Lecture 8: Bifurcations

(course book, ch. 7.1, 7.2, Strogatz, ch. 3)

BIFURCATION = A qualitative (topological) change in the dynamics of a system as a parameter is changed.
i.e. a fixed point appears / disappears / changes stability / limit cycle appears.

Bifurcations in 1D

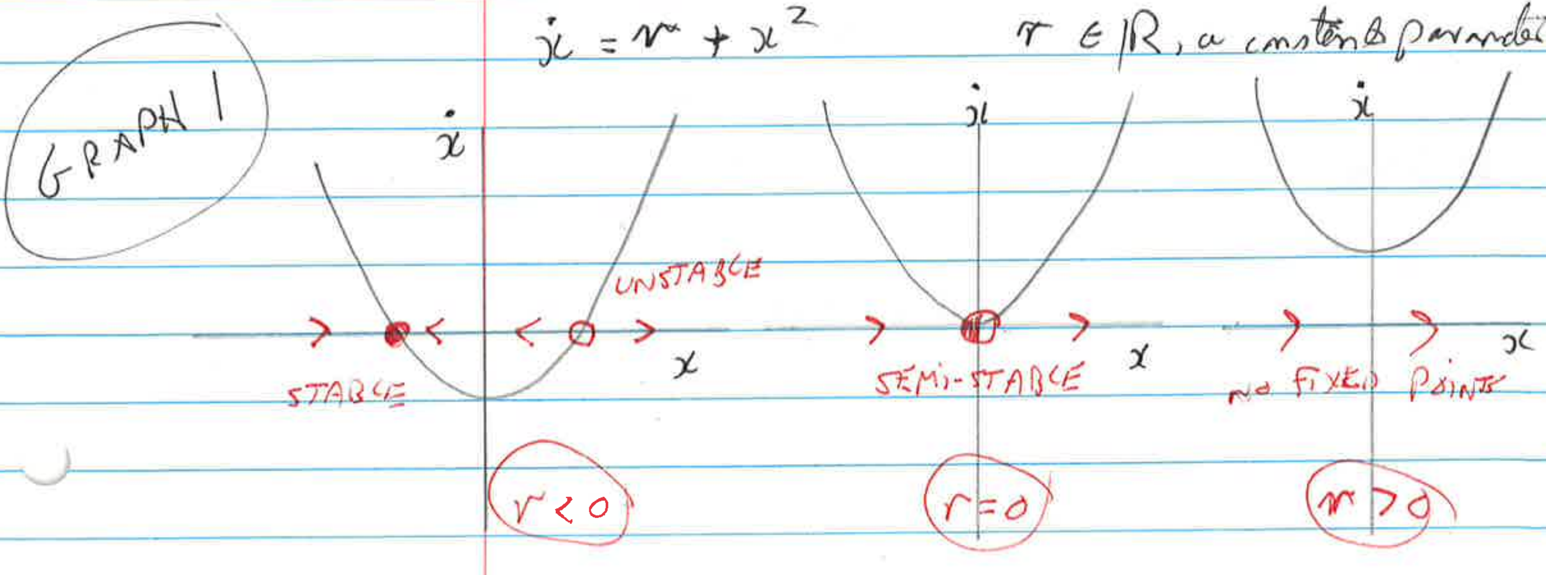
Consider a 1D dynamical system that depends on a parameter r:

$$\dot{x} = f(x, r)$$

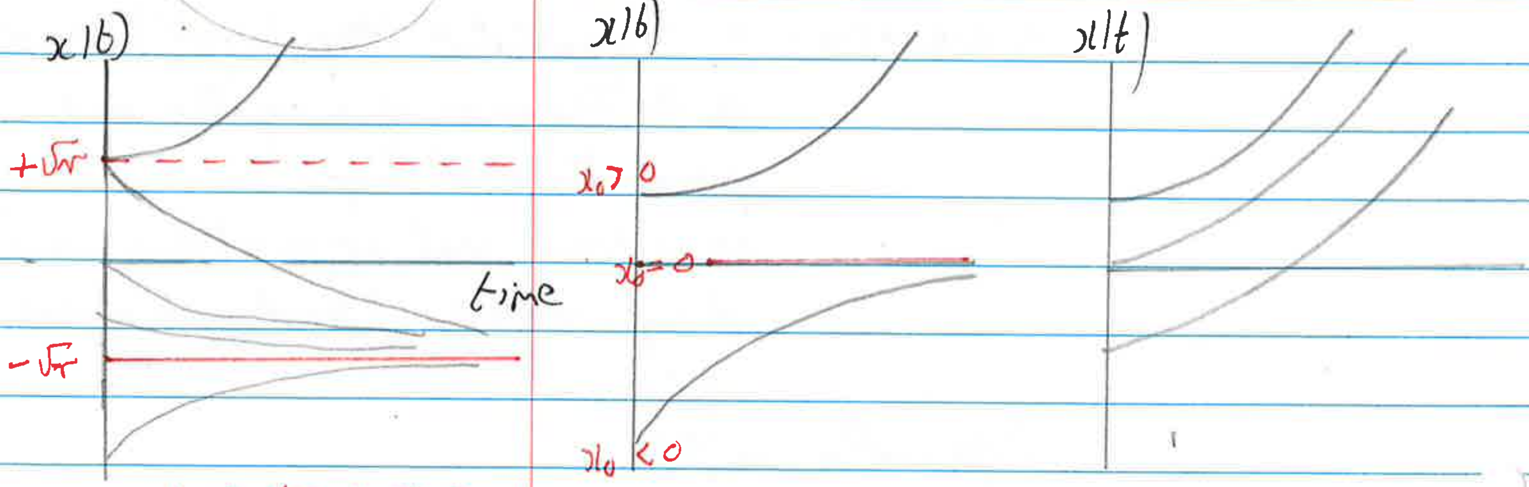
Question: Can fixed points appear / disappear in any way?

Type 1) Saddlenode bifurcation

$$\dot{x} = r + x^2 \qquad r \in \mathbb{R}, \text{ a constant parameter}$$



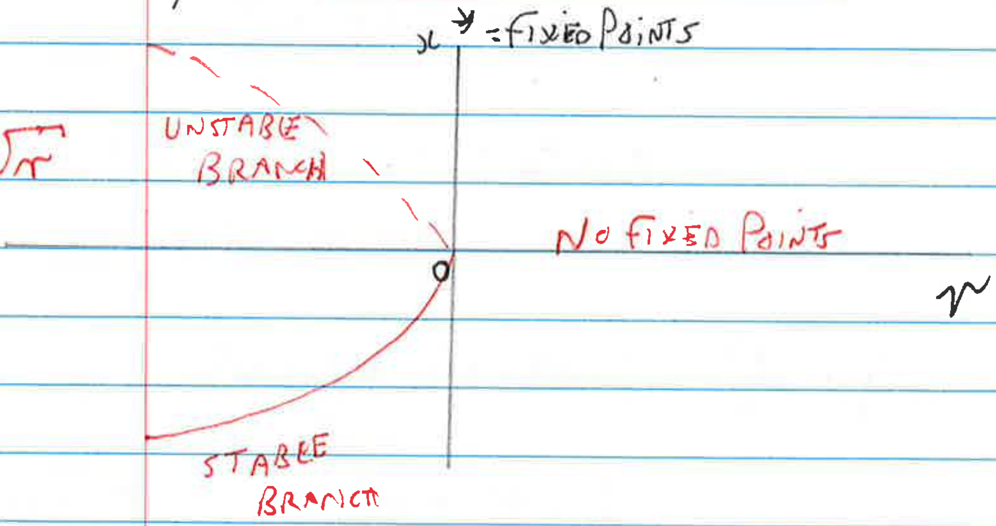
GRAPH 2



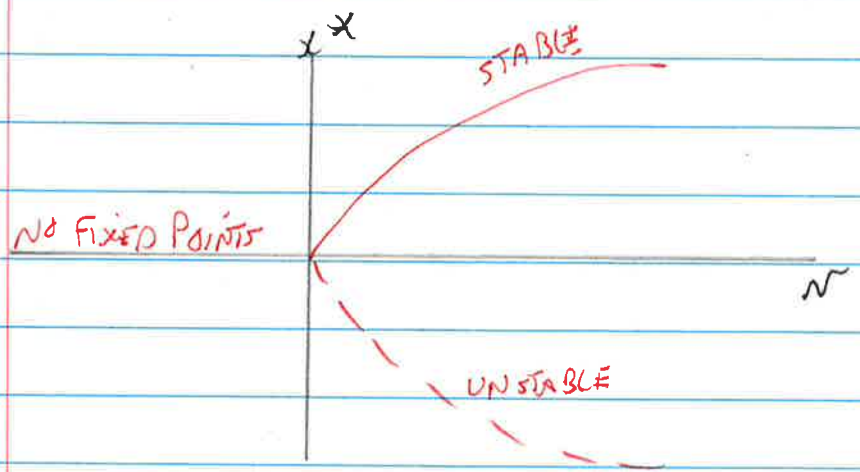
NB monotonic trajectories

We can summarise the behaviour in a single graph: $\boxed{\text{BIFURCATION DIAGRAM}}$

$x^* \sim \pm\sqrt{r}$



If $\dot{x} = r - x^2$?



CONCLUSION: 2 fixed points appear/disappear as r changes

The bifurcation diagram shows how the fixed points change as the parameter r is varied: it shows complete behaviour of the system, so is a useful summary of the dynamics.

cp Full solution: $\dot{x} = -r + x^2$

$$x(t) = \sqrt{r}\left[\frac{x_0(1+e^{2\sqrt{r}t}) + \sqrt{r}(1-e^{-2\sqrt{r}t})}{x_0(1-e^{-2\sqrt{r}t}) + \sqrt{r}(1+e^{2\sqrt{r}t})}\right]$$

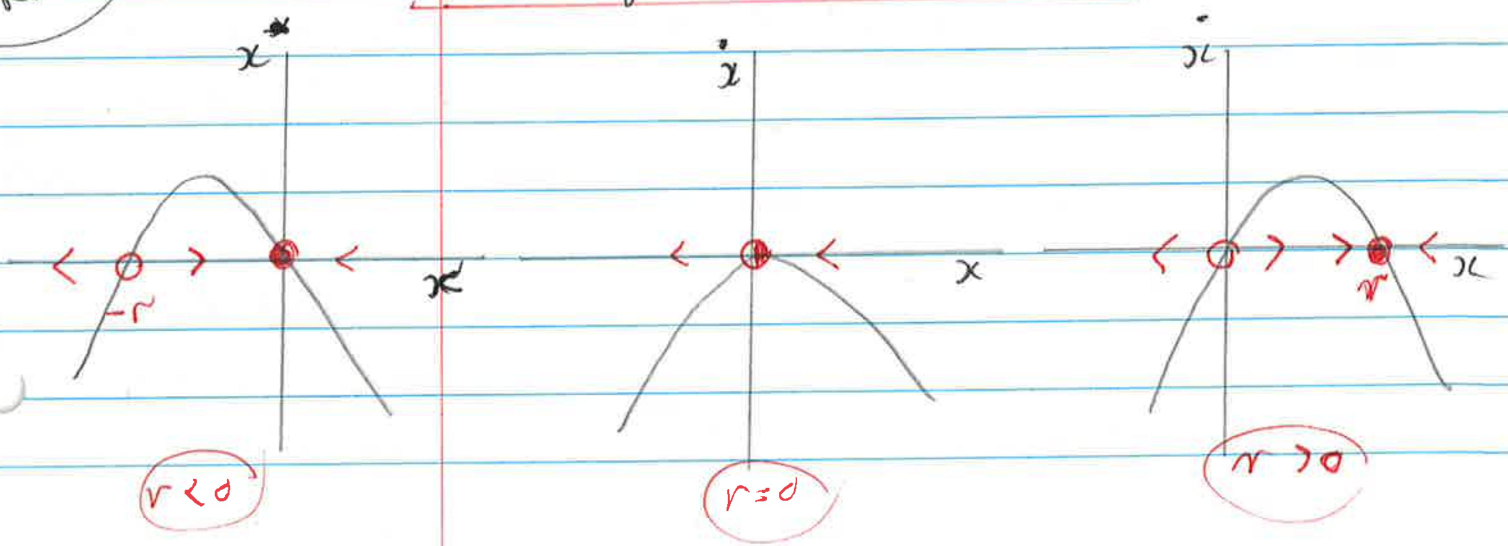
Type 2) Transcritical Bifurcation

In some cases, a fixed point must exist for all values of a parameter (e.g. pop^n model $\dot{N} = N \cdot R(N)$). But the stability of the fixed point can change as a parameter changes.

Consider: $\dot{x} = rx - x^2 = x(r - x)$

2 fixed points: $x^* = 0, r$

GRAPH 1

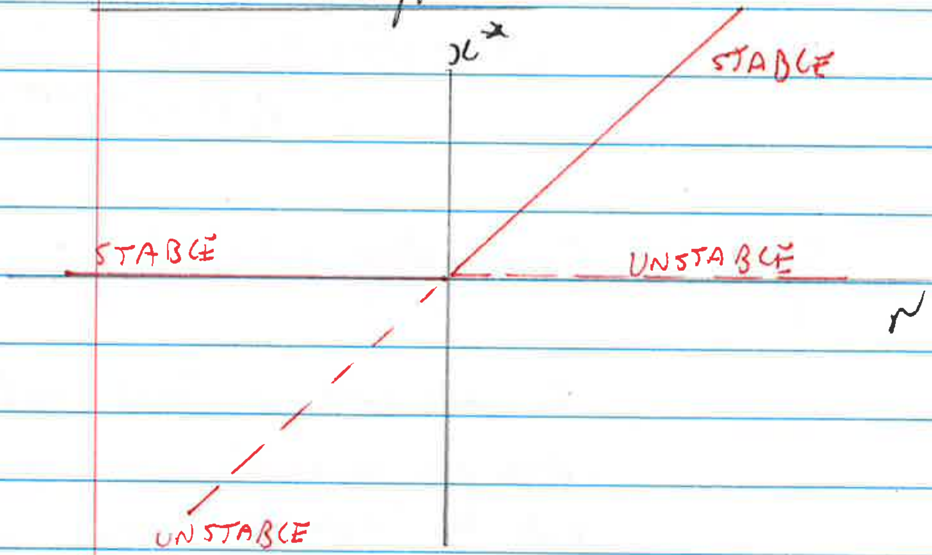


2 Comments:

- The fixed point doesn't move, but it changes stability as r changes.

- The order of the stability of the two FPs does not change - the right-hand one is always stable here (not always the case!)

Bifurcation Diagram



What is the stability? Recall $\dot{\eta} = f'(x^*) \cdot \eta$

And here,

$$f'(x^*) = r - 2x\Big|_{x^*}$$

$\therefore\ x^* = 0 \Rightarrow f'(0) = r$ so >0 if $r>0$ unstable

$=0$ if $r=0$ semi-st.

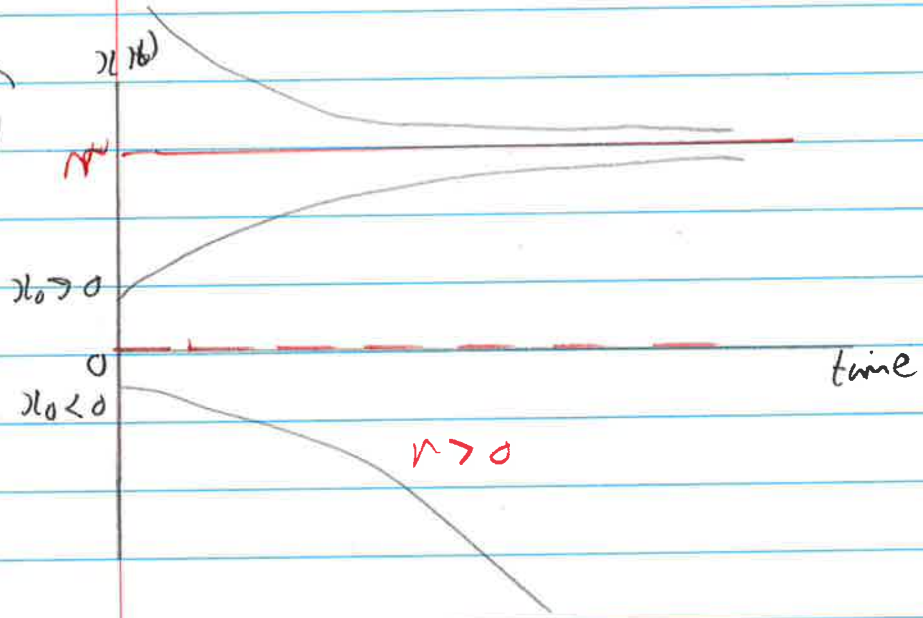
<0 if $r<0$ stable

$x^* = r \Rightarrow f'(r) = -r$ so stable if $r>0$

unstable if $r<0$

semistable if $r=0$

Graph 2



compare the analytic solution:

$$\dot{x} = x(r-x)$$

$$\Rightarrow \frac{dx}{x(r-x)} = dt$$

$$\text{But } \frac{1}{x(r-x)} = \frac{A}{x} + \frac{B}{r-x} = \frac{A(r-x)+Bx}{x(r-x)}$$

$$\left.\begin{aligned} \Rightarrow Ar &= 1 \\ -A+B &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 1/r \\ B &= 1/r \end{aligned}$$

$$\therefore \int_{x_0}^{x} \frac{dx}{x(r-x)} = \int_{x_0}^{x} \frac{dx}{rx} + \int_{x_0}^{x} \frac{dx}{r(r-x)} = t$$

$$\Rightarrow \left[\frac{\ln x}{r}\right]_{x_0}^{x} - \left[\frac{\ln(r-x)}{r}\right]_{x_0}^{x} = t$$

$$\Rightarrow \frac{1}{r}.\ln\left(\frac{x}{x_0}\right) - \frac{1}{r}.\ln\left(\frac{r-x}{r-x_0}\right) = t$$

$$\Rightarrow \ln\left[\frac{x \cdot (r-x_0)}{x_0 \ (r-x)}\right] = rt$$

$$\Rightarrow \frac{x}{x_0} = \frac{(r-x)}{(r-x_0)} e^{rt}$$

$$\Rightarrow x(r-x_0) = x_0(r-x) e^{rt}$$

$$\Rightarrow x\left[r - x_0 + x_0 e^{rt}\right] = r x_0 e^{rt}$$

$$\therefore x(t) = \frac{r x_0 e^{rt}}{r + x_0(e^{rt}-1)}$$

check 1) $x_0 = 0 \Rightarrow x(t) = 0 \ \forall$ time

2) As $t \to \infty \Rightarrow x(t) \to r$ if $x_0 > 0$,

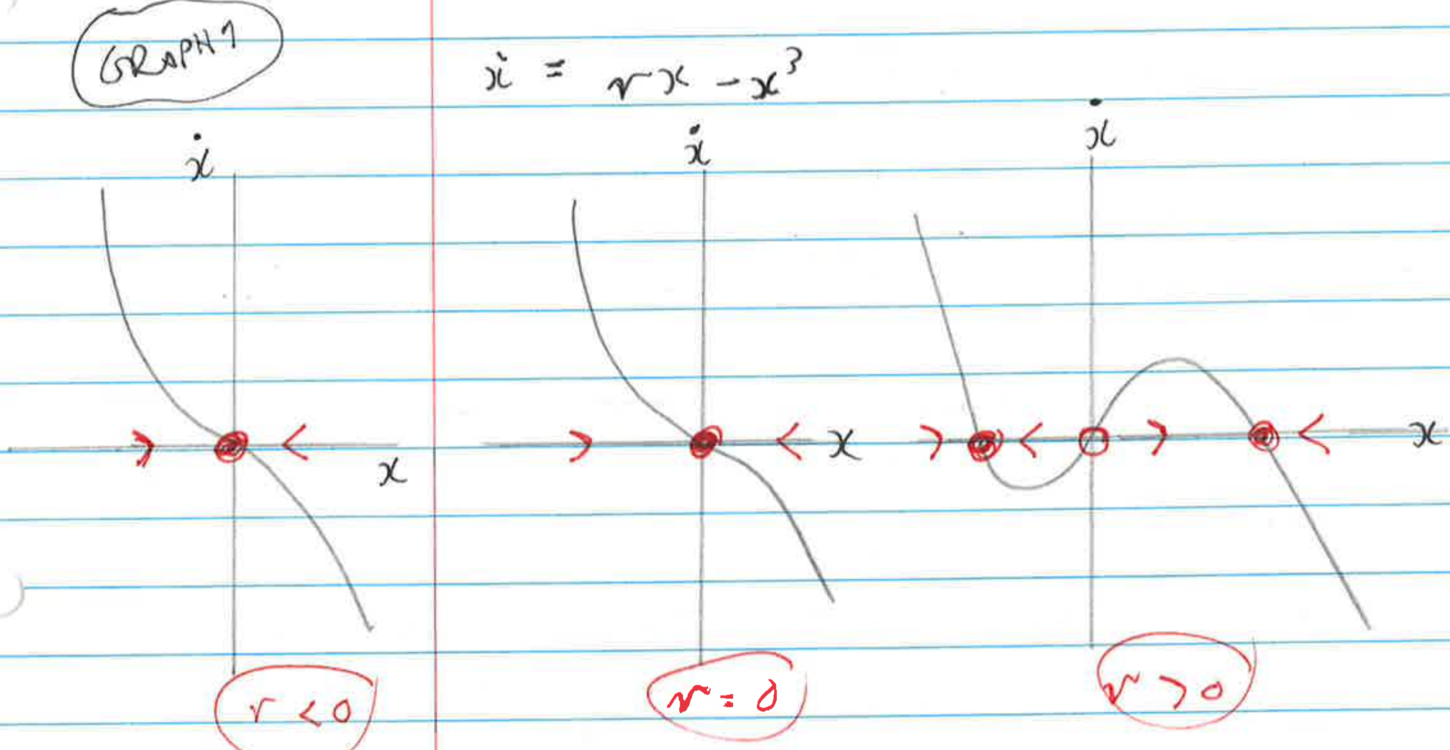
Type 3) Pitchfork Bifurcation

Another type of bifurcation occurs in systems that have a symmetry, e.g. $x \to -x$, spatial reflection symmetry.

There are two types:

Supercritical pitchfork bifurcation

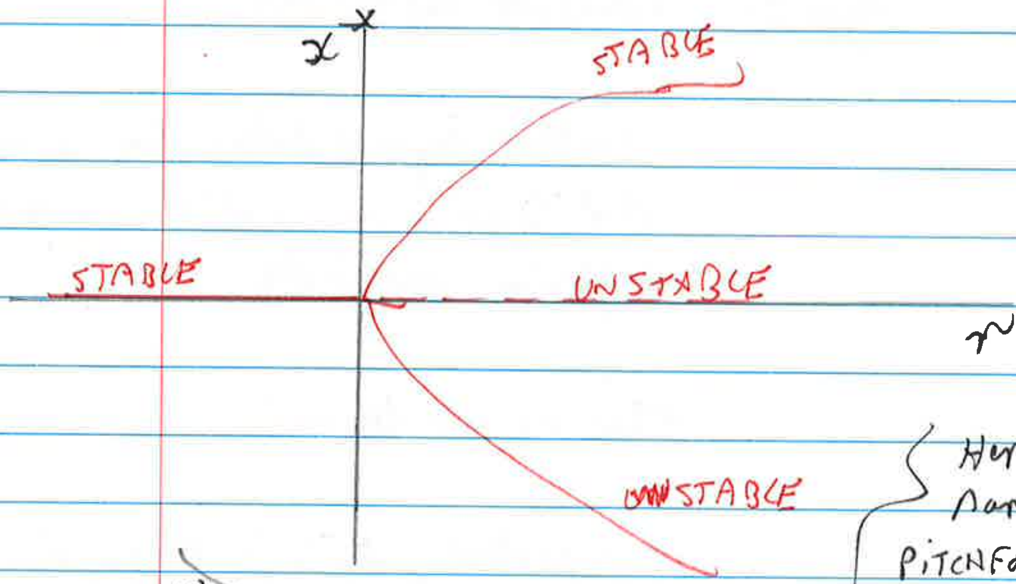
$$\dot{x} = rx - x^3$$



What are the fixed points?

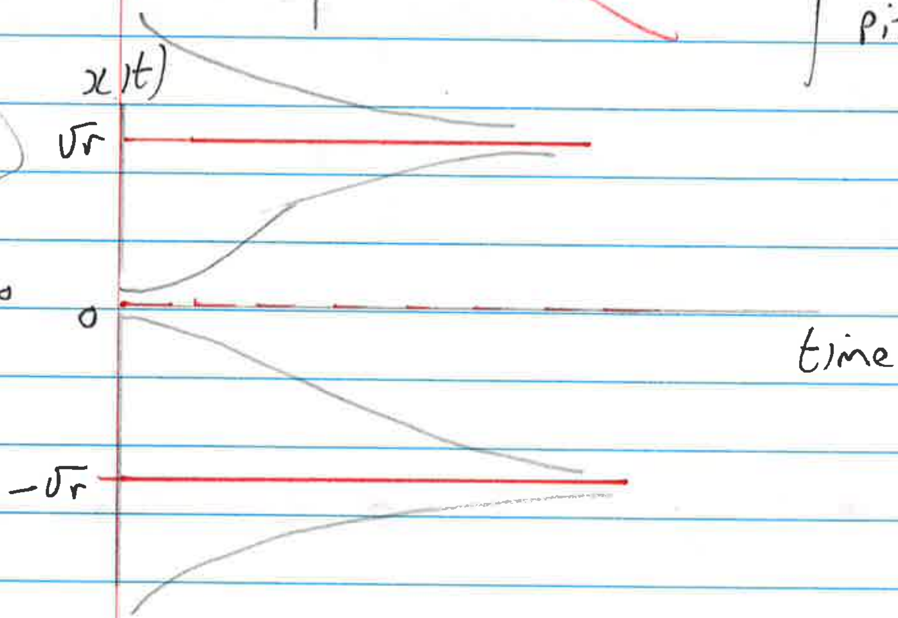
$\dot{x} = x(r - x^2)$ $\therefore x^* = 0, \pm\sqrt{r}$ if $r>0$.

Bifurcation Diagram



Hence the name: PITCHFORK

GRAPH 2



NB $\dot{x} \to 0$ as $x \to 0$, so initial slope is $\sim r x$

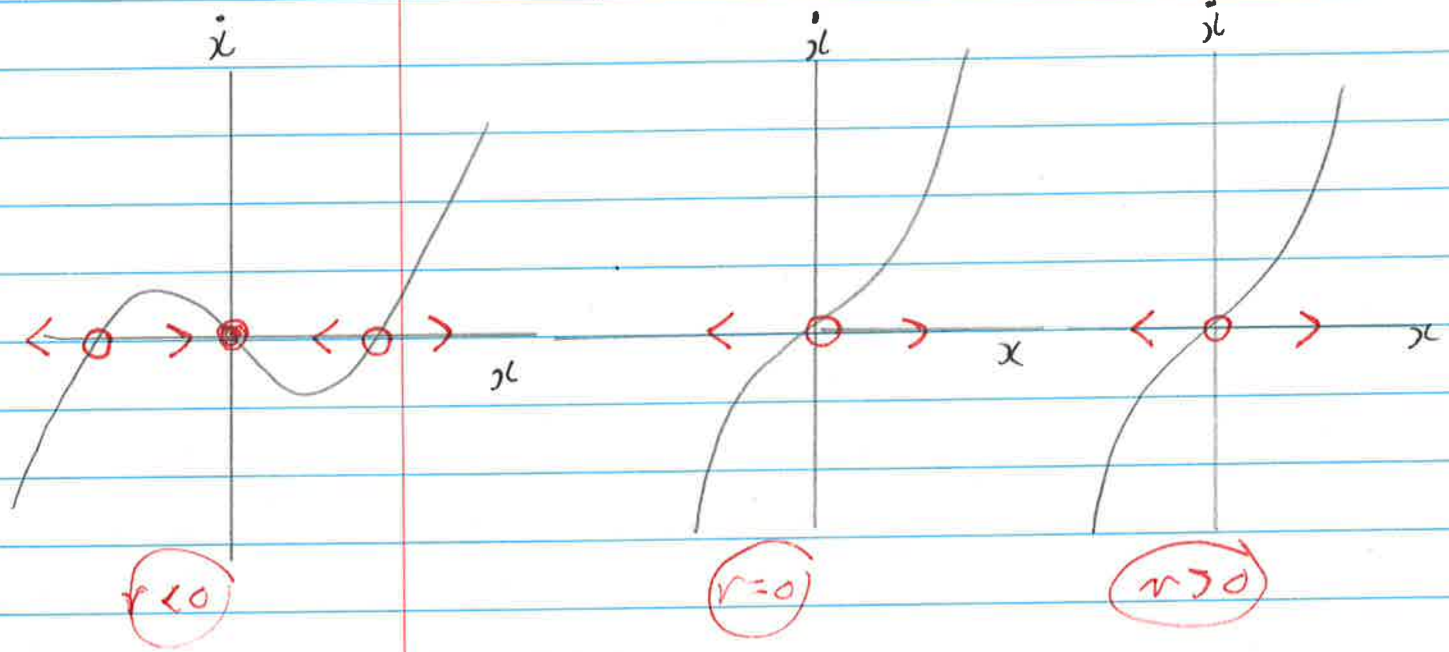
Analytic solution is: ~2 pages of algebra!

$$x(t)^2 = \frac{r x_0^2 e^{2rt}}{r + x_0^2 (e^{2rt} - 1)}$$

Subcritical Pitchfork Bifurcation

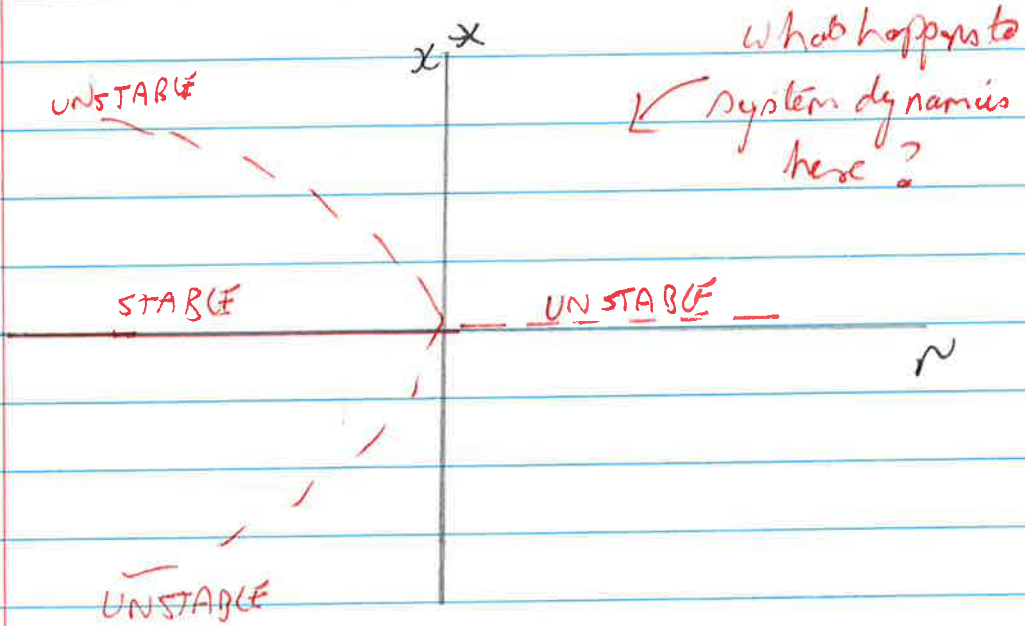
In the previous case, the $-r^3$ term stabilised the system. What if it destabilises it?

$$\dot{x} = rx + x^3$$



Fixed points are: $x^* = 0, \pm\sqrt{|r|}$ if $r<0$

BIFURCATION DIAGRAM



Summary

In a dynamical system governed by $\dot{x} = f(x, r)$, fixed points cannot change arbitrarily:

- Saddle-node bifurcation CREATES / DESTROYS a PAIR OF F.P.s

- Transcritical bifurcations cause a change of stability but not position

- Pitchfork bifurcations cause a stable fixed point to become unstable as r changes; if it is SUPERCRITICAL new stable fixed points appear continuously as r passes through the transition point; if it is subcritical, the unstable non-zero fixed points disappear, leaving only the unstable fixed point behind.

Bifurcations in 2D

Bifurcations also happen in 2D (and higher), but typically the change occurs in a 1D sub-space of the phase portrait, while flow in the other dimensions is just attraction or repulsion.

EXAMPLE:

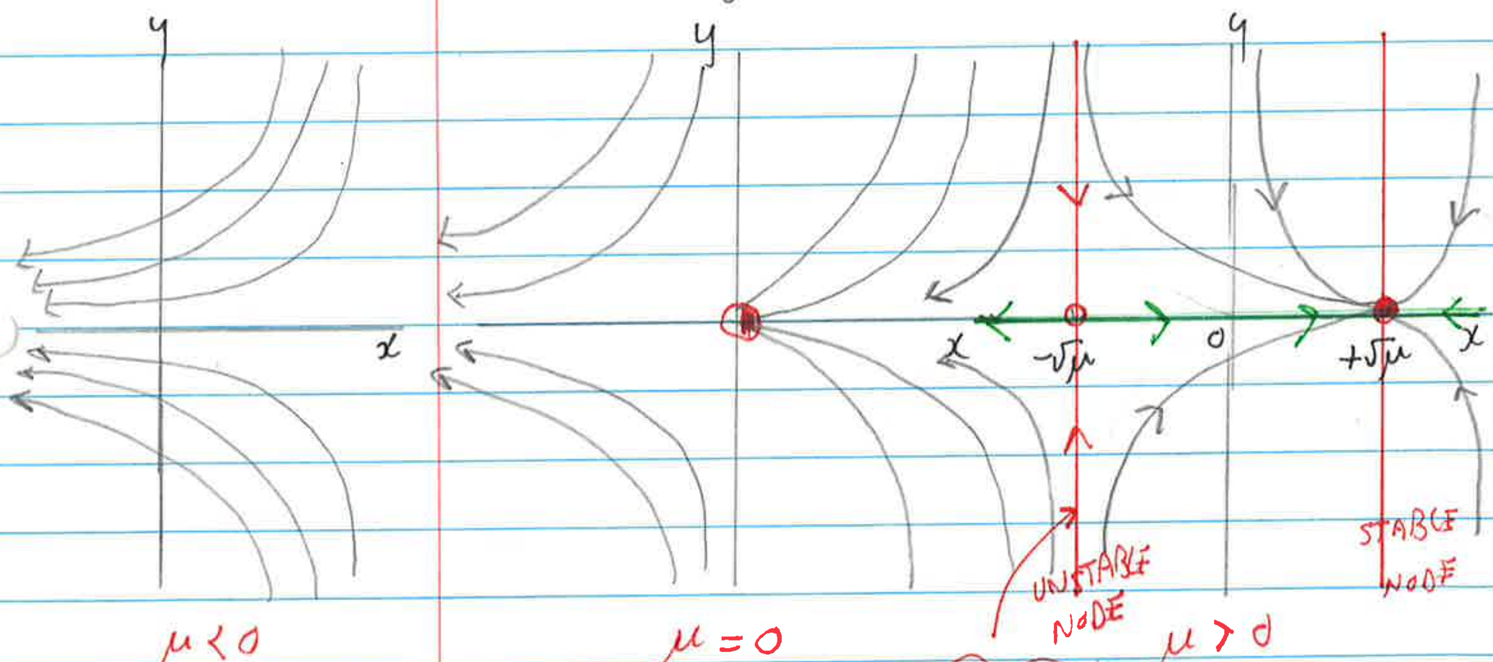
$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$

i.e. Saddlenode bifurcation in X, exponential decay in Y

What is the phase portrait?

nullclines? $\dot{x} = 0 \Rightarrow x = \pm\sqrt{\mu}$ if $\mu > 0$

$\dot{y} = 0 \Rightarrow y = 0$, i.e. X axis



$\mu < 0$
no fixed points

$\mu = 0$
Semi-stable node
$x^* = (0, 0)$

$\mu > 0$
$x^* = (\pm\sqrt{\mu}, 0)$

NB Diagram shows the case for $2\sqrt{\mu} < 1$. It looks different if $2\sqrt{\mu} > 1$

Visualisation

Imagine starting with $\mu < 0$, so there are no fixed points. Let μ increase to become above zero, then two fixed points appear.

Reversing the change in μ, the fixed points approach each other and annihilate at $\mu = 0$.