + A_2' e^{ika} & (1) \\ \rho B_1 e^{-\rho a} = ikA_2 e^{-ika} - ikA_2' e^{ika} \end{cases}$$

$$\begin{cases} A_2 e^{ika} + A_2' e^{-ika} = B_3' e^{-\rho a} & (1) \\ ikA_2 e^{ika} - ikA_2' e^{-ika} = -\rho B_3' e^{-\rho a} \end{cases}$$

$x\ (ik)\ \ in\ (1)$

$$\begin{cases} ikB_1 e^{-\rho a} = ikA_2 e^{-ika} + ikA_2' e^{ika} \\ \rho B_1 e^{-\rho a} = ikA_2 e^{-ika} - ikA_2' e^{ika} \end{cases}$$

$$\begin{cases} ikA_2 e^{ika} + ikA_2' e^{-ika} = ikB_3' e^{-\rho a} \\ ikA_2 e^{ika} - ikA_2' e^{-ika} = -\rho B_3' e^{-\rho a} \end{cases}$$

$$(ik+\rho) B_1 e^{-\rho a} = 2ikA_2 e^{-ika}$$

$$(ik-\rho) B_1 e^{-\rho a} = 2ikA_2' e^{ika}$$

$$(ik-\rho) B_3' e^{-\rho a} = 2ikA_2 e^{ika}$$

$$(ik+\rho) B_3' e^{-\rho a} = 2ikA_2' e^{-ika}$$

A
$$\begin{cases} A_2 = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_1 \\ A_2' = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_1 \end{cases}$$

B
$$\begin{cases} A_2 = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_3' \\ A_2' = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_3' \end{cases}$$

$\psi_1(x) = B_1 e^{\rho x}$ $\quad$ $\psi_2(x) = A_2 e^{ikx} + A_2' e^{-ikx}$ $\quad$ $\psi_3(x) = B_3' e^{-\rho x}$

$+$

$$A \quad \begin{cases} \psi_1(-a) = \psi_2(-a) \\ \psi_1'(-a) = \psi_2'(-a) \end{cases}$$

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$$(ik-\rho)B_3' e^{-\rho a} = 2ikA_2 e^{ika}$$

$$(ik+\rho)B_3' e^{-\rho a} = 2ikA_2' e^{-ika}$$

$$B \quad \begin{cases} A_2 = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_3' \\ A_2' = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_3' \end{cases}$$

$4 equations - 4 unknowns$

$$\psi_1(x) = B_1 e^{\rho x}$$

$$\psi_2(x) = A_2 e^{ikx} + A_2' e^{-ikx}$$

$$\psi_3(x) = B_3' e^{-\rho x}$$

**A**

$$\begin{cases} A_2 = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_1 \\ A_2' = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_1 \end{cases}$$

$4 equations - 4\ unknowns$

**B**

$$\begin{cases} A_2 = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_3' \\ A_2' = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_3' \end{cases}$$

$$\psi_1(x) = B_1 e^{\rho x}$$

$$\psi_2(x) = A_2 e^{ikx} + A_2' e^{-ikx}$$

$$\psi_3(x) = B_3' e^{-\rho x}$$

*A*

$$\begin{cases} A_2 = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_1 \\ A_2' = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_1 \end{cases}$$

*B*

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$$\begin{cases} \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_1 = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_3' \\ \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_1' = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_3' \end{cases}$$

$$\begin{cases} \dfrac{B_3'}{B_1} = \dfrac{ik+\rho}{ik-\rho} e^{2ika} \\ \dfrac{B_3'}{B_1} = \dfrac{ik-\rho}{ik+\rho} e^{-2ika} \end{cases}$$

$$\frac{ik+\rho}{ik-\rho} e^{2ika} = \frac{ik-\rho}{ik+\rho} e^{-2ika}$$

$$e^{4ika} = \left(\frac{ik-\rho}{ik+\rho}\right)^2 = \left(\frac{\rho-ik}{\rho+ik}\right)^2$$

(1) $$e^{2ika} = \frac{\rho-ik}{\rho+ik}$$

(2) $$e^{2ika} = -\frac{\rho-ik}{\rho+ik}$$

$$\psi_1(x) = B_1 e^{\rho x}$$

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$$\psi_3(x) = B_3' e^{-\rho x}$$

(1) $$e^{2ika} = \frac{\rho - ik}{\rho + ik}$$

$$e^{2ika} = \frac{\rho + ik}{\rho + ik} \cdot \frac{\rho - ik}{\rho + ik} = \frac{\rho^2 + k^2}{(\rho + ik)^2}$$

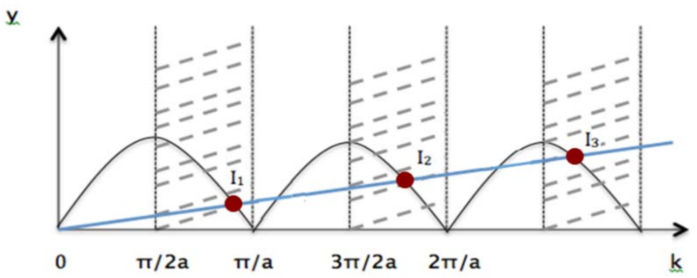
$$\rho + ik = \sqrt{\rho^2 + k^2}\, e^{-ika}$$

$$tg(-ka) = \frac{k}{\rho} > 0$$

$$1 + cotg^2(ka) = \frac{1}{\sin^2(ka)}$$

$$1 + \frac{\rho^2}{k^2} = \frac{1}{\sin^2(ka)}$$

$$\begin{cases} |\sin(ka)| = \dfrac{k}{\sqrt{k^2 + \rho^2}} = \dfrac{k}{k_0} \\ tg(ka) < 0 \end{cases}$$




$$z = x + iy = re^{i\varphi} = \sqrt{x^2 + y^2}\, e^{i\varphi}$$

$$\varphi = arctg\,\frac{y}{x}$$

$$\frac{y}{x} = tg\varphi$$

$$k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$$

(2) $$e^{2ika} = -\frac{\rho - ik}{\rho + ik}$$

$$e^{2ika} = \frac{\rho - ik}{\rho - ik} \cdot \frac{ik - \rho}{\rho + ik} = \frac{(k + i\rho)^2}{\rho^2 + k}$$

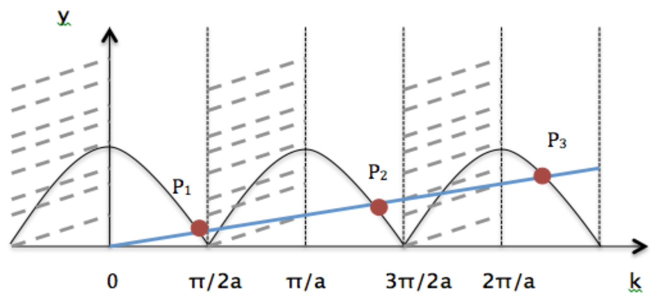
$$k + i\rho = \sqrt{k^2 + \rho^2}\, e^{ika}$$

$$tg(ka) = \frac{\rho}{k} > 0$$

$$1 + tg^2(ka) = \frac{1}{\cos^2(ka)}$$

$$1 + \frac{\rho^2}{k^2} = \frac{1}{\cos^2(ka)}$$

$$\begin{cases} |\cos(ka)| = \dfrac{k}{\sqrt{k^2 + \rho^2}} = \dfrac{k}{k_0} \\ tg(ka) > 0 \end{cases}$$



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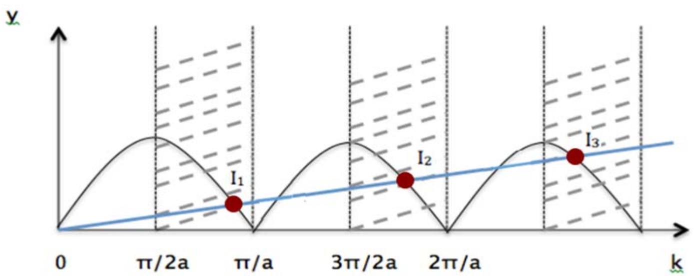


$$z = x + iy = re^{i\varphi} = \sqrt{x^2 + y^2}\, e^{i\varphi}$$

$$\varphi = arctg\,\frac{y}{x}$$

$$\frac{y}{x} = tg\varphi$$

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(2) $$e^{2ika} = -\frac{\rho - ik}{\rho + ik}$$

$$e^{2ika} = \frac{\rho - ik}{\rho - ik} \cdot \frac{ik - \rho}{\rho + ik} = \frac{(k + i\rho)^2}{\rho^2 + k}$$

$$k + i\rho = \sqrt{k^2 + \rho^2}\, e^{ika}$$

$$tg(ka) = \frac{\rho}{k} > 0$$

$$1 + tg^2(ka) = \frac{1}{\cos^2(ka)}$$

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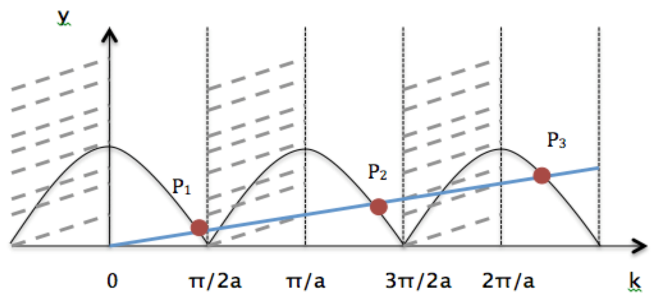

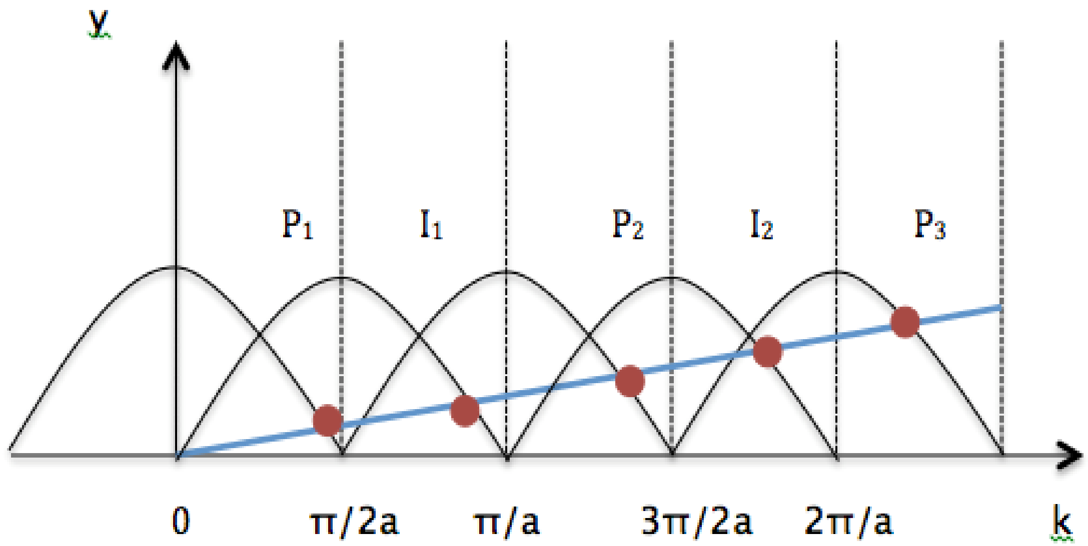
y
P1
I1
P2
I2
P3
0
π/2a
π/a
3π/2a
2π/a
k

Solution to the eigenfunctions…

$$\psi_1(x) = B_1 e^{\rho x}$$

$$\psi_2(x) = A_2 e^{ikx} + A_2' e^{-ikx}$$

$$\psi_3(x) = B_3' e^{-\rho x}$$

(1) $$e^{2ika} = \frac{\rho - ik}{\rho + ik}$$

(2) $$e^{2ika} = -\frac{\rho - ik}{\rho + ik}$$

A $$\begin{cases} A_2 = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_1 \\ A_2' = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_1 \end{cases}$$

B $$\begin{cases} A_2 = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_3' \\ A_2' = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_3' \end{cases}$$

$$\begin{cases} \psi_1(x) = B_1 e^{\rho x} \\ \psi_2(x) = A_2 e^{ikx} + A_2 e^{-ikx} = 2A_2 \cos(kx) \\ \psi_3(x) = B_1 e^{-\rho x} \end{cases}$$

# Summary

$$\psi_1(x) = B_1 e^{\rho x}$$

$$\psi_2(x) = A_2 e^{ikx} + A_2' e^{-ikx}$$

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A $\begin{cases} \psi_1(-a) = \psi_2(-a) \\ \psi_1'(-a) = \psi_2'(-a) \end{cases}$

B $\begin{cases} \psi_2(a) = \psi_3(a) \\ \psi_2'(a) = \psi_3'(a) \end{cases}$

A $\begin{cases} A_2 = \dfrac{ik+\rho}{2ik} e^{(ik-\rho)a} B_1 \\ A_2' = \dfrac{ik-\rho}{2ik} e^{-(ik+\rho)a} B_1 \end{cases}$

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$$\frac{ik+\rho}{ik-\rho} e^{2ika} = \frac{ik-\rho}{ik+\rho} e^{-2ika}$$

$$e^{4ika} = \left(\frac{ik-\rho}{ik+\rho}\right)^2 = \left(\frac{\rho-ik}{\rho+ik}\right)^2$$

$$e^{2ika} = \frac{\rho-ik}{\rho+ik} \quad (1)$$

$$(2) \quad e^{2ika} = -\frac{\rho-ik}{\rho+ik}$$

$$\begin{cases} |\sin(ka)| = \dfrac{k}{\sqrt{k^2+\rho^2}} = \dfrac{k}{k_0} \\ tg(ka) < 0 \end{cases}$$

$$\begin{cases} \psi_1(x) = B_1 e^{\rho x} \\ \psi_2(x) = A_2 e^{ikx} + A_2 e^{-ikx} = 2A_2 \cos(kx) \\ \psi_3(x) = B_1 e^{-\rho x} \end{cases}$$

$$\begin{cases} |\cos(ka)| = \dfrac{k}{\sqrt{k^2+\rho^2}} = \dfrac{k}{k_0} \\ tg(ka) > 0 \end{cases}$$