Exercise 9 solution: Van der Pol oscillator

11/11/25

$$\dot{x} = y$$
$$\dot{y} = -x + \mu(1 - x^2)y$$

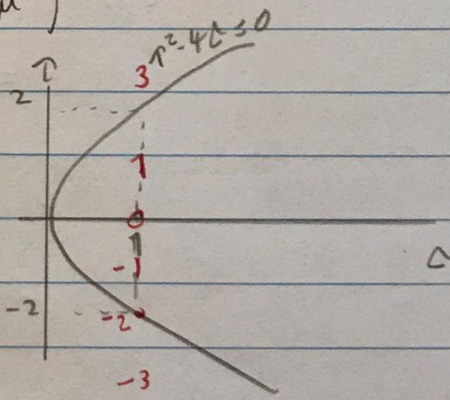
$\mu \in \mathbb{R}$
model defined everywhere
(unless otherwise stated)

1) Fixed point $(0, 0)$

$$\underline{\underline{J}} = \begin{pmatrix} 0 & 1 \\ -1 - 2\mu x y & \mu(1 - x^2) \end{pmatrix}$$

so $\underline{\underline{J}}(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$

$$\tau = \mu$$
$$\Delta = 1$$



2) $\Delta = 1 > 0$ so it cannot be a saddlepoint because that requires $\Delta < 0$.

3) We need τ, Δ for each point in: $-3, -2, -1, 0, 1, 3$

T	Δ	Type	stability
-3	1	Node	Stable
-2	1	star/Deg. node	"
-1	1	spiral	"
0	1	Centre	neutrally stable
1	1	spiral	unstable
3	1	node	unstable

For $\mu = -2$, $\Delta = 1$ $\quad \underline{J} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

Eigenvalue: $\begin{vmatrix} 0-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow (-\lambda)(-2-\lambda) + 1 = 0$

$\Rightarrow \lambda^2 + 2\lambda + 1 = 0$

$\therefore \lambda = \dfrac{-2 \pm \sqrt{4-4}}{2} = \underline{-1}$

only one eigenvalue

Eigenvector

$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow v_1 + v_2 = 0 \quad \therefore \underline{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General solution is:

$$\underline{x} = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We have a single eigenvalue, so it is a degenerate node.

But this cannot be the whole solution, but we don't need it for this question.

4) $\mu = -1$

$$\Rightarrow \underline{\underline{J}}(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \therefore \tau = -1, \ \Delta = 1 \quad \therefore \text{stable spiral}$$

nullclines?

$\dot{x} = 0 \Rightarrow y = 0$ i.e. x axis

$\dot{y} = 0 \Rightarrow -x - (1 - x^2)y = 0$

$$\therefore y = \frac{-x}{1 - x^2}$$

Direction of vector field:

What is $\dot{y}$ along $\dot{x} = 0$ i.e. x axis

$\dot{y} = -x$ so < 0 for $x > 0$, and vice versa and what is $\dot{x}$ along $y=0$, i.e. along $y = \frac{-x}{1-x^2}$

$$\dot{x} = y = \frac{-x}{1-x^2}$$

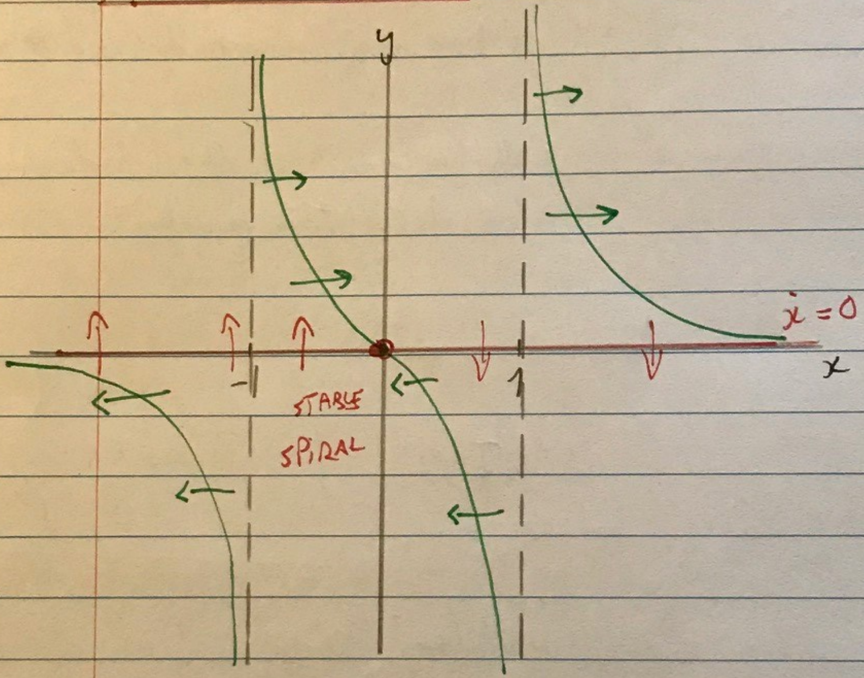
Slope of nullcline at $x=0$:

$$y = \frac{-x}{1-x^2}$$

$$\Rightarrow y' = \frac{(1-x^2).-1 + x.-2x}{(1-x^2)^2}$$

$$= \frac{-1-x^2}{(1-x^2)^2}$$

$$\lim_{x\to 0} y' = -1$$



5) It changes from stable node $\rightarrow$ stable spiral $\rightarrow$ centre $\rightarrow$ unstable spiral $\rightarrow$ unstable node

6) $\dot{x} = y + \mu x$

$\dot{y} = -x + \mu(1-x^2)y$

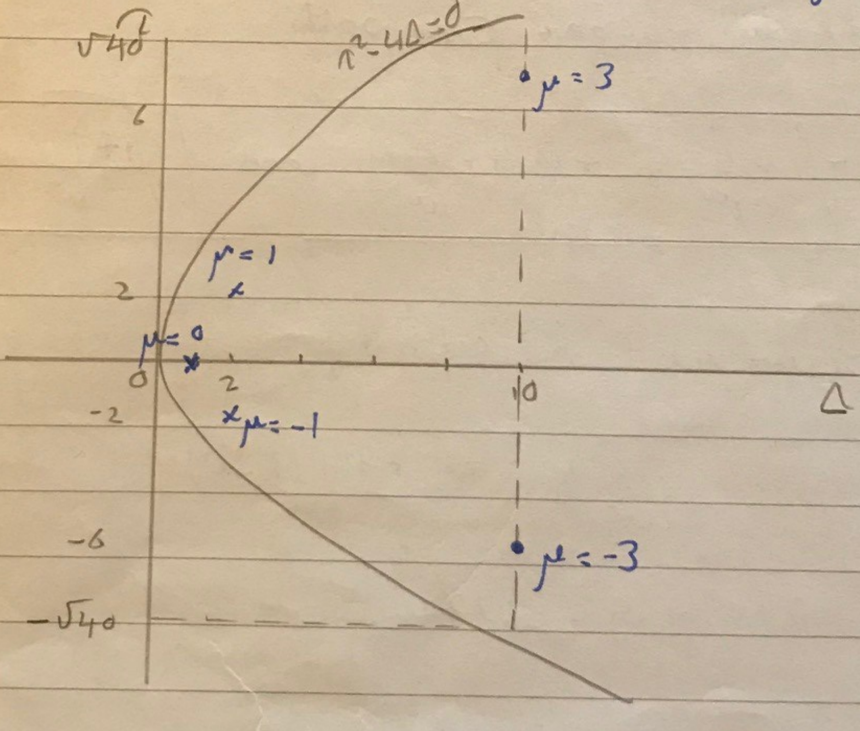
$$\underline{\underline{J}} = \begin{pmatrix} \mu & 1 \\ -1-2\mu x y & \mu(1-x^2) \end{pmatrix}$$

For the fixed point $(0,0)$, we have $\underline{\underline{J}} = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$

$\therefore \tau = 2\mu$ $\Rightarrow \tau^2 - 4\Delta = 4\mu^2 - 4(\mu^2+1) = -4 \; \forall \mu$ $\therefore$ it is a spiral for all μ

$\Delta = \mu^2 + 1$ $\therefore$ It can never be a saddle point as $\Delta > 0$ for all μ.

$\mu = -3$	$\tau = -6$	$\Delta = 10$
$\mu = -1$	$\tau = -2$	$\Delta = 2$
$\mu = 0$	$\tau = 0$	$\Delta = 1$
$\mu = 1$	$\tau = 2$	$\Delta = 2$
$\mu = 3$	$\tau = 6$	$\Delta = 10$



6.2) other fixed points must satisfy:

$$\mu x + y = 0 \qquad \Rightarrow \quad y = -\mu x$$

$$-x + \mu(1 - x^2)y = 0$$

$$\Rightarrow -x + \mu(1 - x^2) . -\mu x = 0$$

$$\Rightarrow \mu^2 x(1 - x^2) = -x$$

$\therefore x = 0$ or $\quad \mu^2(1 - x^2) = -1$

$$\therefore \mu^2 + 1 = \mu^2 x^2$$

$$\therefore x^2 = 1 + \frac{1}{\mu^2} \quad \text{or} \quad x = \pm\sqrt{1 + \frac{1}{\mu^2}}$$

There are two non-zero fixed points:

$$\underline{\left(-\sqrt{1 + \frac{1}{\mu^2}}, +\mu\sqrt{1 + \frac{1}{\mu^2}}\right)} \quad \text{and} \quad \underline{\left(\sqrt{1 + \frac{1}{\mu^2}}, -\mu\sqrt{1 + \frac{1}{\mu^2}}\right)}$$

6.3) For $\mu = 0.1$, $\dot{x} = 0.1x + y$

$$\dot{y} = -x + 0.1(1 - x^2)y$$

Nullclines: $\dot{x} = 0 \Rightarrow y = -0.1x$

$$\dot{y} = 0 \Rightarrow y = \frac{10x}{1 - x^2}$$

6.2

For the second model, we have:

$$\underline{J} = \begin{pmatrix} \mu & 1 \\ -1-2\mu xy & \mu(1-x^2) \end{pmatrix}$$

For the FP $(x^*, y^*) = \left(-\sqrt{1+\frac{1}{\mu^2}}, \mu\sqrt{1+\frac{1}{\mu^2}}\right)$

$$\tau = \mu + \mu(1-x^{*2})$$

$$= \mu + \mu\left[1 - \left(-\left(1+\frac{1}{\mu^2}\right)^{\frac{1}{2}}\right)^2\right]$$

$$= \mu + \mu\left[1 - \left(1+\frac{1}{\mu^2}\right)\right] = \mu - \mu = 0$$

$$\Delta = \mu^2(1-x^2) + (1+2\mu xy)$$

$$= \mu^2\left[1 - \left(1+\frac{1}{\mu^2}\right)\right] + \left(1 + 2\mu\cdot -\mu\left(1+\frac{1}{\mu^2}\right)\right)$$

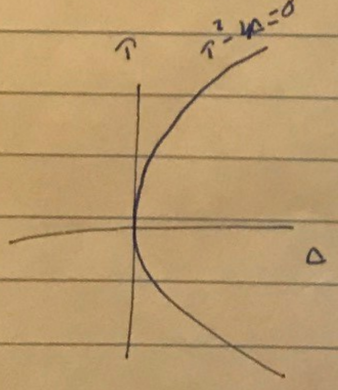
$$= -1 + 1 - 2\mu^2\left(1+\frac{1}{\mu^2}\right)$$

$$= \cancel{2} - 2\mu^2 = -2(1+\mu^2)$$

$\therefore$ ~~If $\mu^2 < \frac{1}{2}$ it is a centre, but if $\mu^2 > 1/2$, it's a~~ saddle point.

For $\mu = 0.1$
$\tau = 0$
$\Delta = 1 - 0.02 = 0.98$
$(x^*, y^*) = (10.05, 1.05)$

For $\mu = 1$, $\tau = 0$
$\Delta = -4$
$(x^*, y^*) = (-\sqrt{2}, \sqrt{2})$



$$f_x(x^*, y^*) = \left(\sqrt{1 + \frac{1}{\mu^2}}, -\mu\sqrt{1 + \frac{1}{\mu^2}} \right)$$

$$\tau = \mu + \mu(1 - x^{*2})$$

$$= \mu + \mu\left(1 - \left(1 + \frac{1}{\mu^2}\right)\right) = \mu + \mu \cdot \frac{-1}{\mu^2} = 0$$

$$\Delta = \mu^2(1 - x^2) + 1 + 2\mu x y$$

$$= \mu^2\left(1 - \left(1 + \frac{1}{\mu^2}\right)\right) + 1 + 2\mu\left(1 + \frac{1}{\mu^2}\right)^{\frac{1}{2}} \cdot -\mu\left(1 + \frac{1}{\mu^2}\right)^{\frac{1}{2}}$$

$$= \cancel{-1} + \cancel{1} - 2\mu^2\left(1 + \frac{1}{\mu^2}\right)$$

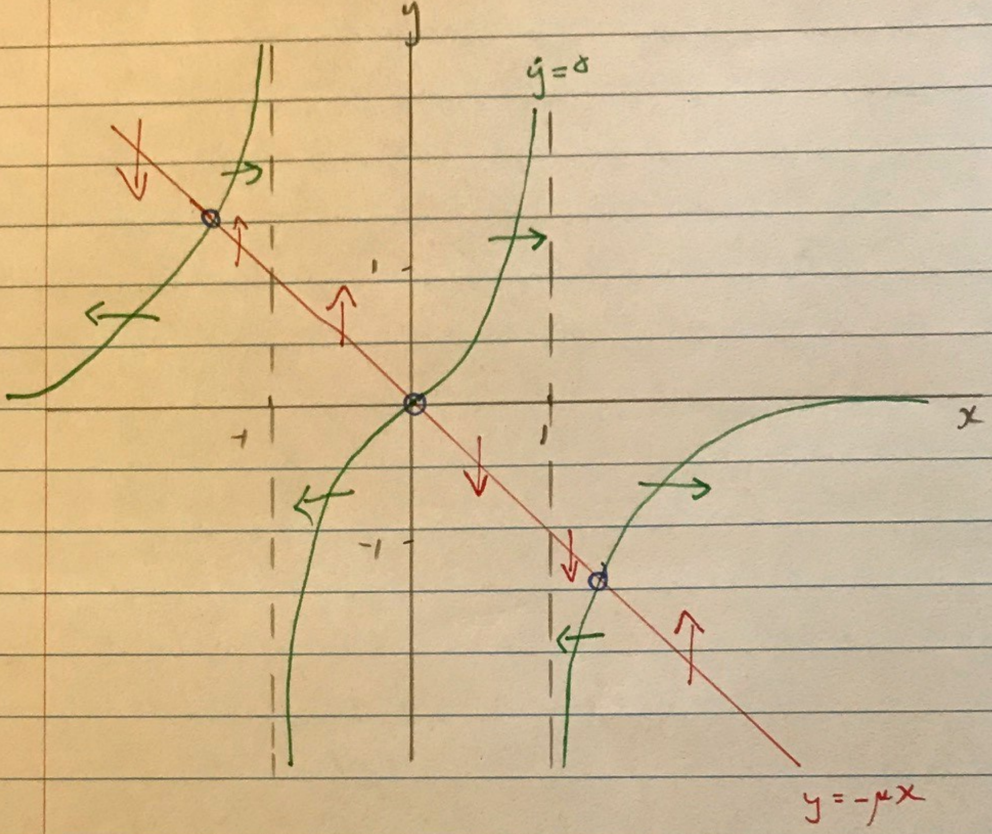
$$= -2 - 2\mu^2 = -2(1 + \mu^2)$$

Where are the nullclines?

$\dot{x} = 0 \Rightarrow y = -\mu x$ i.e. straight line through origin

$\dot{y} = 0 \Rightarrow -x + \mu(1 - x^2)y = 0$

$$\therefore y = \frac{x}{\mu(1 - x^2)}$$



What is $\dot x$ along $\dot y = 0$

$$\dot x = \mu x + y = \mu x + \frac{x}{\mu(1-x^2)} = x\left(\mu + \frac{1}{\mu(1-x^2)}\right)$$

if $0 < x < 1 \Rightarrow \dot x > 0$

if $-1 < x < 0 \Rightarrow \dot x < 0$

if $x > 1$ and less than x^*,

and $\dot y$ along $\dot x = 0 \Rightarrow \dot y = -x + \mu(1-x^2)y = -x + (1-x^2)\cdot -\mu x$

$$= -x - \mu x(1-x^2)$$

$$= -(1+\mu)x + \mu x^3$$ so if $x \gg 0$ $\dot y > 0$ [illegible]