Exercise 5: Models for competitive exclusion, microbial ecology

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Week 5: Graded Series

Course: BIO-341 *Dynamical systems in biology*

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SSV, BA5, 2025 (Graded exercise series)

All working by hand and calculations need to be shown

1 Linear phase portraits (10 pts)

Consider the following linear system:

$$\dot{x} = y$$
$$\dot{y} = -2x - 3y$$

1. Find the type of the fixed point(s) and specify its/their stability. (by hand)

2. Find the eigenvalues and eigenvectors of the matrix associated with this system. (by hand)

3. Use 2. to write the general solution of the system. (by hand)

4. Plot the phase portrait in the (x, y)**-plane.** (by hand) Plot:

- the fixed point(s)
- the nullclines and the direction of the vector field along them
- the eigenvectors labelled fast / slow if a node, or stable / unstable manifold if a saddlepoint
- a few representative trajectories, with arrows on them showing the flow in all four quadrants.

2 Competitive exclusion in microbial ecology (20 pts)

In this model, two interacting populations (e.g., bacteria L, M (e.g., *Escherichia coli* and *Lactobacillus acidophilus*) compete for the same resources in the same environment (for example in the gut).

A general model that describes this interaction takes the form :

$$\frac{dL}{dt} = L\,(4 - L - 2M)$$
$$\frac{dM}{dt} = M\,(3 - L - M)$$

with $L, M \geq 0$.

1. Find the nullclines. Draw them in a plot in which L is the X axis, and M the Y axis. Indicate the location of the fixed points (only) on the plot, and the direction of the vector field on all regions of the nullclines.(Do this and all following questions by hand)

2. Calculate the Jacobian matrix as a function of L, M. Evaluate it at all fixed points, and state the type and stability of the fixed points.

3. For each fixed point found in Q2, find the following:

- the eigenvalues and associated eigenvectors (keep only 2 decimal places in calculations)
- the general solution for trajectories near each fixed point, labelling the fast / slow eigenvectors if it's a node, and stable / unstable manifolds if a saddlepoint

4. Draw the phase portrait for the system in a new (large) plot. Make sure you clearly show the following:

- the location and stability of each fixed point (solid dot for stable node, open dot for unstable node and saddlepoint)
- fast / slow eigenvectors, and stable / unstable manifolds
- the full nullcline curves labelled with the direction of the vector field along them
- representative trajectories starting in all areas of the first quadrant (labelled with arrows showing the direction of the vector field), e.g., beginning at large L/small M, large M/small L, large L/large M, etc. Avoid overcrowding any area, but show enough that the flow is clear around all fixed points.