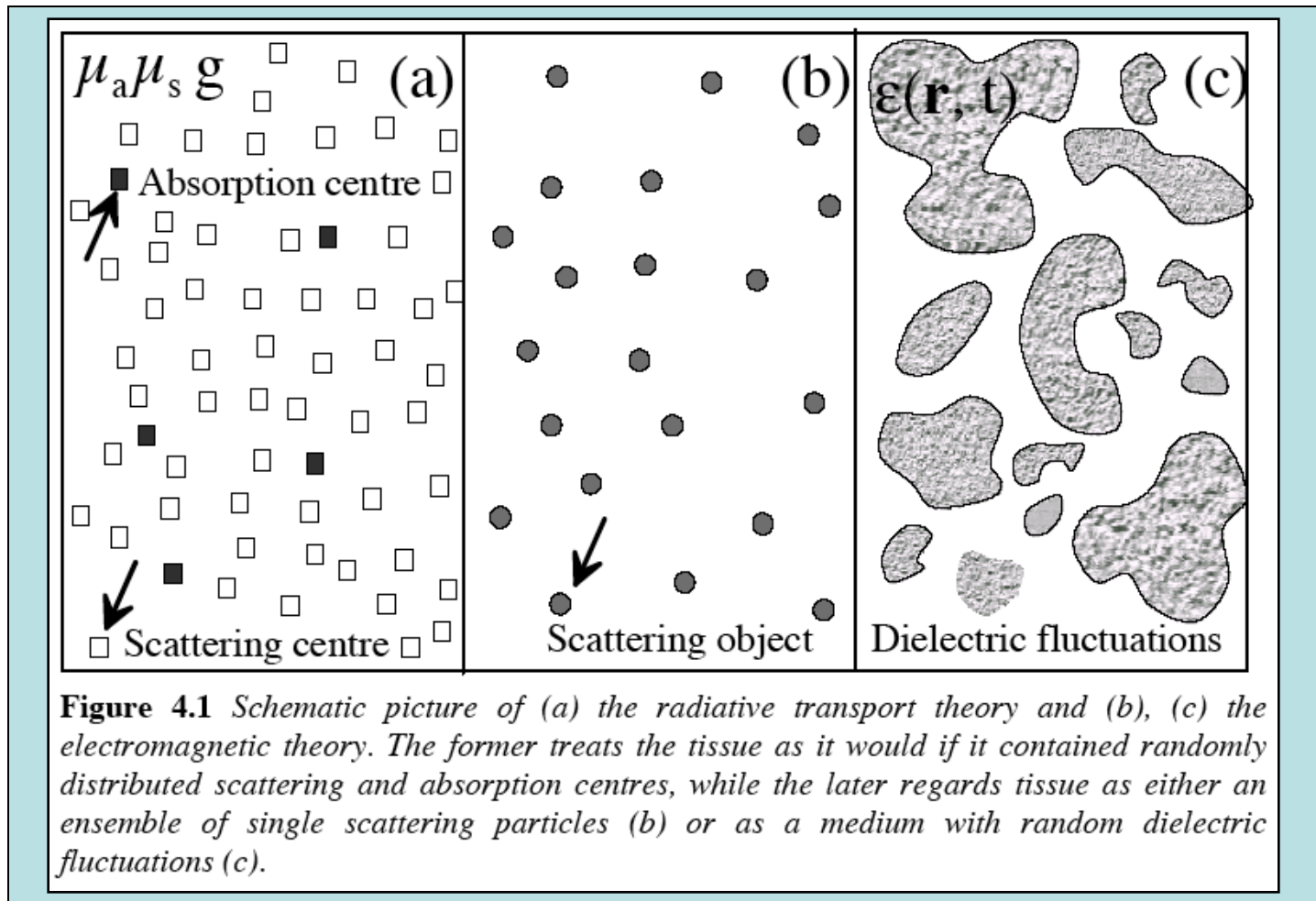


## **5.2 The Radiative Transport Equation (RTE)**

# Dualism of light

- **Wave picture:** light is considered as an **electromagnetic wave** and modeled by the **Maxwell equations**
- **Particle picture:** light is considered as a stream of energetic particles - photons and modeled by **energy conservation** - the **transport equation**

# Schematic picture of the radiative transport theory



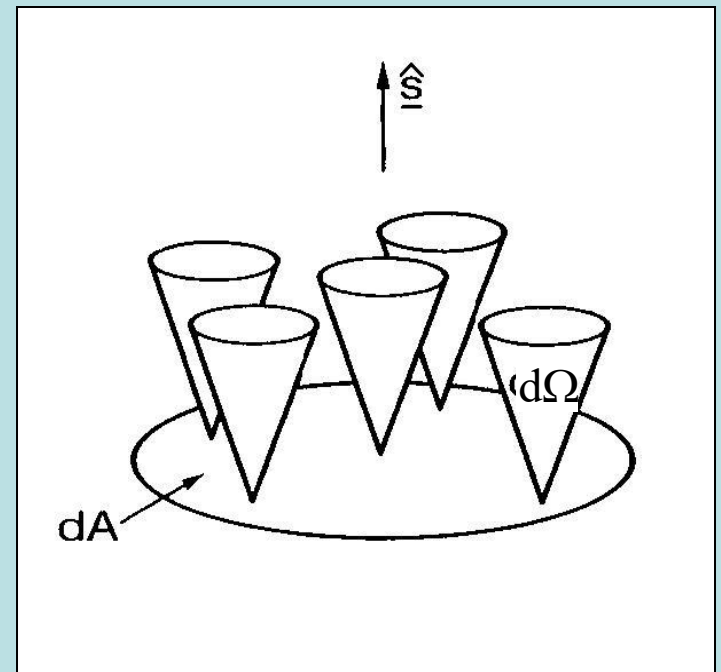
# How do we reach an expression for light transport in tissue?

To get an expression that could be solved either analytically or numerically, we start to look into **conservation of energy** in a small volume of tissue

# Radiance

**Radiance**,  $L(\mathbf{r}, \mathbf{s})$ , is the quantity used to describe the propagation of photon power.

**Definition:** **Radiance** is the power [W] that passes through or is emitted from a particular area  $dA$  [m<sup>2</sup>] and falls within a given solid angle  $d\Omega$  [steradian] in a specified direction  $\mathbf{s}$ .



# The photon distribution function $N(\mathbf{r}, \mathbf{s}, t)$

**Definition:** The **photon distribution function**  $N(\mathbf{r}, \mathbf{s}, t)d^3r d\Omega$  is the number of photons in the volume  $d^3r$  within  $d\Omega$  with the direction  $\mathbf{s}$  at time  $t$ .

The unit of  $N(\mathbf{r}, \mathbf{s}, t)$  is [photons  $\text{m}^{-3}\text{sr}^{-1}$ ].

The **radiance** is obtained by multiplying  $N$  by the photon energy and the velocity of light in the medium:

$$L(\mathbf{r}, \mathbf{s}, t) = N(\mathbf{r}, \mathbf{s}, t)h\nu \cdot c$$

# Radiant Energy Fluence Rate $F(\mathbf{r},t)$

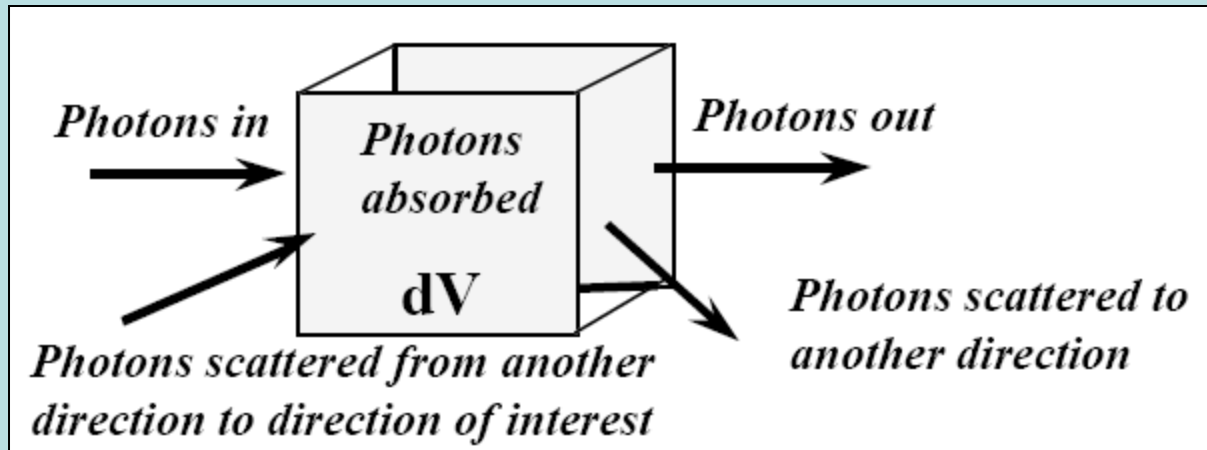
The integral of the radiance over all directions, called the **fluence rate**  $F(\mathbf{r})$  [W/m<sup>2</sup>], has more practical significance than the radiance itself.

$$F(\mathbf{r}, t) = \int_{4\pi} L(\mathbf{r}, \mathbf{s}, t) d\Omega$$

This is because an absorption chromophore located at  $\mathbf{r}$  can absorb photons irrespectively of their direction of propagation

# Radiative Transport Equation RTE (I)

- Consider a small volume  $dV$  and a direction  $s$ .
- Conservation of energy yields that photons can only be added to or subtracted from the photon distribution function in specific interactions.



# Radiative Transport Equation

Radiance  $L$  ( $\text{W}\cdot\text{m}^{-2}\cdot\text{sr}^{-1}$ )

Change of radiance  $L$

$$\frac{1}{c} \frac{\partial L(\mathbf{r}, \mathbf{s}, t)}{\partial t} + \mathbf{s} \cdot \nabla L(\mathbf{r}, \mathbf{s}, t) = -(\mu_a(\mathbf{r}) + \mu_s(\mathbf{r})) L(\mathbf{r}, \mathbf{s}, t)$$

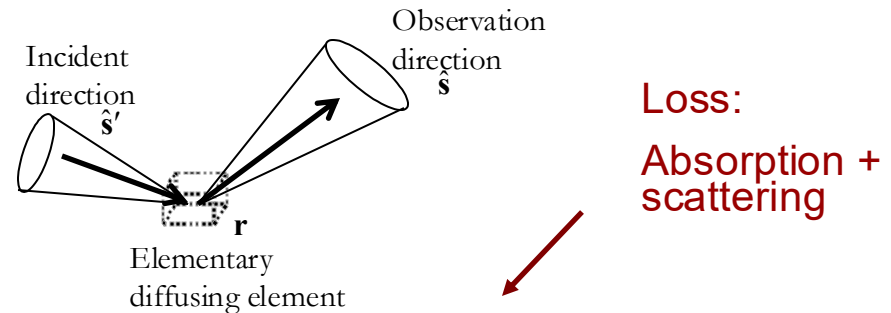
Gains/losses at boundaries

$$+ \mu_s(\mathbf{r}) \int_{4\pi} p(\mathbf{s}', \mathbf{s}) L(\mathbf{r}, \mathbf{s}', t) d^2 s' + q(\mathbf{r}, \mathbf{s}, t)$$

Gains: scattering + source

Integro-differential equation difficult to solve

→ Use the diffusion approximation equation.



# First Term

Change of photon distribution

$$\int_V \frac{\partial N(\mathbf{r}, \mathbf{s}, t)}{\partial t} dV$$

# Gauss Theorem

**Gauss' theorem** relates the flow of a vector field through a surface to the behavior of the vector field inside the surface. More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence of the region inside the surface.

Intuitively, it states that *the sum of all sources minus the sum of all sinks gives the net flow out of a region.*

$$\int_V \nabla f dV = \oint_S f dS$$

# Second Term

Photons lost at boundaries with Gauss Theorem :

$$-c \int_V \mathbf{s} \cdot \nabla N(\mathbf{r}, \mathbf{s}, t) dV = -c \oint_S \mathbf{s} \cdot N(\mathbf{r}, \mathbf{s}, t) dS$$

# Third term

Loss (II): Photons scattered  
from direction  $\mathbf{s}$  to any other direction  $\mathbf{s}'$

$$- \int_V c\mu_s(\mathbf{r})N(\mathbf{r}, \mathbf{s}, t)dV$$

# Forth term

Loss (III): Photons absorbed coming from direction  $\mathbf{s}$

$$- \int_V c\mu_a(\mathbf{r})N(\mathbf{r}, \mathbf{s}, t)dV$$

# Fifth Term

Gain (I): Photons gained through scattering from any direction  $\mathbf{s}'$  into direction  $\mathbf{s}$ .

$$+ \int_{V_v} c\mu_s(\mathbf{r})dV \int_{4\pi} p(\mathbf{s}', \mathbf{s})N(\mathbf{r}, t, \mathbf{s}')ds'$$

# Sixth Term

Gain (II): Photons gained through a light source  $q$

$$+ \int_V q(\mathbf{r}, \mathbf{s}, t) dV$$

# Final Transport Equation

$$\frac{1}{c} \frac{\partial L(\mathbf{r}, \mathbf{s}, t)}{\partial t} = -\mathbf{s} \cdot \nabla L(\mathbf{r}, \mathbf{s}, t) - (\mu_a + \mu_s) L(\mathbf{r}, \mathbf{s}, t) + \mu_s \int_{4\pi} p(\mathbf{s}', \mathbf{s}) L(\mathbf{r}, \mathbf{s}') d\Omega' + q(\mathbf{r}, \mathbf{s}, t)$$

because:

$$L(\mathbf{r}, \mathbf{s}) = N(\mathbf{r}, \mathbf{s}) h\nu \cdot c$$

# Steady State Transport Equation

$$\mathbf{s} \cdot \nabla L(\mathbf{r}, \mathbf{s}) = \frac{\partial L(\mathbf{r}, \mathbf{s})}{\partial s} =$$
$$-(\mu_a + \mu_s)L(\mathbf{r}, \mathbf{s}) + \mu_s \int_{4\pi} p(\mathbf{s}', \mathbf{s}) L(\mathbf{r}, \mathbf{s}') d\Omega' + q(\mathbf{r}, \mathbf{s})$$

# How to Solve RTE ?

## Computer-based

### Numerical Techniques

Discretization of equation  
finite-difference and finite element method  
(i.e. with COMSOL)  
**difficult to implement, fast results,**  
**accuracy depends on discretization (Mesh)**

### Monte Carlo Simulations

Stochastic solution methods. Many single photon are “randomly” propagated (i.e. with TRACEPRO)  
Indirect solution of the RTE.  
**easy to implement, “time consuming”**  
**very accurate**

## Analytical

### Diffusion Approximation

Approximating RTE to find simple partial differential equations.  
**easy to implement**  
**fast results**  
**limited accuracy**

### Kubelka Munk Approximation

1-D Model based on the study of the propagation of diffuse flux.  
**easy to implement**  
**fast results**  
**limited accuracy**  
**1-D geometry and diffuse flux**

# Approximation Criteria

- **Scattering dominant regime (red + near infrared)**

$$\mu'_s \gg \mu_a$$

Diffusion approximation can be used to describe light transport

- **Intermediate regime (visible)**

$$\mu'_s \approx \mu_a$$

Most difficult to handle rigorously

→ generally use Monte Carlo methods

# Diffusion Approximation

- For a dense medium of primarily scatterers, the transport equation can be simplified to the **Diffusion Equation**
- Can be solved **analytically** for special cases or more generally by **numerical techniques**
- Validity of diffusion equation is limited to tissue cases where the light has been highly scattered ( $\mu'_s \gg \mu_a$ ); i.e.,  $\mu'_s$  should be at least 10 times greater than  $\mu_a$

# Monte Carlo Approximation

- For situations where Diffusion theory breaks down, the most useful method is **Monte Carlo modeling**
  - Computational technique which simulates multiple scattering trajectories of individual photons through a turbid medium
  - Each interaction is governed by the random processes of absorption and scattering