Exercise 1: Growth models in 1D

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Course: BIO-341 *Dynamical systems in biology*

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[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from ipywidgets import interact
```

1 Growth models in 1D

1.1 Linear model for population growth

Consider a population of N birds with birth and death rates n and m. Arrival of new individuals through migrations occurs at rate $a > 0$. This can be translated into the simple model:

$$\frac{dN}{dt} = F(N) = (n - m)N + a \tag{1}$$

1) Write down what type of equation this is, e.g. first order, second order, linear, non-linear, etc.

2) Solve this equation analytically using the Ansatz: $N(t) = Ae^{\lambda t} + B$. Express A, B and λ in function of the rates and the population size $N_0 = N(t = 0)$; explicitly write the solution $N(t)$.

3) Solve the equation using an alternative method, such as the separation of variables.

4) Qualitative analysis

1) Draw $F(N)$ in function of N for the two cases (i) $n < m$ and (ii) $n > m$. *Note: you are also expected to draw and solve such simple problems by hand.*
2) What is the main qualitative difference between the two cases and how does this affect the long time $t \to \infty$ behavior of the solution that you found above?
3) Here you can verify what you answered under 2) numerically. Generate some representative plots for $N(t)$ where you vary the parameters and initial conditions of the model.

5) Discuss why this equation is good or bad at describing real populations.

1.2 Integration by separation of variables

1) Solve the following differential equations to obtain $x(t)$ by the method of separation of variables. Use the initial condition $x(t) = x_0$ when $t = 0$.

1. $\frac{dx}{dt} = xe^{-2t}$
2. $\frac{dx}{dt} = 4x^2 - 1$

1.3 The non-autonomous Gompertz model for tumor growth

A surprisingly accurate model for the growth of a tumor of volume N is given by the following differential equation

$$\frac{dN}{dt} = r(t)N(t) \tag{2}$$

with $r(t) = r_0 e^{-at}$ and initial size $N(0) = N_0$. In other words, the population grows with a time dependent rate $r(t)$, which decreases exponentially in time with a rate a.

1) Give a plausible explanation for the proposed behavior of $r(t)$. Why should the growth rate decrease with time?

2) What is the meaning of r_0?

3) Show that the solution for $N(t)$ in function of the 3 parameters N_0, r_0, a can be written as $N(t) = N_0 e^{\frac{r_0}{a}(1-e^{-at})}$.
Hint: Use the method of separation of variables.

4) Study the solution:

1. Show that for very short times the population grows linearly like $N(t) = N_0(1 + r_0 t)$.
2. Show that for very long times $N(t) \cong N_{max}(1 - \frac{r_0}{a}e^{-at})$.

Hint: Use the Taylor approximation $e^x \approx 1 + x$ (valid for small x) for the inner or the outer exponential when appropriate.

5) Sketch the solution. Indicate N_{max}. How does the N approaches N_{max}?