Exercise 9: Bifurcation in the van der Pol oscillator

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Course: Dynamical systems in biology (BIO 341)

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Consider the following two-dimensional model (the van der Pol oscillator):

$dx/dt = y$

$dy/dt = -x + \mu (1 - x^2) y$

where μ is a real constant (positive or negative). This question explores some of the conditions for there to be a bifurcation at $\mu = 0$, in which a stable spiral changes to an unstable spiral and a limit cycle appears as the parameter μ is varied (but we are not looking at the limit cycle in this problem.)

1) The origin (0,0) is a fixed point for any value of μ. Find the Jacobian at the origin, and evaluate its trace and determinant in terms of μ.

2) State why that the origin cannot be a saddlepoint for any value of μ.

3) On the Tau-Delta plot, mark the location of the fixed point for the following values of μ = -3, -2, -1, 0, 1, 3. Label each point with the type/stability of the fixed point. For the case $\mu = -2$, find the eigenvalue(s) and eigenvector(s) and determine if the fixed point is a star or degenerate node.

4) For the case $\mu = -1$, draw the phase portrait including the nullclines (with the direction of the other vector field dx/dt and dy/dt along them) and two or three typical trajectories (not too many trajectories!). Can you infer the type of the fixed point from the phase portrait?

5) How does the fixed point change character as μ increases from negative to positive values?

6) Now consider the similar equations:

$dx/dt = y + \mu x$

$dy/dt = -x + \mu (1 - x^2) y$

6.1) Find the Jacobian for the fixed point at the origin, and the fixed point's type and stability for μ = -3, -1, 0, 1, 3. Label each point again with its type/stability.

6.2) Are there other fixed points than the origin for this model? If so, find their coordinates (x^*, y^*) as a function of μ. Set μ = 0.1 and find the coordinates of any non-zero fixed points.

6.3) For an arbitrary value of μ, but not very different from 1, draw the phase portrait including: the nullclines, fixed point(s), and direction of the vector field on the nullclines (no trajectories). Can you identify the type and stability of the non-zero fixed points from your diagram? If so, say what they are.