Shifting a 2D Fixed point to the origin

28/6/24

Suppose $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{c}$ $\qquad \underline{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ = constant

Can we transform this so that $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$?

Let $u = x + \alpha$ and find α, β to make c_1, c_2 disappear.

$v = y + \beta$

Clearly $\dot{u} = \dot{x}$ and let $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\dot{v} = \dot{y}$

$$\Rightarrow \quad \dot{u} = ax + by + c_1 = a(u-\alpha) + b(v-\beta) + c_1$$

$$\dot{v} = cx + dy + c_2 = c(u-\alpha) + d(v-\beta) + c_2$$

$$\Rightarrow \quad \dot{u} = au + bv - a\alpha - b\beta + c_1$$

$$\dot{v} = cu + dv - c\alpha - d\beta + c_2$$

So, we want to find α, β so that: $-a\alpha - b\beta + c_1 = 0$

$$-c\alpha - d\beta + c_2 = 0$$

solve for α: $\alpha = \dfrac{-b\beta + c_1}{a}$ and $\alpha = \dfrac{-d\beta + c_2}{c}$

$$\Rightarrow \quad (-b\beta + c_1) \cdot c = (-d\beta + c_2) \cdot a$$

$\Rightarrow (ad - bc)\beta = ac_2 - cc_1$

$$\boxed{\therefore \beta = \frac{(ac_2 - cc_1)}{(ad - bc)}} \qquad \text{or} \qquad \beta = \frac{ac_2 - cc_1}{\det \underset{=}{A}}$$

$$\therefore \alpha = \frac{-b\beta + c_1}{a} = \frac{-b}{a}\left(\frac{ac_2 - cc_1}{(ad-bc)}\right) + \frac{c_1}{a}$$

$$= \frac{-abc_2 + b\cancel{c}c_1 + c_1(ad - b\cancel{c})}{a(ad-bc)}$$

$$= \frac{-\cancel{a}bc_2 + \cancel{a}dc_1}{\cancel{a}(ad-bc)} = \frac{dc_1 - bc_2}{ad - bc}$$

$$\boxed{\therefore \alpha = \frac{dc_1 - bc_2}{ad - bc}} \qquad \text{or} \qquad \alpha = \frac{dc_1 - bc_2}{\det \underset{=}{A}}$$

e.g. $\underset{=}{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad \underset{\sim}{c} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$ $\quad$ so $a = 1,\ b = 2,\ c = 3,\ d = 4,\ c_1 = 5,\ c_2 = 10$

$\det \underset{=}{A} = -2$ and $\alpha = \dfrac{4.5 - 2.10}{-2} = 0$

$$\beta = \frac{1.10 - 3.5}{-2} = 2.5$$

$\therefore u = x$

$v = y + 2.5$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} u \\ v - 2.5 \end{pmatrix} + \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{matrix} u + 2v - 5 + 5 \\ 3u + 4v - 10 + 10 \end{matrix} = \begin{matrix} u + 2v \\ 3u + 4v \end{matrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{QED}$$