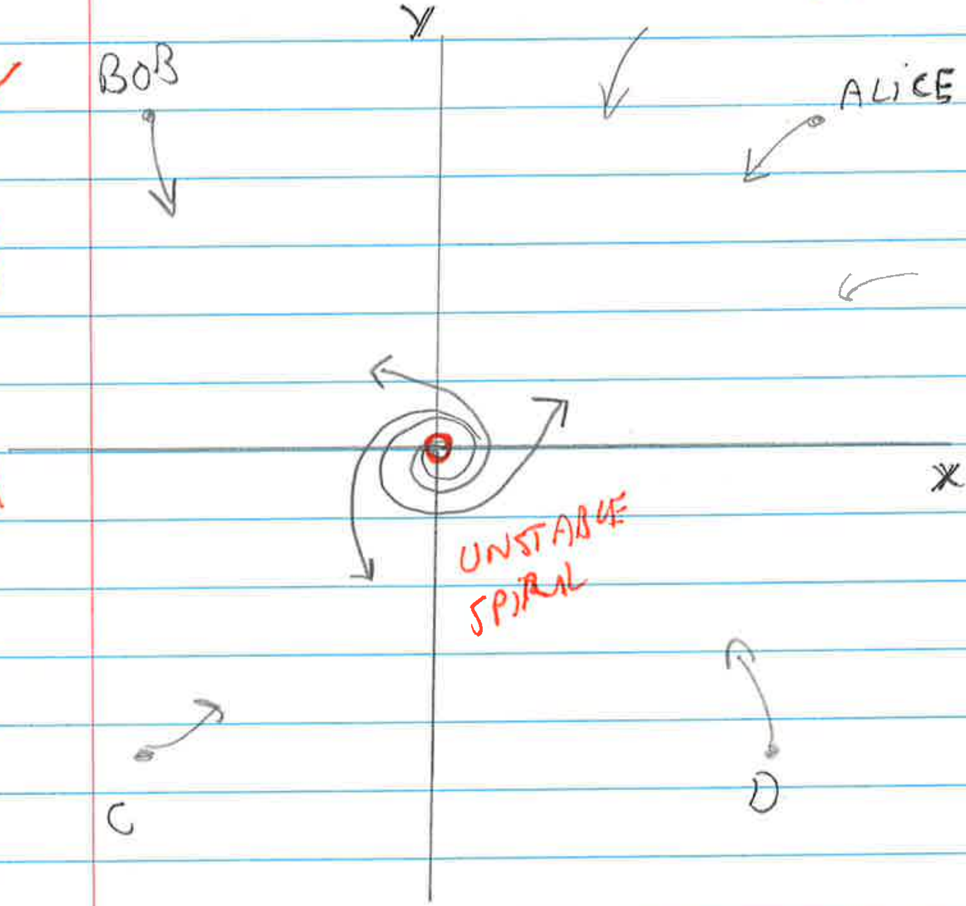
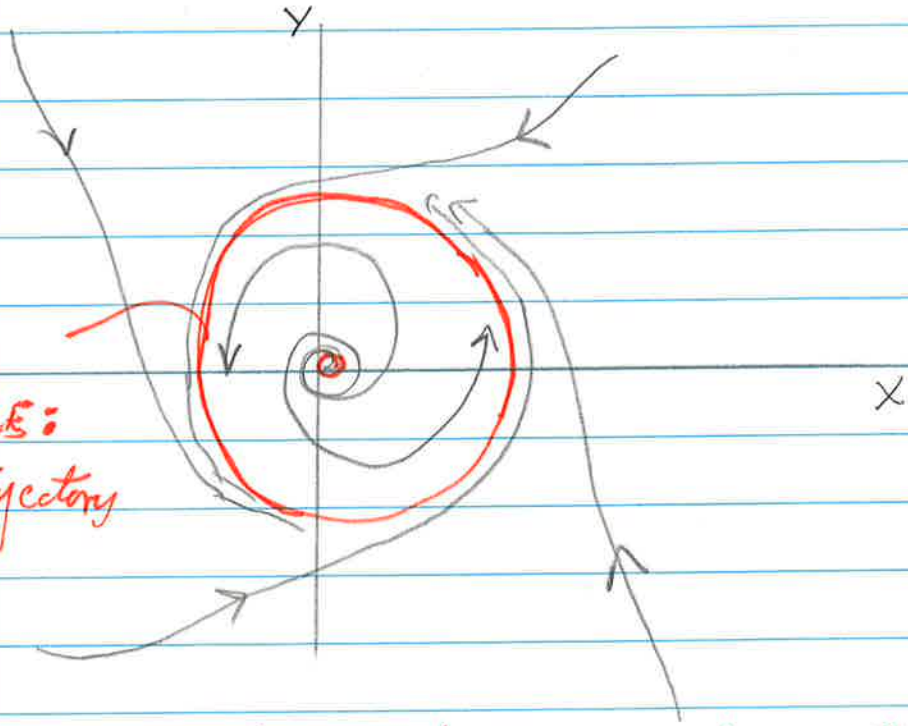
Lecture 7: Limit Cycles

(course notes: Ch. 5.2, Strogatz ch. 7)

All trajectories point inwards



How can we complete this phase portrait?

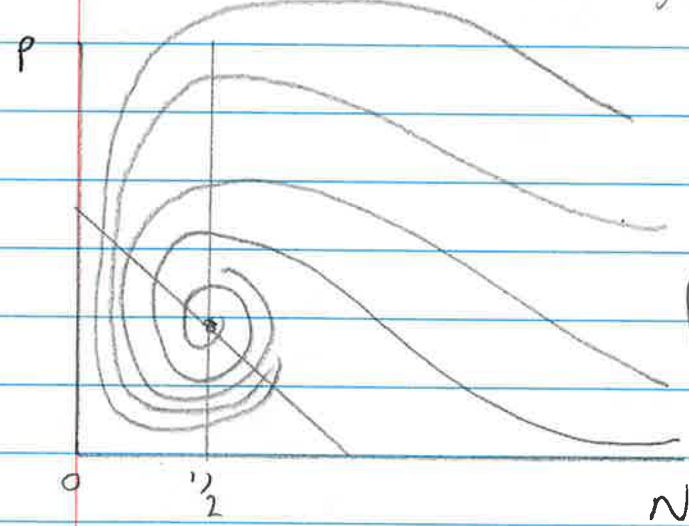


This is a STABLE LIMIT CYCLE:

≡ An isolated, closed trajectory

What do they represent? Self-oscillating systems, e.g. heart, fireflies, periodic chemical reactions. NOTE not equilibrium! PTO

~~What is an unstable limit cycle?~~



∞ number of
trajectories gets squeezed
into $[0, \frac{1}{2})$ on N axis

LECTURE 7: LIMIT CYCLES

(course notes: 5.1.2, Strogatz ch 7)

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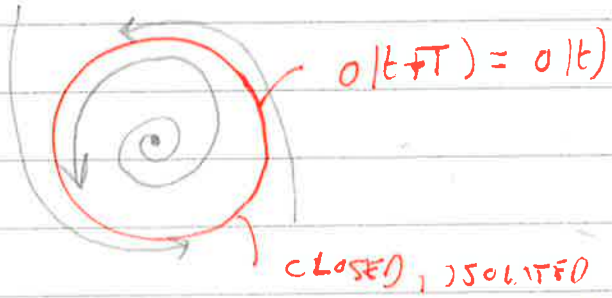
DEFINITION: A limit cycle is an ISOLATED, CLOSED trajectory.



ISOLATED ⇒ Nearby trajectories are NOT closed (cp. centre which are closed)
They either spiral towards the limit cycle (STABLE LC)
or away from it (UNSTABLE LC)

CLOSED ⇒ A limit cycle is periodic with period T, so the variables
repeat their values over time, i.e, oscillate.

NB A limit cycle is not always a circle!

NOTES

1) Limit cycles only exist in 2D or higher non-linear systems

2) Centres are closed/periodic but not isolated, i.e. nearby
trajectories are closed (and also centres)

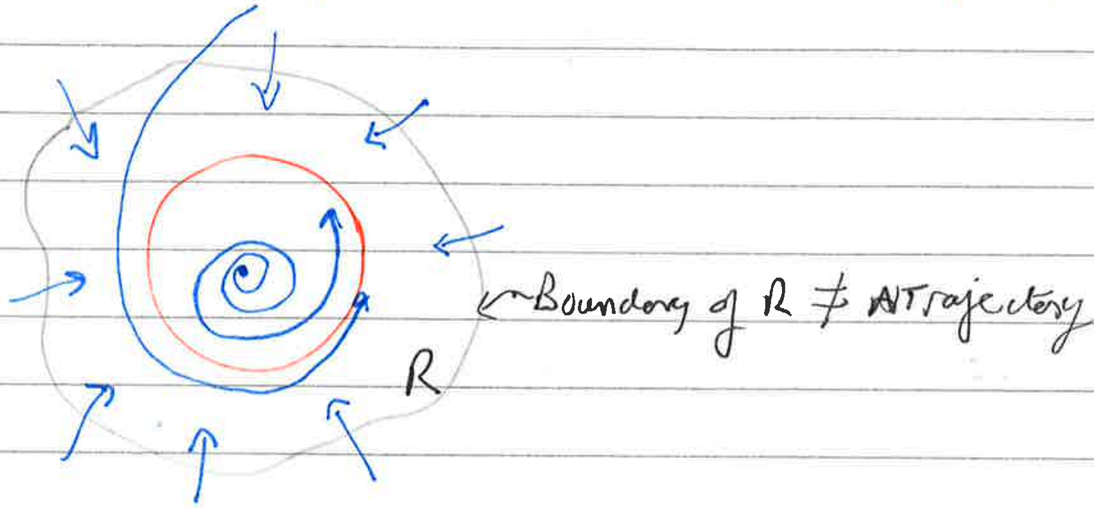
3) If you perturb a centre you move to a different trajectory,
but a perturbed LC returns to the LC

∴ Limit cycles are robust

POINCARÉ - BENDIXSON THEOREM

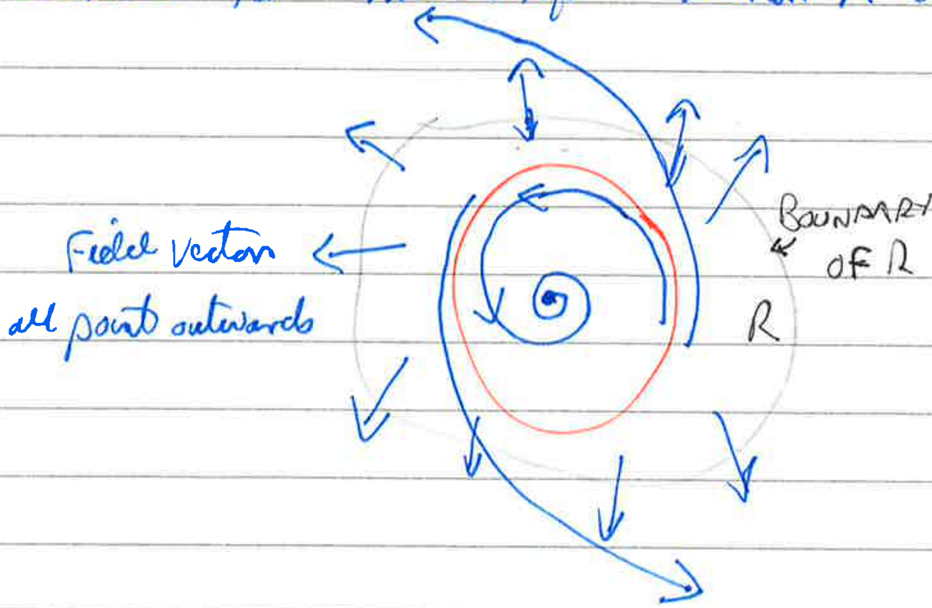
A stable limit cycle exists in a bounded region R if:

- all field vectors on the boundary of R point onto its interior
- there is an unstable node or spiral within R (NOT a saddle point)



An unstable limit cycle exists in a bounded region R if:

- all field vectors on the boundary of R point outwards
- there is a stable node or spiral within R (not a saddle point)



N.B. Semistable limit cycles also exist. NBB Boundary must be closed, even a tiny gap destroys the theorem.

EXAMPLE: Strogatz 7.1.5

$$\dot{x} = -y + x(1-x^2-y^2)$$

consider what each term does

$$\dot{y} = x + y(1-x^2-y^2)$$

ROTATION   Forces trajectories towards $x^2+y^2=1$

the fixed point is obviously $x=y=0$, i.e. the origin.

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

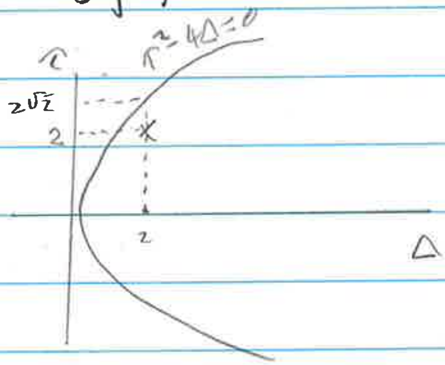
What type is it?   which has $\tau = 0$, $\Delta = 1$ $\therefore$ centre

Recipe

- Find $\underline{J}$
- Evaluate at fixed point
- classify using T, Δ
- Find eigenvalues/vectors
- " " general trajectory

$$\underline{J} = \begin{pmatrix}1-y^2-3x^2 & -1-2xy\\ 1-2xy & 1-x^2-3y^2\end{pmatrix}$$

$$\Rightarrow \underline{J}(0,0) = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$



$\therefore \tau = 2$
$\Delta = 2$

$\therefore$ unstable spiral at $(0,0)$.

What are the nullclines? We'll come back to that

Eigenvalues

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$$\therefore \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Eigenvectors : $\lambda = 1+i$

$$\begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

Note that these must agree or you've made a mistake!

$\Rightarrow -iv_1 - v_2 = 0$ $\quad \therefore v_2 = -iv_1 \quad \therefore v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$v_1 - iv_2 = 0$

$\lambda = 1-i$

$$\begin{pmatrix} 1-(1-i) & -1 \\ 1 & 1-(1-i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$\Rightarrow i v_1 - v_2 = 0 \qquad \therefore v_2 = i v_1 \qquad \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$v_1 + i v_2 = 0$

$\therefore$ General trajectory is:

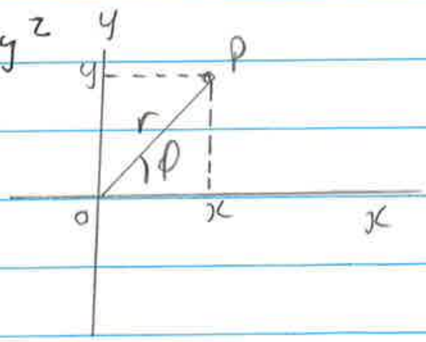
$$\underline{x}(t) = c_1 e^{(1+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{(1-i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

NB. $\mathrm{Re}\, \lambda_i > 0 \Rightarrow$ unstable spiral.

How does this produce REAL trajectories? See moodle Lecture 6.

How do we plot the trajectories? Convert to plane polar coordinates.

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \quad \begin{aligned} r^2 &= x^2 + y^2 \\ \mathrm{Tan}\,\phi &= \frac{y}{x} \end{aligned}$$



Recall original equations:

$$\dot{x} = -y + x(1 - x^2 - y^2)$$

$$\dot{y} = x + y(1 - x^2 - y^2)$$

Substitute for x, y in terms of r, ϕ:

$$\dot{x} = -r\sin\phi + r\cos\phi\,(1-r^2)$$
$$\dot{y} = r\cos\phi + r\sin\phi\,(1-r^2)$$

and from $r^2 = x^2 + y^2$
$$\tan\phi = y/x$$

$$\Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$

$$\Rightarrow r\dot{r} = r\cos\phi\,(-r\sin\phi + r\cos\phi\,(1-r^2)) + r\sin\phi\,(r\cos\phi - r\sin\phi\,(1-r^2))$$

$$= -r^2\cancel{\cos\phi\sin\phi} + r^2\cos^2\phi\,(1-r^2) + r^2\cancel{\sin\phi\cos\phi} + r^2\sin^2\phi\,(1-r^2)$$

$$= r^2(1-r^2)$$

$$\therefore \dot{r} = r(1-r^2)$$

and from $\tan\phi = y/x$

$$\Rightarrow \sec^2\phi\,\dot{\phi} = \frac{x\dot{y} - \dot{y}x}{x^2}$$

$$= \frac{r\cos\phi\,(r\cos\phi + r\sin\phi\,(1-r^2)) - r\sin\phi\,(-r\sin\phi + r\cos\phi\,(1-r^2))}{r^2\cos^2\phi}$$

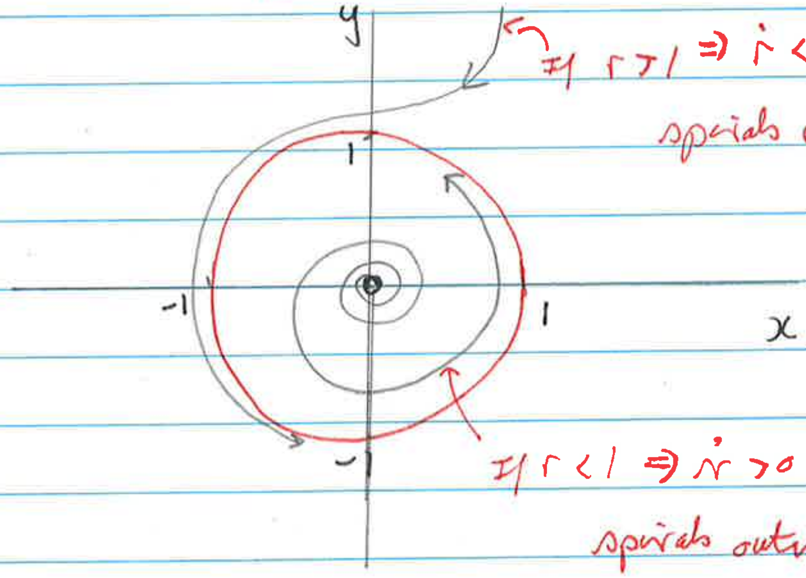
$$\Rightarrow \sec^2\phi \cdot \dot{\phi} = \frac{r^2\cos^2\phi + r^2\cos\phi\sin\phi(1-r^2) + r^2\sin^2\phi - r^2\sin\phi\cos\phi(1-r^2)}{r^2\cos^2\phi}$$

$$= \frac{1}{\cos^2\phi} = \sec^2\phi$$

$$\therefore \dot{\phi} = 1$$

Do you recognise this?

The solution is: $\boxed{\begin{aligned}\dot{r} &= r(1-r^2)\\ \dot{\phi} &= 1\end{aligned}}$



If $r > 1 \Rightarrow \dot{r} < 0 \Rightarrow$ spirals inwards

If $r < 1 \Rightarrow \dot{r} > 0 \Rightarrow$ spirals outwards

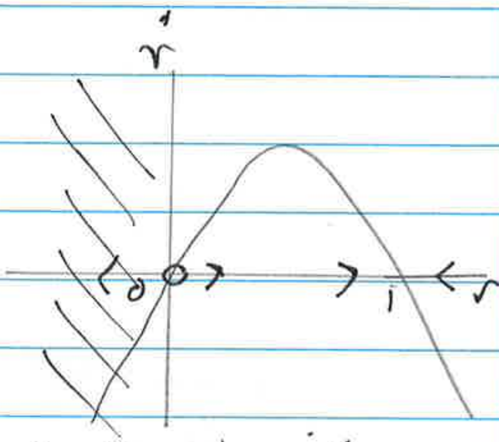
$\therefore$ this is a STABLE limit cycle.

Trapping Region

A trapping region is any closed curve with radius > 1 here.

cp Logistic equation: $\dot{x} = x(1-x)$

Here we have a logistic equation in the polar coordinate r.



Not ~~realistic~~ physically meaningful for radius.

Nullclines?

See introductory slides

$\dot{x} = -y + x(1-x^2-y^2) = 0$

$\dot{y} = x + y(1-x^2-y^2) = 0$

$\Rightarrow y^3 - (1-x^2)y - x = 0$ i.e. A cubic equation.

What do limit cycles represent?

- self-oscillating systems eg heart, fireflies, airplane wing oscillations, periodic chemical reactions

Selkov Model of Glycolysis

(course notes 5.2 ; Strogatz Example 7.3.2, p 207)

Glycolysis (turns) glucose into pyruvate, and part of this uses ATP to drive a reaction, producing ADP. It's called the glycolysis cycle, so we expect it to be periodic.

Consider
$$\dot{x} = -\underbrace{x}_{\text{degradation}} + \underbrace{ay}_{\text{promotion}} + x^2 y \qquad a, b > 0$$

CONSERVED

$$\dot{y} = \underbrace{b}_{\text{source}} - \underbrace{ay}_{\text{degradation}} - \underbrace{x^2 y}_{\text{competition / suppression}}$$

$x = [ADP]$
$y = [ATP]$

cp Selkov p81 ex 11 with x, y swapped.

$\boxed{\text{Can we find a limit cycle in these equations?}}$

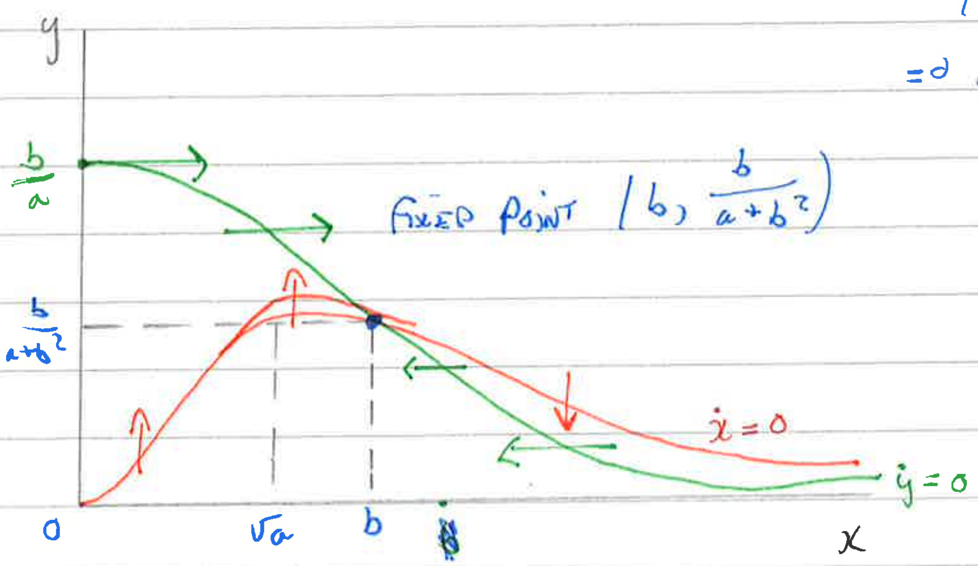
Where are the nullclines?

$\dot{x} = 0 \Rightarrow -x + ay + x^2 y = 0 \quad \therefore \underline{y = \dfrac{x}{a+x^2}} \quad \Rightarrow y' = \dfrac{a - x^2}{(a+x^2)^2}$

> 0 when $x = 0$

$\dot{y} = 0 \Rightarrow b - ay - x^2 y = 0 \quad \therefore \underline{y = \dfrac{b}{a+x^2}} \quad \Rightarrow y' = \dfrac{-2bx}{(a+x^2)^2}$

$= 0$ when $x = 0$



The fixed point is where the nullclines cross $\Rightarrow x = b$ and $y = \dfrac{b}{a+b^2}$

note that I've made an assumption here about a, b so that the peak of the x nullcline occurs at an x value less than the fixed point, namely

$$\sqrt{a} < b \quad \text{or} \quad a < b^2$$

what is the sign of $\dot{y}$ along the x nullcline? $y = \dfrac{x}{a+x^2}$

Substituting this into $\dot{y}$ gives:

$$\dot{y} = b - ay - x^2 y = b - (\cancel{a+x^2}) \cdot \frac{x}{\cancel{a+x^2}} = b - x$$

$\therefore$ $\dot{y} > 0$ when $x < b$, and $\dot{y} < 0$ when $x > b$

and the sign of $\dot{x}$ along $\dot{y} = 0$, i.e. $y = \dfrac{b}{a+x^2}$?

$$\Rightarrow \dot{x} = -x + (\cancel{a+x^2}) \cdot \frac{b}{\cancel{a+x^2}} = b - x$$

$\therefore$ $\dot{x} > 0$ when $x < b$, and $\dot{x} < 0$ when $x > b$

How do we show there is a limit cycle in this model?

1) There is a bounded region (Trapping region) in which field vectors point inwards (or outwards)

2) There is an unstable (or stable) fixed point in the trapping region.

consider 2) first. We know a fixed point is at $\left(b, \frac{b}{a+b^2}\right)$, what is its type?

$$\underline{\underline{J}} = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -(a+x^2) \end{pmatrix} = \begin{pmatrix} -1+\frac{2b^2}{a+b^2} & a+b^2 \\ \frac{-2b^2}{a+b^2} & -(a+b^2) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-a+b^2}{a+b^2} & a+b^2 \\ \frac{-2b^2}{a+b^2} & -(a+b^2) \end{pmatrix}$$

$$\therefore T = \operatorname{Tr} \underline{\underline{J}}\left(b, \frac{b}{a+b^2}\right) = \frac{-a+b^2}{a+b^2} - (a+b^2) = \frac{-a+b^2-(a+b^2)^2}{a+b^2}$$

$$= \frac{-a+b^2-a^2-2ab^2-b^4}{(a+b^2)}$$

$$\therefore T = -\left[\frac{b^4+(2a-1)b^2+a^2+a}{(a+b^2)}\right]$$

$$\Delta = \left(\frac{-a+b^2}{\cancel{a+b^2}}\right) \cdot -(\cancel{a+b^2}) + (\cancel{a+b^2}) \cdot \left(\frac{2b^2}{\cancel{a+b^2}}\right)$$

$$= a - b^2 + 2b^2$$

$$\therefore \Delta = a + b^2$$

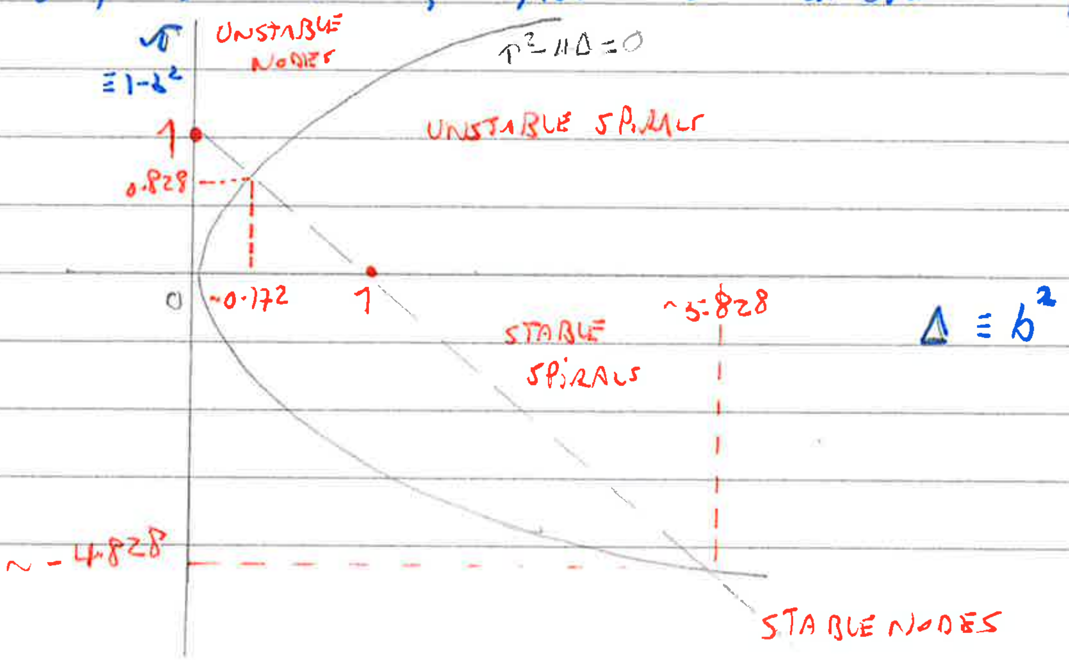
Given that $a, b > 0$, we can see that $\Delta > 0 \ \forall a, b$, so definitely not a saddlepoint. But because τ, Δ depend on two parameters a, b, it's not easy to see the type of the fixed point.

Look at a simpler case: $a = 0$ (why not $b = 0$?)

$$\therefore \tau = -\left[\frac{b^4 - b^2}{b^2}\right] = 1 - b^2$$

$$\Delta = b^2$$

As we vary b, how do τ, Δ move around in the τ-Δ plot?



Where does the straight line cross $\tau^2 - 4\Delta = 0$?

$$\tau^2 - 4\Delta = (1-b^2)^2 - 4b^2 = 1 - 2b^2 + b^4 - 4b^2$$

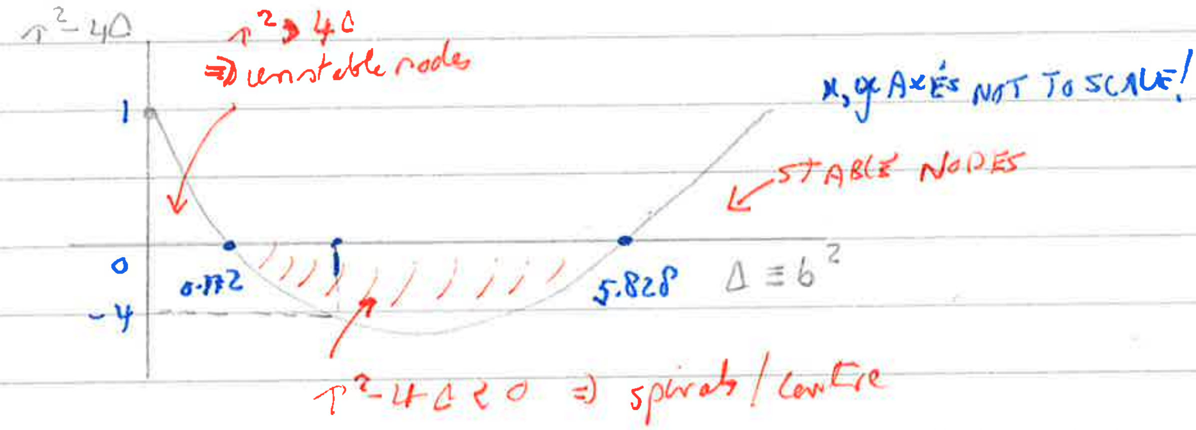
$$= 1 - 6b^2 + b^4 = 0$$

this small for small b, but dominates at large b

consider $x = b^2$, then the equation is: $x^2 - 6x + 1 = 0$

with solutions: $x = \dfrac{6 \pm \sqrt{36-4}}{2} = \dfrac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$

$\sim 5.828,\ 0.172$



As we allow b to vary from 0 upwards, the fixed point type changes from unstable node to unstable spiral, centre, etc.
(start drawing the bifurcation diagram on page 7)

Now to the full case : $a > 0$ What is the type of the Fixed point?

NB for a STABLE LIMIT CYCLE we want an unstable node or spiral

consider $\tau = -\left[\dfrac{b^4 + (2a-1)b^2 + a^2 + a}{(a+b^2)}\right]$, $\Delta = a + b^2 > 0$

Let $x = b^2$ again;

$$\Rightarrow \tau = -\left[\frac{x^2 + (2a-1)x + a^2 + a}{a + x}\right]$$

$$\Delta = a + x$$

For a given value of a, we can again explore what happens to τ, Δ as we vary b. In particular – where does τ change sign? (i.e. F.P. change from unstable to stable or vice versa)

We must solve : $x^2 + (2a-1)x + a^2 + a = 0$ NB. $a + x > 0$

$$\therefore x = \frac{1 - 2a \pm \sqrt{(2a-1)^2 - 4a(1+a)}}{2}$$

$$= \frac{1 - 2a \pm \sqrt{\cancel{4a^2} - 4a + 1 - 4a - \cancel{4a^2}}}{2}$$

$$x = \frac{1 - 2a \pm \sqrt{1 - 8a}}{2}$$

$$\Rightarrow \boxed{b^2 = \frac{1 - 2a \pm \sqrt{1 - 8a}}{2}}$$

and this must be real & positive.

$a \le 1/8$ and when $a = 1/8$, $b^2 = \dfrac{1 - 2 \cdot 1/8}{2} = 3/8$ uniquely

and when $a = 0$ then $b^2 = 0$ or 1 are (start drawing Figure here)
where τ changes sign.

So, for a given value of a, there is a range of values of b for which $\tau > 0$, and outside this range $\tau < 0$.

e.g. $a = 0.1 \Rightarrow b^2 = \dfrac{1 - 0.2 \pm \sqrt{1 - 0.8}}{2} = 0.6236, 0.1764$

$\Rightarrow b = 0.79, 0.42$

b^2	τ
0	-1.1
0.1764	0
0.2	0.033...
0.5	0.067...
0.6236	0
1	-0.28

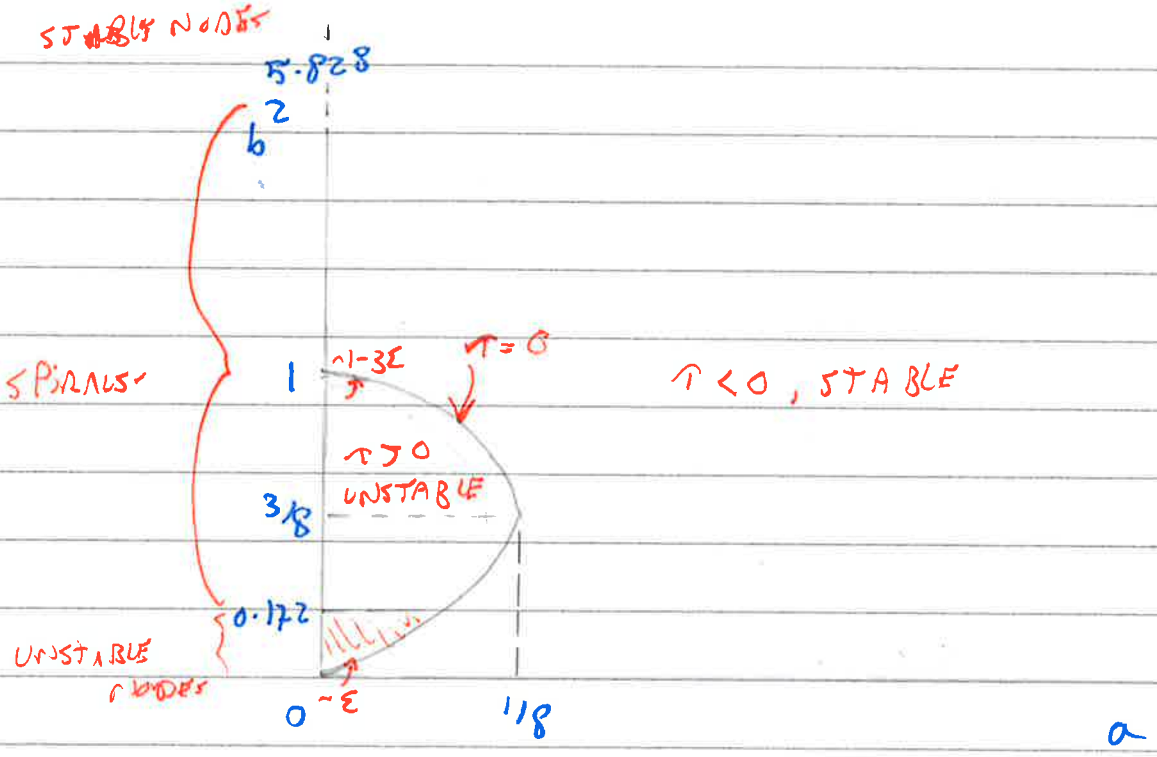
NB $b^2 = 0 \Rightarrow \tau = -(1 + a)$

The bounding curve is defined by: $b^2 = \dfrac{1 - 2a \pm \sqrt{1 - 8a}}{2}$

Let $a = \varepsilon \approx 0$

$\Rightarrow b^2 = \dfrac{1 - 2\varepsilon \pm (1 - 8\varepsilon)^{\frac{1}{2}}}{2} = \dfrac{1 - 2\varepsilon \pm (1 - 4\varepsilon + \ldots O(\varepsilon^2))}{2}$

$= \dfrac{1 - 2\varepsilon \pm (1 - 4\varepsilon)}{2} \doteq 1 - 3\varepsilon, \ \varepsilon$



For a given value of a: $0 \le a \le 1/8$, b^2 has a range

$$\frac{1-2a-\sqrt{1-8a}}{2} < b^2 < \frac{1-2a+\sqrt{1-8a}}{2}$$

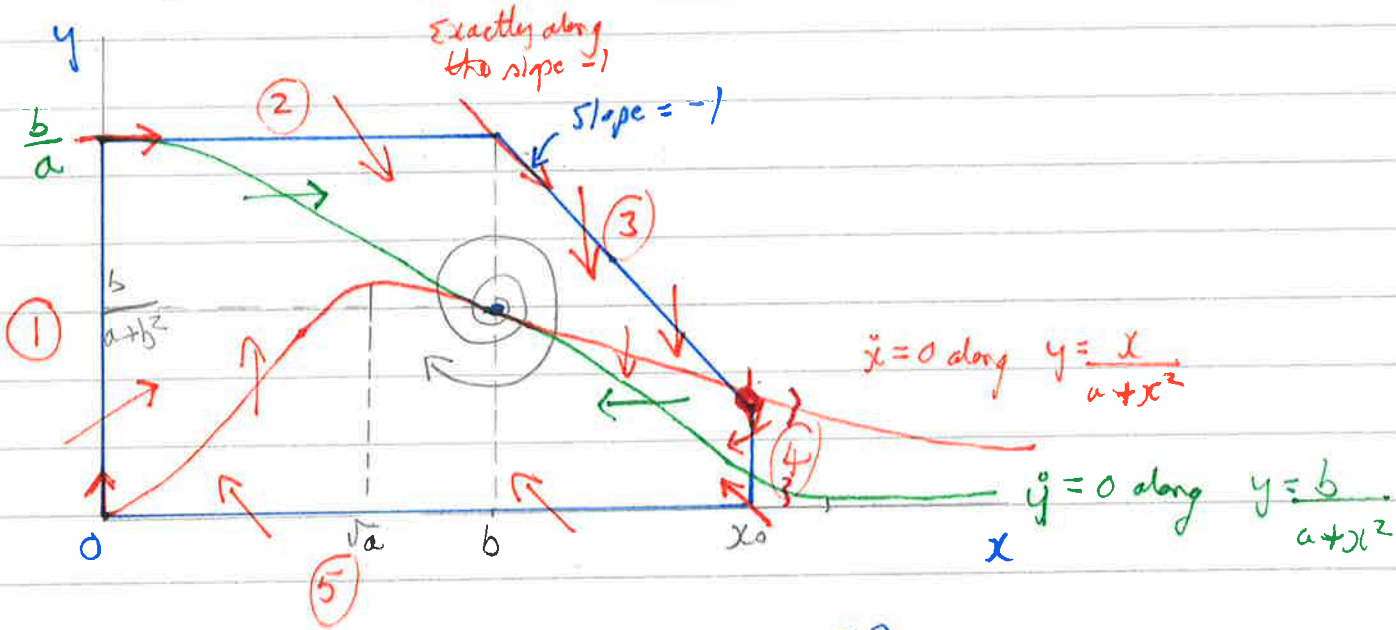
What have we learned?

For a given $a < 1/8$, as we vary b we can go from a stable spiral to an unstable spiral as b^2 increases.

We now need to show there is a trapping region in order to apply the PBT, and prove the existence of a stable limit cycle.

Recall: $\dot{x} = -x + ay + x^2 y$
$\dot{y} = b - ay - x^2 y$

$a, b > 0$



Consider the 5 regions shown: which way do the field vectors point?

1) $x = 0$, $y = 0$ to b/a

$\Rightarrow \dot{x} = ay > 0$

$\dot{y} = b - ay > 0$ for $y < b/a$, and $\dot{y} = 0$ when $y = \frac{b}{a}$

and $\dot{y} = b$ when $y = 0$, $\dot{x} = 0$ when $x, y = 0$

$$\boxed{\begin{aligned}\dot{x} &= -x + ay + x^2y \\ \dot{y} &= b - ay - x^2y.\end{aligned}}$$

2) $0 \leq x \leq b$, $y = b/a$

$$\dot{x} = -x + (a+x^2)\frac{b}{a} = b - x + \frac{bx^2}{a} > 0 \text{ for } x \leq b$$

$$\dot{y} = b - a\left(\frac{b}{a}\right) - x^2\frac{b}{a} = -\frac{b}{a}x^2 < 0$$

check $x = b$, $\left.\begin{aligned}\dot{x} &= \frac{b^3}{a} \\ \dot{y} &= \frac{-b^3}{a}\end{aligned}\right\} \Rightarrow \frac{dy}{dx} = -1$

4) Vertical line at a value of x s.t. $x > b$ and $y = 0$ up to $y = \frac{x}{a+x^2}$ The x nullcline

Recall the equation for the x nullcline: $y = \frac{x}{a+x^2}$

At the bottom when $x \geq b$, $y = 0$

$\left.\begin{aligned}\dot{x} &= -x < 0 \\ \dot{y} &= b > 0\end{aligned}\right\}$ so inward to the region.

As y increases up to $y = \frac{x}{a+x^2}$, then $\dot{x} = -x + (a+x^2)y$ but $y < \frac{x}{a+x^2}$

$\Rightarrow \dot{x} < 0 \ \forall y$ up to $\frac{x}{a+x^2}$ and 0 when $y = \frac{x}{a+x^2}$

and $\dot{y} = b - (a+x^2).y$, so $\dot{y} = b$ when $y = 0$ up to $b - x$ when $y = \frac{x}{a+x^2}$
which is < 0 as $x > b$.

5) This is the x axis $(y=0)$ from $x=0$ to $x>b$.

we've done the origin.

At $x = x > b$ $y = 0$ $\Rightarrow \dot{x} = -x < 0$, $\dot{y} = b > 0$ } true for all $0 < x \leq$ large x.

3) This is the tricky one!

This is a straight line with slope -1 from $(b, \frac{b}{a+b^2})$ to the x nullcline at $x > b$, i.e. $y = \frac{x}{a+x^2}$

consider first what are $\dot{x}, \dot{y}$ for $x, y \gg 0$?

$$\left.\begin{aligned} \dot{x} &\to x^2 y \\ \dot{y} &\to -x^2 y \end{aligned}\right\} \Rightarrow \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -1$$ i.e. field vectors point downwards at $45°$.

Now look at $\dot{x}, \dot{y}$ along the straight line: what is $\frac{dy}{dx}$ along here?

$$\boxed{\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b-(a+x^2)y}{-x+(a+x^2)y} = -\frac{[(a+x^2)y-b]}{[(a+x^2)y-x]}}$$

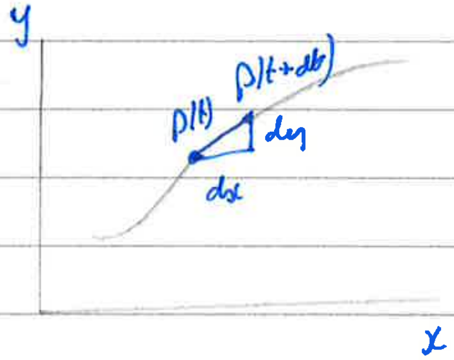
1) Note that $y > \frac{x}{a+x^2}$ along the line because we are always above the curve $y = \frac{x}{a+x^2}$

so the denominator is always greater than zero.

2) $x > b$, so denominator is smaller than numerator $\Rightarrow \frac{dy}{dx} < -1$ i.e. more negative

So, the field vectors point down at an angle $\theta > 45^\circ$ i.e. inwards to the region.

Why is $y' = \frac{\dot{y}}{\dot{x}}$?



$\dot{x} = f(x, y)$

$\dot{y} = g(x, y)$

$$\left.\begin{aligned} \Rightarrow dx &\sim f\,dt \\ dy &\sim g\,dt \end{aligned}\right\} \Rightarrow \frac{dy}{dx} = \frac{g}{f} = \frac{\dot{y}}{\dot{x}}$$

Selkov model

Lecture 7