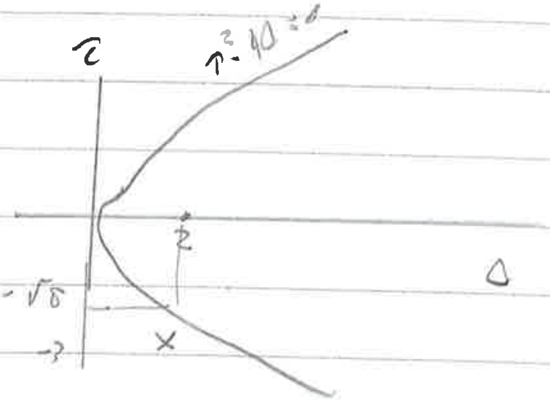
Graded Exercise 1 2025

12/9/25

$$\dot{x} = y$$
$$\dot{y} = -2x - 3y$$

$\therefore M = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ and $\tau = -3$, $\Delta = 2$



$\therefore$ It is a stable node

Eigenvalues

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(3+\lambda) + 2 = 0$

$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$

$\therefore \lambda = \dfrac{-3 \pm \sqrt{9-8}}{2} = \dfrac{-3 \pm 1}{2} = -1, -2$

Eigenvectors

$\lambda = -1$ $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow v_1 + v_2 = 0$ $\therefore v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= -2 \quad \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow 2v_1 + v_2 = 0 \quad \therefore \underline{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

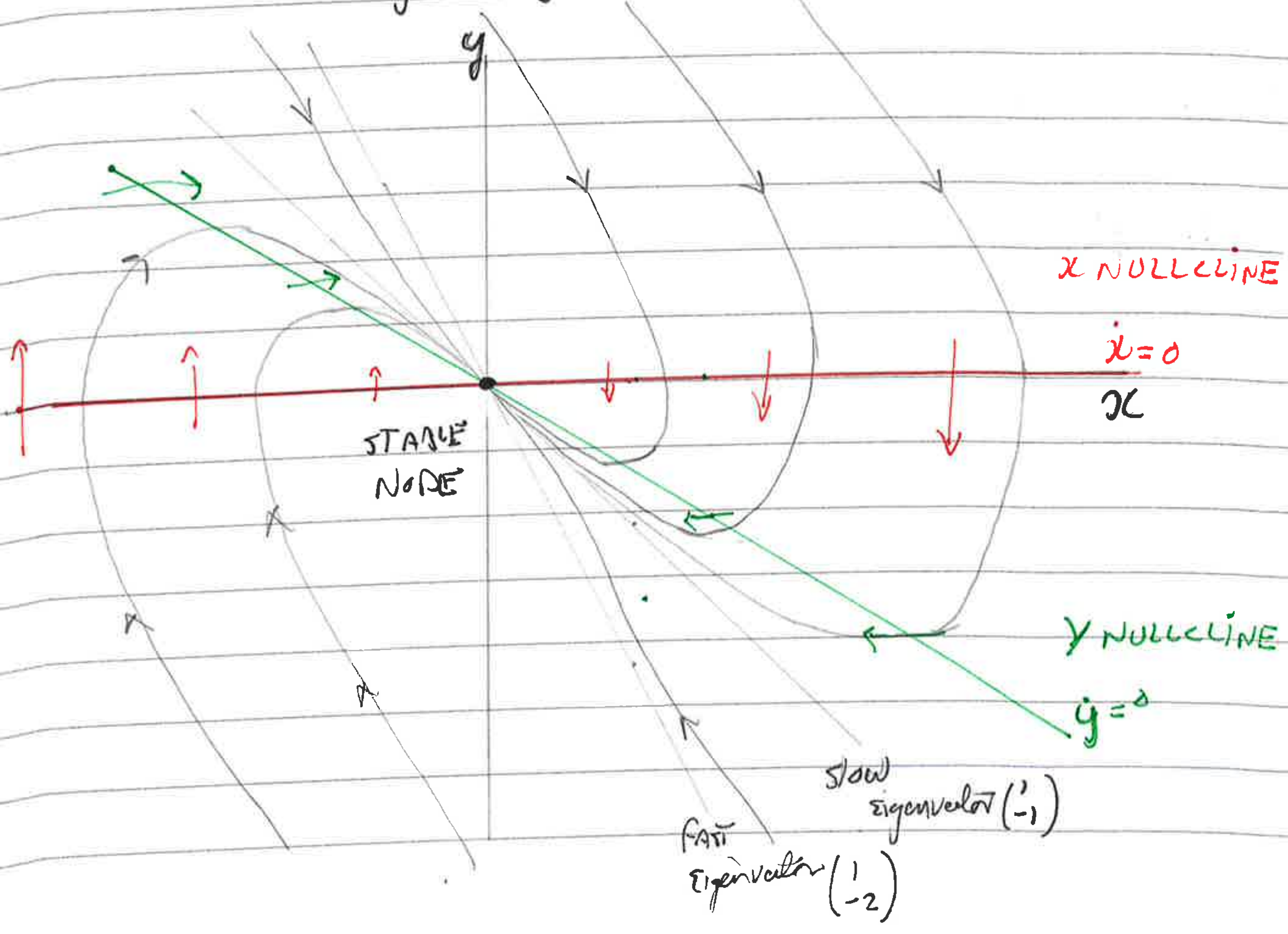
$\therefore$ General solution is: $\underline{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Slow Eigenvector — Fast eigenvector

Null clines? $\dot{x} = 0 \Rightarrow y = 0$ i.e. x axis. And when $y = 0 \Rightarrow \dot{y} = -2x$.

$\dot{y} = 0 \Rightarrow -2x - 3y = 0$ And when $y = -\frac{2}{3}x$, $\dot{x} = -\frac{2}{3}x$.

$\therefore y = -\frac{2}{3}x$



Model of Two Interacting Populations

Graded Exercise 1 2025

29/9/25

$$\dot{L} = L(4-L-2M) \qquad L, M \geq 0$$

$$\dot{M} = M(3-L-M)$$

1) Find the nullclines:

$\dot{L}=0 \Rightarrow L=0$ or $4-L-2M=0 \Rightarrow M = \frac{4-L}{2}$

$\dot{M}=0 \Rightarrow M=0$ or $M = 3-L$

Direction of vector field along nullclines:

What is $\dot{M}$ along $\dot{L}=0$?

$L=0 \Rightarrow \dot{M} = M(3-M) > 0$ for $M<3$
< 0 for $M>3$

$M = \frac{4-L}{2} \Rightarrow \dot{M} = \frac{(4-L)}{2}\left(3-L-\left(\frac{4-L}{2}\right)\right)$

$$= \left(\frac{4-L}{2}\right)\left(1-\frac{L}{2}\right) = \frac{1}{4}(4-L)(2-L)$$

$\therefore \dot{M} > 0$ for $L<2$ and $L>4$ (not relevant as outside first quadrant)

What is $\dot{L}$ along $M=0$?

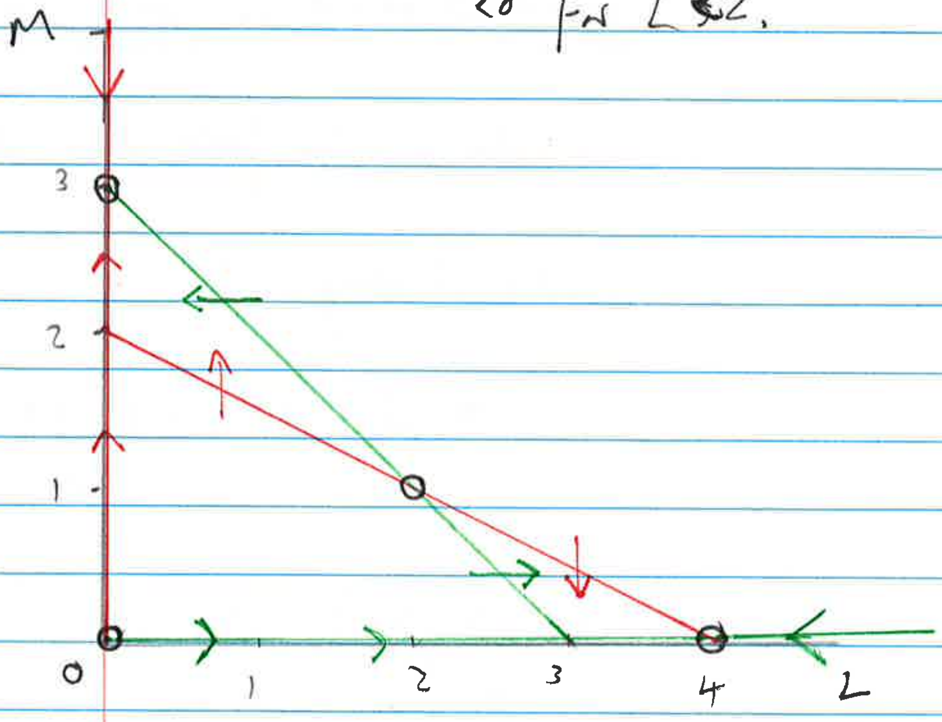
$M=0 \Rightarrow \dot{L} = L(4-L)$ $\quad > 0$ for $L<4$

$\qquad\qquad\qquad\qquad\quad < 0$ for $L>4$

and along $M = 3-L$:

$$\dot{L} = L(4-L-2(3-L)) = L(-2+L)$$

$$= L(L-2) \quad > 0 \text{ for } L>2$$
$$\qquad\qquad\quad < 0 \text{ for } L<2.$$



Location of Fixed points: $(0,0)$

$(4,0)$

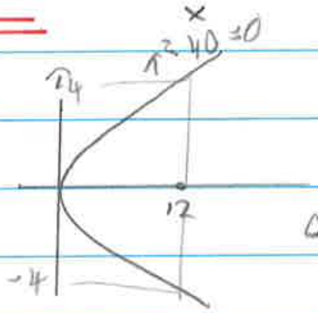
$(0,3)$

$(2,1)$

2 $$\underline{\underline{J}} = \begin{pmatrix} \frac{\partial \dot{L}}{\partial L} & \frac{\partial \dot{L}}{\partial M} \\ \frac{\partial \dot{M}}{\partial L} & \frac{\partial \dot{M}}{\partial M} \end{pmatrix}$$

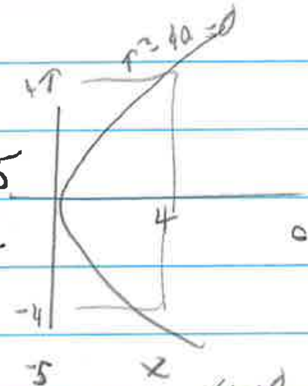
$$= \begin{pmatrix} 4-2L-2M & -2L \\ -M & 3-L-2M \end{pmatrix}$$

A) $(0,0)$ $\underline{\underline{J}} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ $\therefore \tau = 7$, $\Delta = 12$



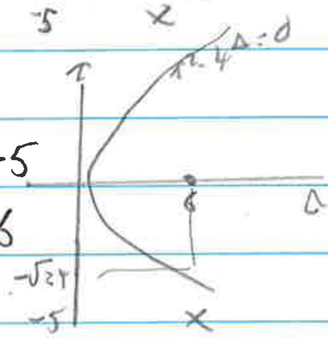
$\therefore$ unstable node

B) $(4,0)$ $\underline{\underline{J}} = \begin{pmatrix} -4 & -8 \\ 0 & -1 \end{pmatrix}$ $\therefore \tau = -5$, $\Delta = 4$



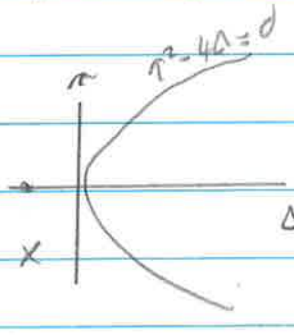
$\therefore$ stable node

c) $(0,3)$ $\underline{\underline{J}} = \begin{pmatrix} -2 & 0 \\ -3 & -3 \end{pmatrix}$ $\therefore \tau = -5$, $\Delta = 6$



$\therefore$ stable node

d) $(2,1)$ $\underline{\underline{J}} = \begin{pmatrix} -2 & -4 \\ -1 & -1 \end{pmatrix}$ $\therefore \tau = -3$, $\Delta = -2$



$\therefore$ Saddlepoint

3) A) $(0,0)$ $\underline{\underline{J}} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} 4-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)(3-\lambda) = 0$

$\therefore \lambda = 3, 4$

Eigenvectors: $\lambda = 3$ $\begin{pmatrix} 4-3 & 0 \\ 0 & 3-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_1 = 0$ and $\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = 4$ $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_2 = 0$ and $\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underline{x}(t) = c_1 e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

slow Eigenvector   Fast Eigenvector

3 B) $(4,0)$ $\underline{J} = \begin{pmatrix} -4 & -8 \\ 0 & -1 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} -4-\lambda & -8 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (4+\lambda)(1+\lambda) = 0$

$\therefore \lambda = -1, -4$

Eigenvectors: $\lambda = -1$ $\begin{pmatrix} -3 & -8 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow -3v_1 - 8v_2 = 0 \quad \therefore v_2 = -\frac{3}{8}v_1$

$\therefore \underline{v_1} = \begin{pmatrix} 1 \\ -3/8 \end{pmatrix}$

$\lambda = -4$ $\begin{pmatrix} 0 & -8 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow -8v_2 = 0 \quad \therefore v_2 = 0 \quad \therefore \underline{v_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore \underline{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -3/8 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Slow eigenvector Fast eigenvector

3 c) $(0,3)$ $\underline{J} = \begin{pmatrix} -2 & 0 \\ -3 & -3 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} -2-\lambda & 0 \\ -3 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow (2+\lambda)(3+\lambda) = 0$

$\therefore \lambda = -2, -3$

Eigenvector: $\lambda = -2$ $\begin{pmatrix} 0 & 0 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow -3v_1 - v_2 = 0 \quad \therefore v_2 = -3v_1$

$\therefore \underline{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\lambda = -3$ $\begin{pmatrix} 1 & 0 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow v_1 = 0 \quad \therefore \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore \underline{x}(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Slow eigenvector Fast eigenvector

3 a) $(2,1)$ $\underline{J} = \begin{pmatrix} -2 & -4 \\ -1 & -1 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} -2-\lambda & -4 \\ -1 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (2+\lambda)(1+\lambda) - 4 = 0$

$\Rightarrow \lambda^2 + 3\lambda - 2 = 0$

$\therefore \lambda = \dfrac{-3 \pm \sqrt{9+8}}{2} = \dfrac{-3 \pm \sqrt{17}}{2} = 0.56, -3.56$

$\lambda = -3.56$, stable manifold

$\begin{pmatrix} -2-(-3.56) & -4 \\ -1 & -1-(-3.56) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1.56 & -4 \\ -1 & 2.56 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow 1.56v_1 - 4v_2 = 0$

$-v_1 + 2.56v_2 = 0$

$\therefore v_2 = \dfrac{1.56}{4} v_1 = 0.39 v_1$

$\therefore \underline{v}_1 = \begin{pmatrix} 1 \\ 0.39 \end{pmatrix}$

The unstable manifold has $\lambda = 0.56$

$$\begin{pmatrix} -2-0.56 & -4 \\ -1 & -1-0.56 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -2.56 & -4 \\ -1 & -1.56 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

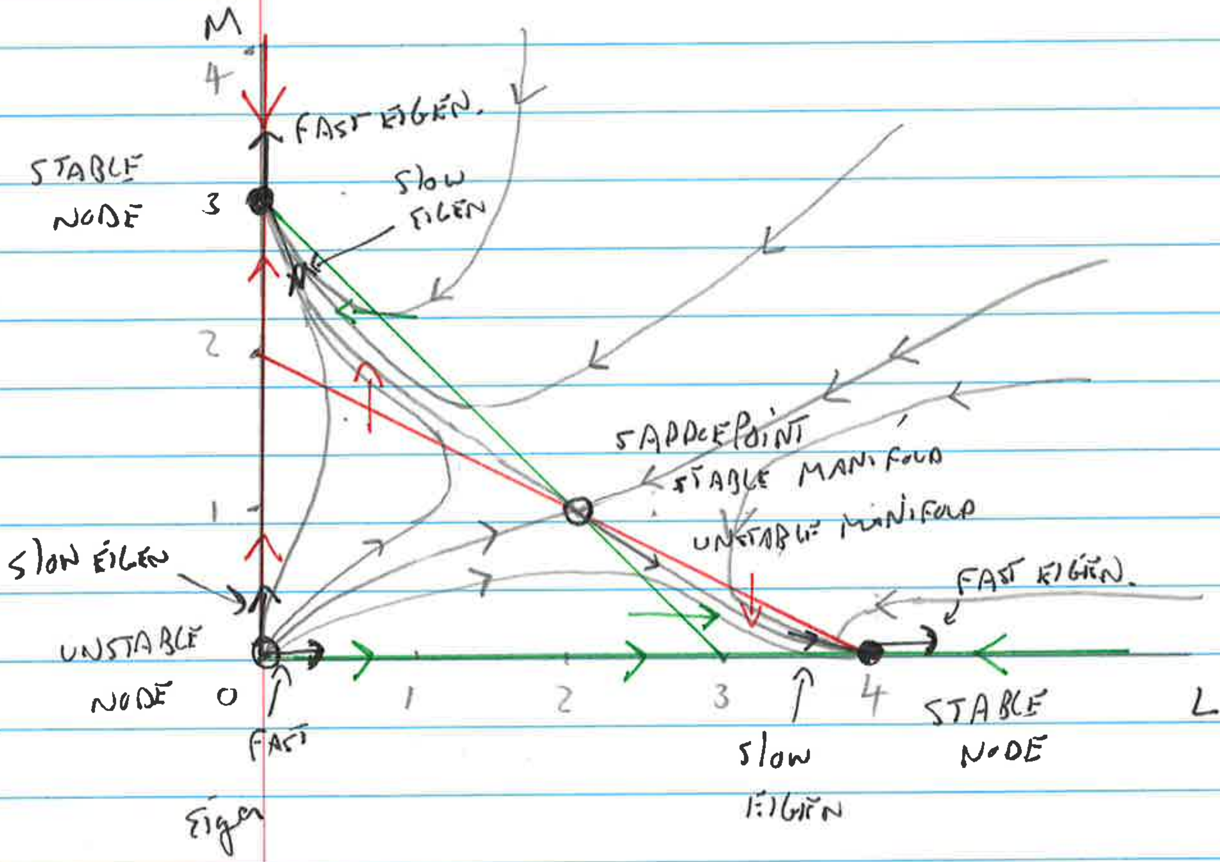
$$\Rightarrow -2.56 v_1 - 4 v_2 = 0$$
$$-v_1 - 1.56 v_2 = 0$$

$$\therefore v_2 = \frac{-2.56}{4} v_1 = -0.64 v_1$$

$$\therefore \underline{v}_2 = \begin{pmatrix} 1 \\ -0.64 \end{pmatrix}$$

$$\therefore \underline{x}(t) = c_1 e^{-3.56t}\begin{pmatrix} 1 \\ 0.39 \end{pmatrix} + c_2 e^{0.56t}\begin{pmatrix} 1 \\ -0.64 \end{pmatrix}$$

STABLE Manifold unstable manifold


M
4
3
2
1
0
1
2
3
4
L
STABLE NODE
FAST EIGEN.
Slow EIGEN
SADDLE POINT
STABLE MANIFOLD
UNSTABLE MANIFOLD
Slow EIGEN
UNSTABLE NODE
FAST EIGEN.
STABLE NODE
FAST Eigen
Slow EIGEN