Lecture 8 Recap

BIFURCATIONS

↓

≡ A qualitative, topological change in the flow of the vector field.

↓

1 D)

Saddlenode

$\dot{x} = r + x^2$

- creates/destroys Fixed points in pairs

Transcritical

$\dot{x} = rx - x^2$

- changes stability but not location of a fixed point

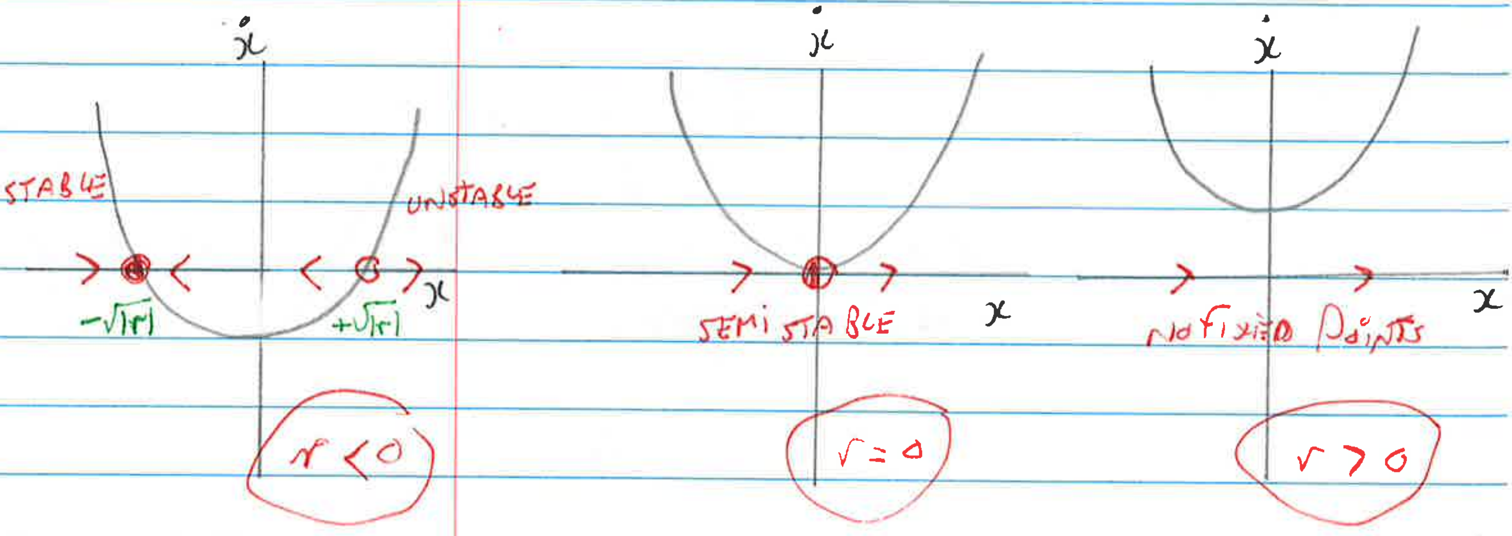
Pitchfork

supercritical:
$\dot{x} = rx - x^3$

Subcritical:
$\dot{x} = rx + x^3$

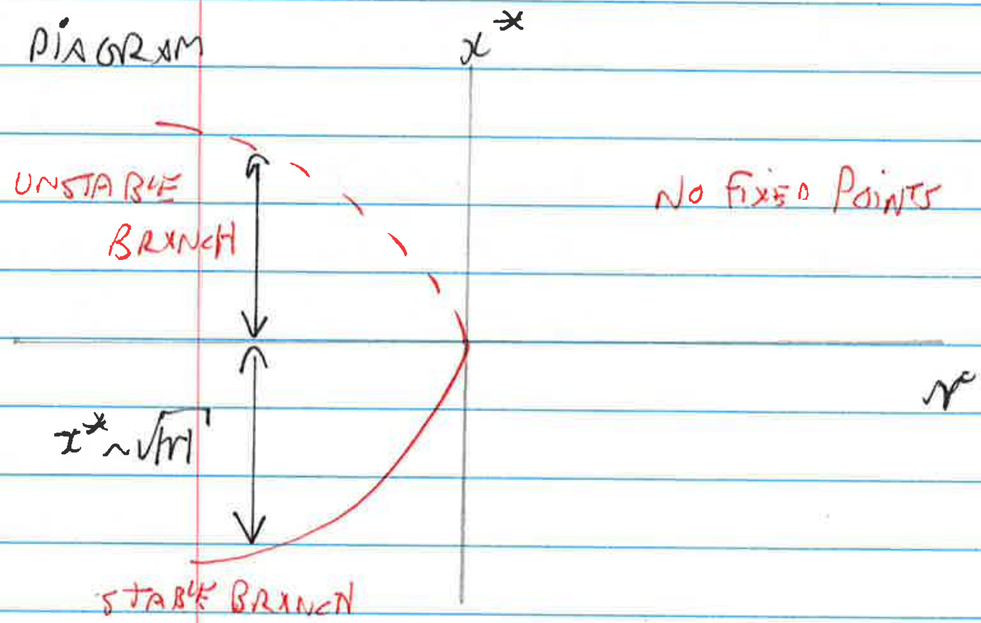
- creates/destroys fixed points and changes their stability

Saddlenode Bifurcation : $\dot{x} = r + x^2$



NB The bifurcation doesn't have to be at $r = 0$!

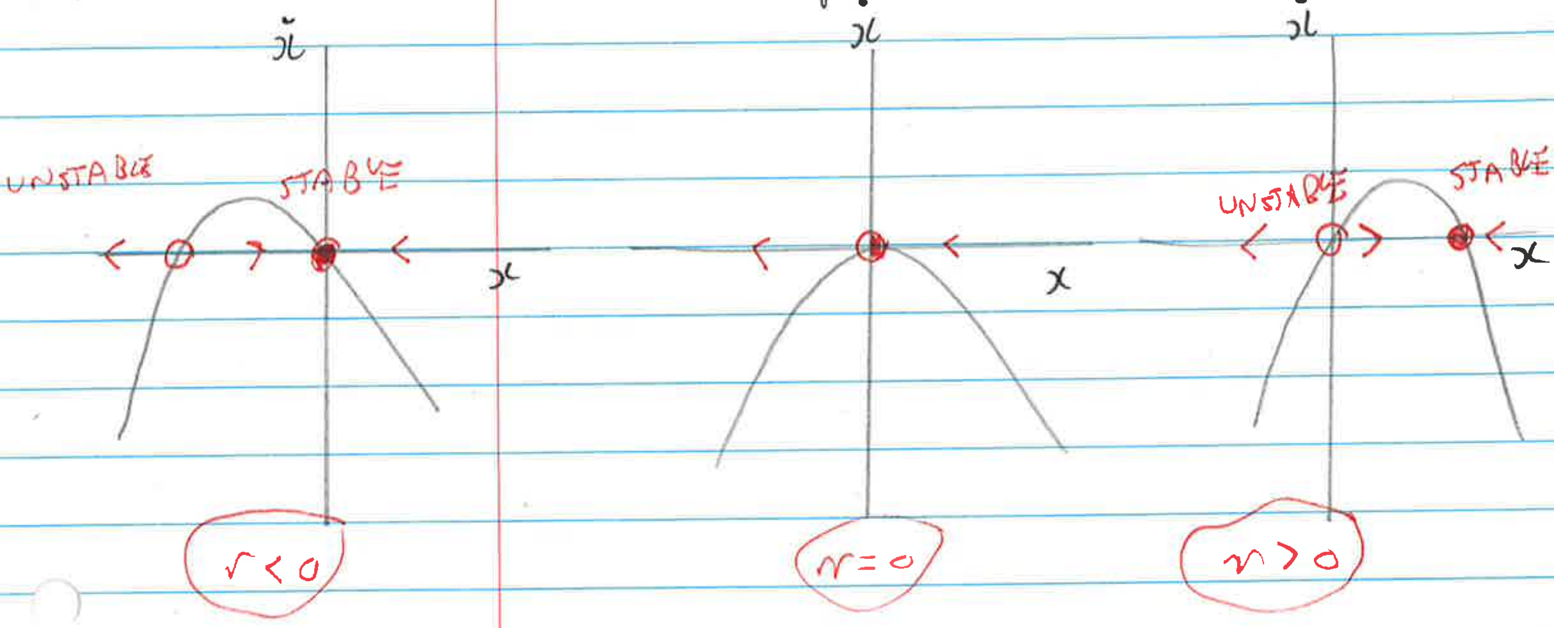
BIFURCATION DIAGRAM



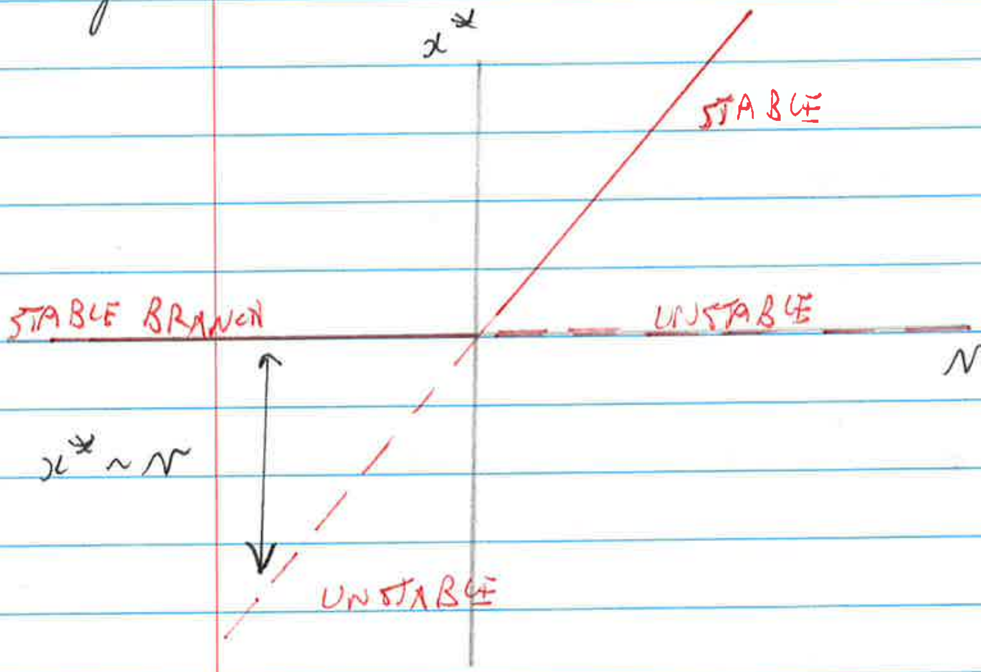
NBB All the interesting action happens at/near fixed points. The bifurcation diagram shows you succinctly what happens and where.

NBBB Adjacent branches must have the opposite stability.

Transcritical Bifurcation: $\dot{x} = rx - x^2$

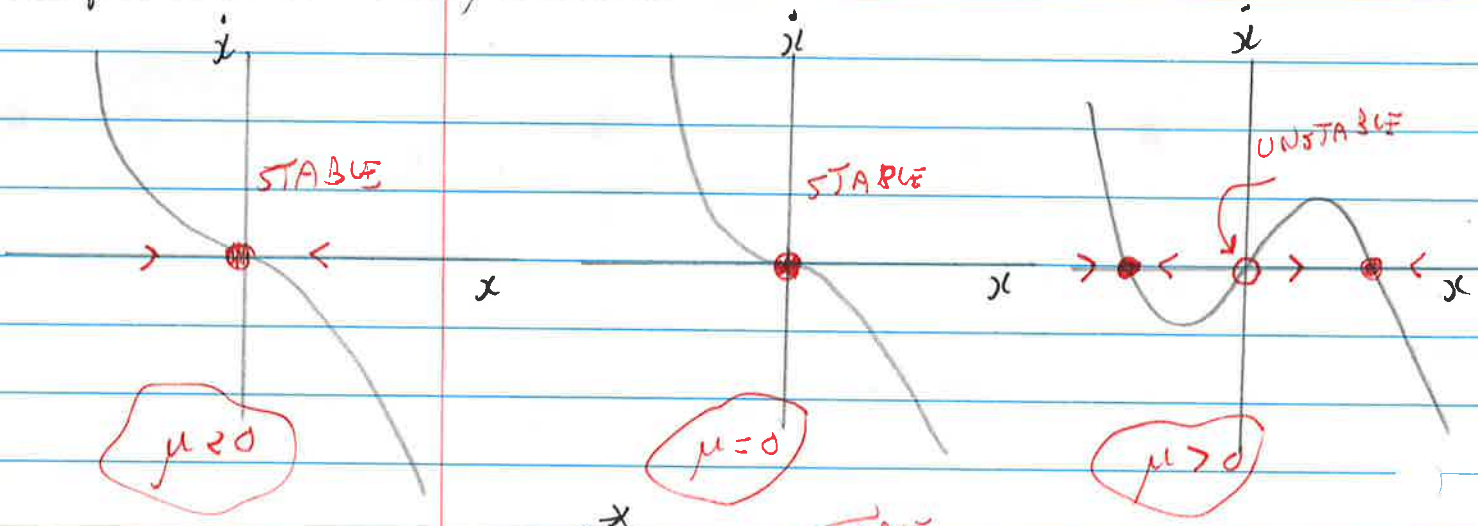


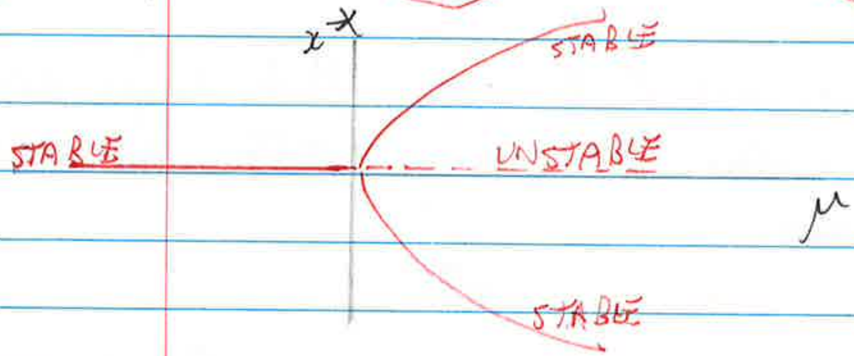
NB It's the fixed point at the origin that changes stability but not location. Notice the order of stability (left = unstable; right = stable) doesn't change.



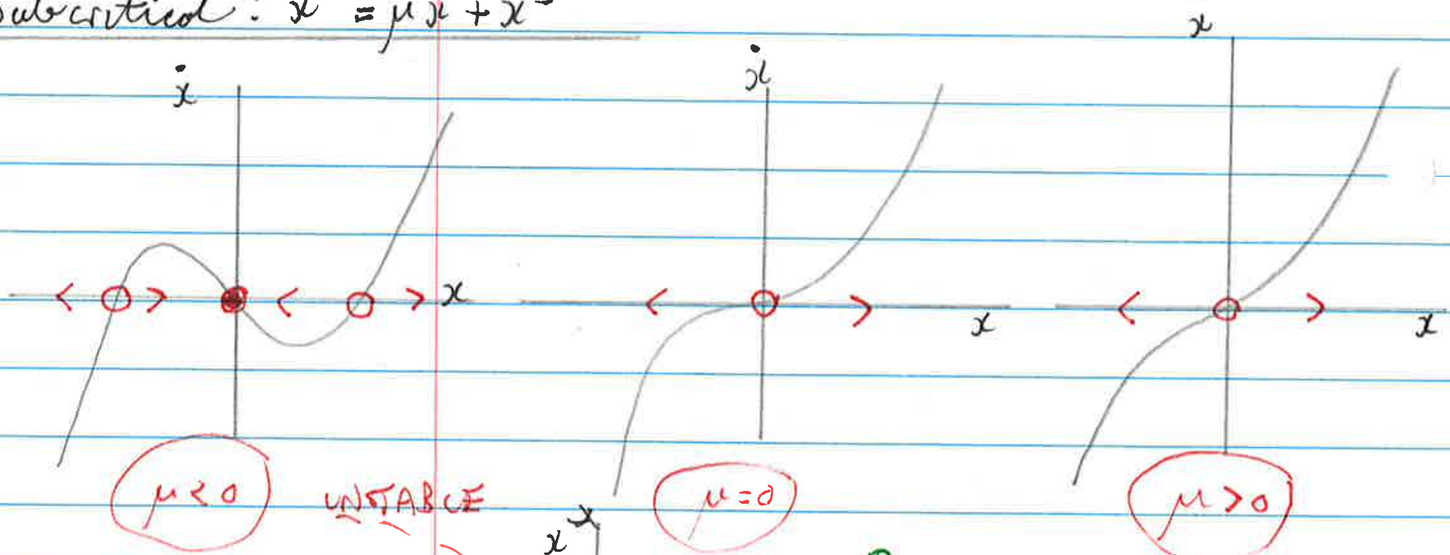
1D Pitchfork Bifurcation

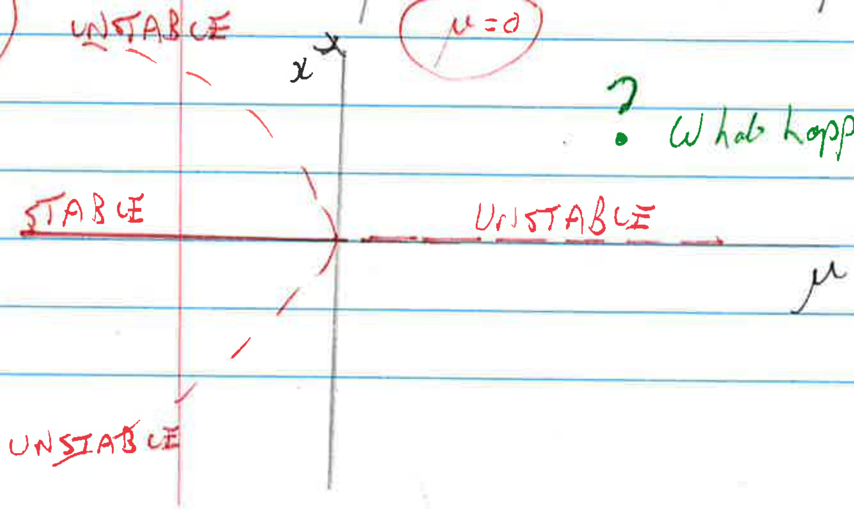
Supercritical: $\dot{x} = \mu x - x^3$





Subcritical: $\dot{x} = \mu x + x^3$





? What happens to the dynamics here?

Hopf Bifurcation in 2D

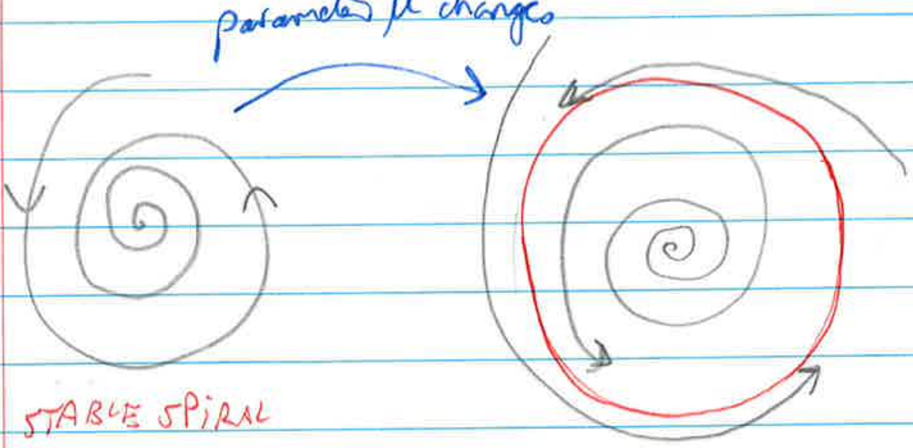
Consider a stable spiral – how can it become unstable as a parameter changes?

cp Linear System with spiral

$$\begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -2 \\ 2 & 2 \end{pmatrix}$$

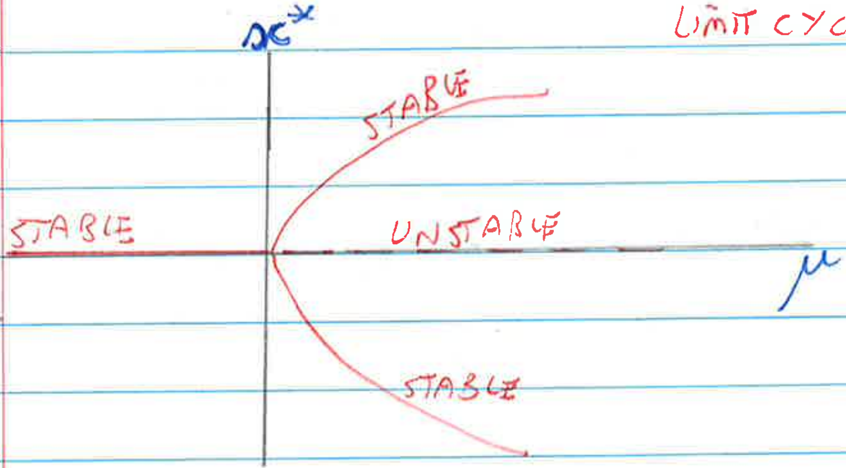
STABLE SP. UNSTABLE SP.

change of stability is all that happens.



Supercritical pitchfork Bifurcation:

$\dot{x} = \mu x - x^3$



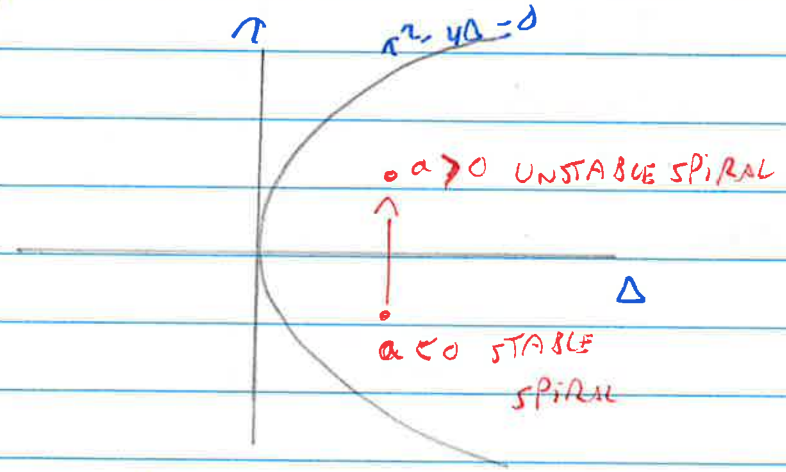
Hopf bifurcation is the 2d equivalent of a pitchfork bifurcation: there are two types. The above is the Supercritical case: { SOFT, CONTINUOUS, SAFE

Recall: $\dot{x} = f(x, y)$ $\quad J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$

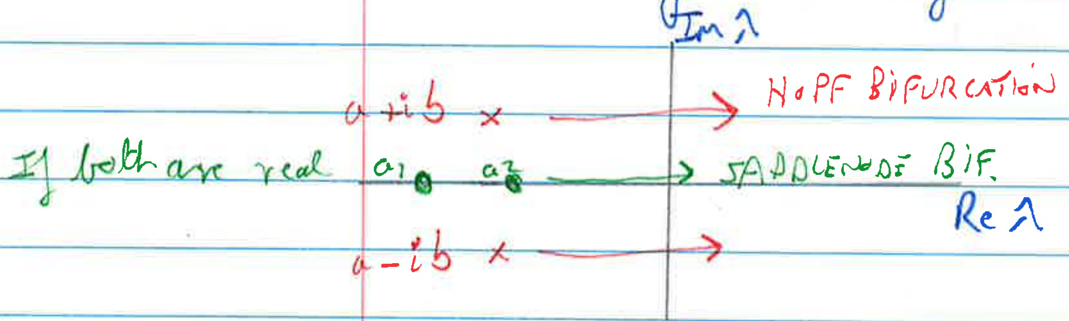
$\dot{y} = g(x, y)$

Eigenvalues at a fixed point are either both Real or complex conjugates.

spiral $\Rightarrow$ $\lambda_{1,2} = a \pm ib$



How do the eigenvalues change?



Re λ $(\equiv a)$ changes from negative to positive as the parameter μ changes.

If Im λ $(\equiv b)$ is zero, both eigenvalues lie on the real axis, and we have a saddlenode bifurcation because one becomes positive before the other.

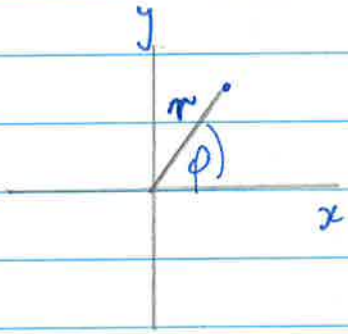
$b \neq 0$ a increases through ϕ

Both eigenvalues cross the imaginary axis, and the stable spiral ($a<0$) becomes unstable ($a>0$)

Is that all that happens? NO.
(but it is in a linear system with a spiral!)

Let's see why.

Generalise the supercritical pitchfork bifurcation to 2d. Use plane polar coordinates.



$$\boxed{\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\phi} &= \omega + b r^2 \end{aligned}}$$

μ = controls stability
ω = frequency of small oscillation
b = couples frequency to amplitude

To find the type & stability of the fixed points we use cartesian coordinates.

Let $x = r\cos\phi$
$y = r\sin\phi$

$$\left\{ \begin{aligned} r^2 &= x^2 + y^2 \\ \tan\phi &= \frac{y}{x} \end{aligned} \right.$$

$\Rightarrow$ $\dot{x} = \dot{r}\cos\phi - r\sin\phi \, . \, \dot{\phi}$

$\dot{y} = \dot{r}\sin\phi + r\cos\phi \, . \, \dot{\phi}$

substitute for $\dot{r}, \dot{\phi}$ from above.

$$\Rightarrow \dot{x} = (\mu r - r^3)\cos\phi - r\sin\phi\,(\omega + b r^2)$$

$$= \mu r\cos\phi - r^3\cos\phi - \omega r\sin\phi - b r^3\sin\phi$$

$$= (\mu - r^2)\,x - y\,(\omega + b r^2)$$

$$= \mu x - \omega y - x(x^2+y^2) - by(x^2+y^2)$$

$$= \mu x - \omega y - (x+by)(x^2+y^2)$$

$$\boxed{\therefore \dot{x} = \underbrace{\mu x - \omega y}_{\text{LINEAR PART}} \underbrace{- (x+by)(x^2+y^2)}_{O(x^3,\,y^3)}}$$

similarly $\dot{y} = (\mu r - r^3)\sin\phi + x(\omega + b r^2)$

$$= y(\mu - r^2) + x(\omega + b r^2)$$

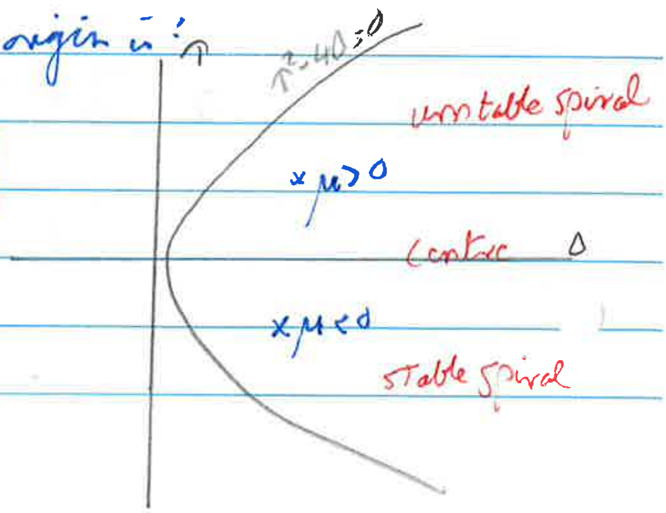
$$\boxed{\dot{y} = \omega x + \mu y + (bx - y)(x^2+y^2)}$$

The Jacobian at the origin is:

$$\underline{\underline{J}} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

$$\therefore \tau = 2\mu$$

$$\Delta = \mu^2 + \omega^2$$



so $T^2 - 4\Delta = 4\mu^2 - 4(\mu^2 + \omega^2)$

$= -4\omega^2 < 0 \quad \forall \mu$

$\therefore$ Always a spiral or centre

Conclusion

As μ goes from negative to positive through the transition, the stable spiral at the origin becomes unstable.

What are the eigenvalues of $\underline{J}$?

$$\begin{vmatrix} \mu - \lambda & -\omega \\ \omega & \mu - \lambda \end{vmatrix} = 0$$

$\Rightarrow (\mu - \lambda)^2 + \omega^2 = 0$

$\Rightarrow \mu^2 - 2\mu\lambda + \lambda^2 + \omega^2 = 0$

$\therefore \lambda^2 - 2\mu\lambda + (\mu^2 + \omega^2) = 0$

$\therefore \lambda = \dfrac{2\mu \pm \sqrt{4\mu^2 - 4(\mu^2 + \omega^2)}}{2} = \dfrac{2\mu \pm \sqrt{-4\omega^2}}{2}$

$\boxed{\lambda_{1,2} = \mu \pm i\omega}$

as expected, eigenvalues cross the Im axis as μ increases from < 0 to > 0.

For a linear system nothing else happens! Limit cycles are inherently non-linear objects.

That was the origin: does anything else happen?

Consider the radial equation:

$$\dot{r} = \mu r - r^3 = r(\mu - r^2) = f(r,\mu)$$

Roots: $r^* = 0$ or $r^* = \pm\sqrt{\mu}$

But $r < 0$ is meaningless in plane polar coords.

$$\therefore r^* = 0 \text{ or } +\sqrt{\mu}$$

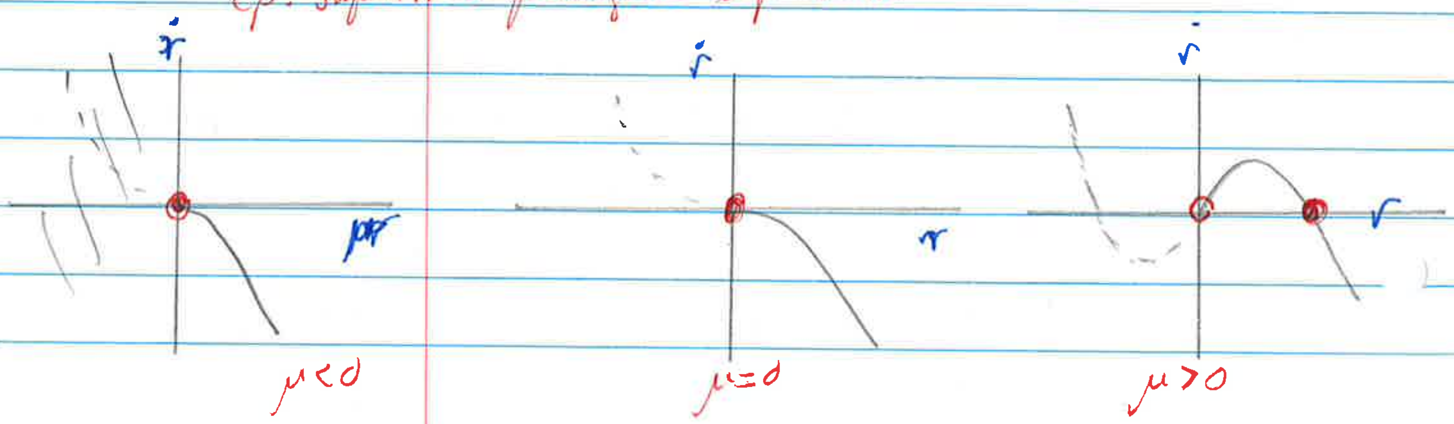
stability?

$$f'(r,\mu) = \mu - 3r^2$$

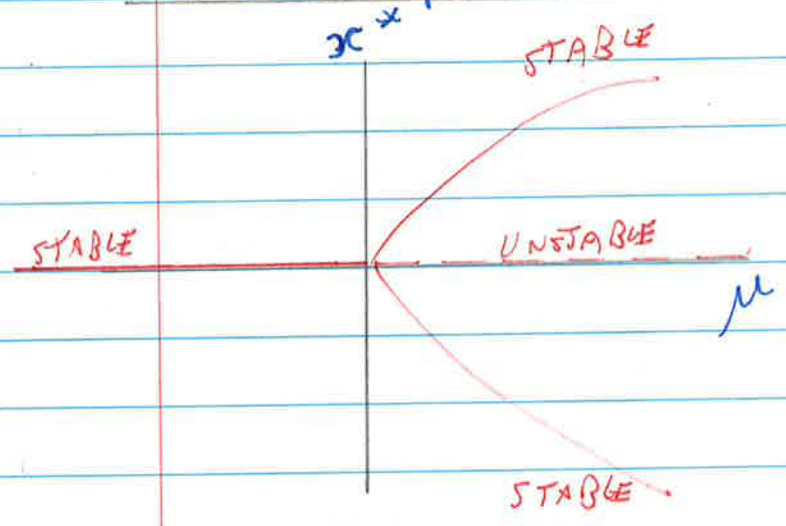
ORIGIN: $\Rightarrow f'(0,\mu) = \mu$ i.e. stable if $\mu < 0$, unstable $\mu > 0$

NON-ZERO FP: and $f'(\sqrt{\mu},\mu) = -2\mu$ i.e. unstable if $\mu < 0$, and stable for $\mu > 0$.

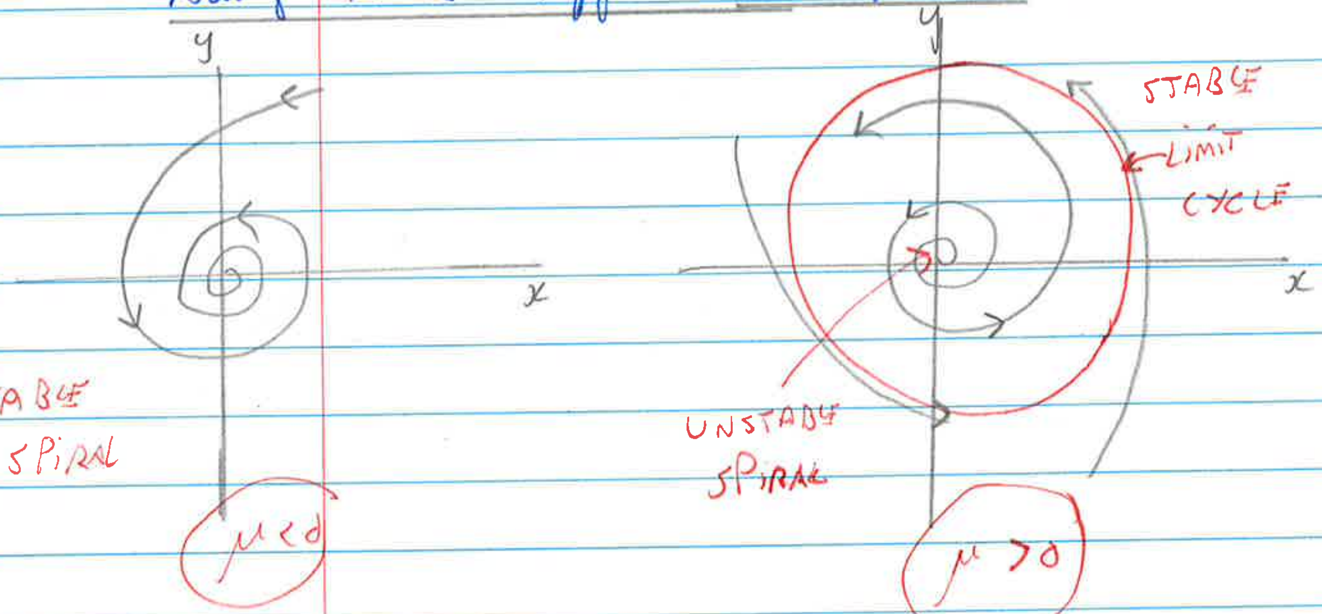
cp. supercritical pitchfork bifurcation.



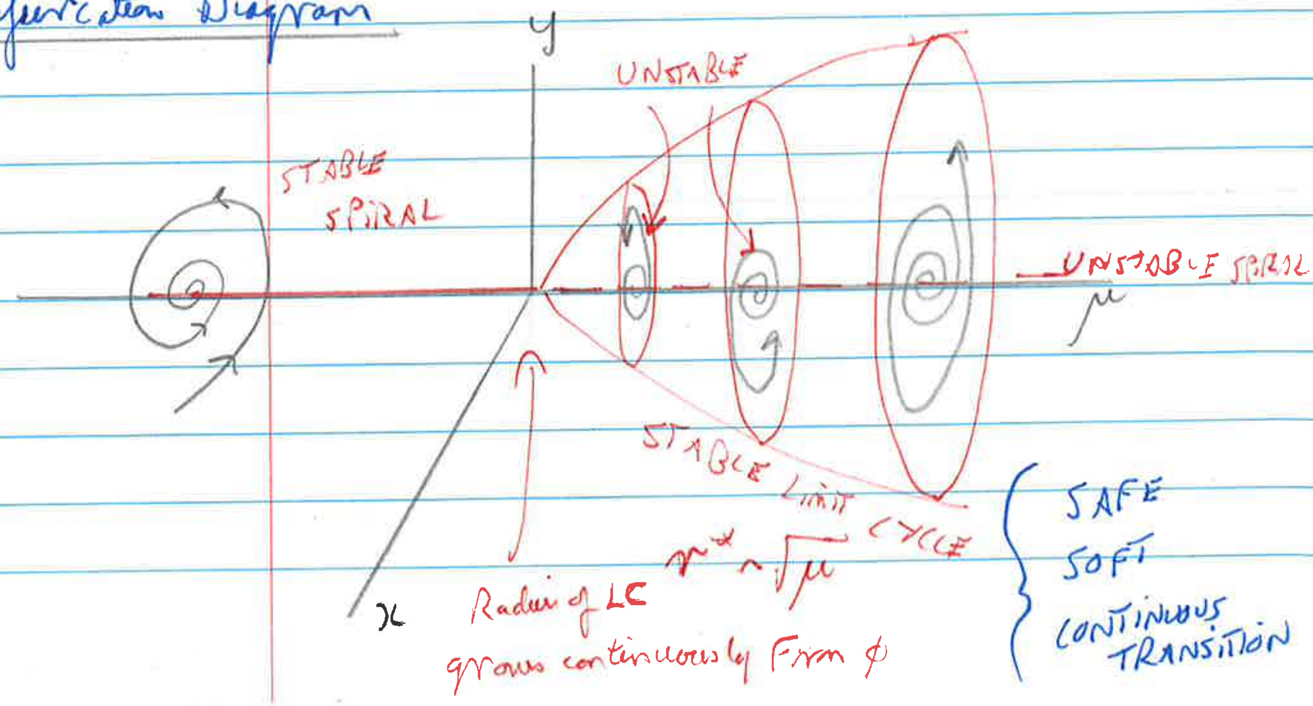
1d Pitchfork Bifurcation



Now for the 2d Hopf case: Phase Portrait



Bifurcation Diagram



SAFE
SOFT
CONTINUOUS
TRANSITION

Summary

As μ increases through zero (from below), the stable limit cycle at the origin becomes unstable, and a stable limit cycle appears; its radius grows continuously from zero proportional to $\sqrt{\mu}$.

The frequency of the limit cycle is ω for small amplitude.

Subcritical Hopf Bifurcation

what happens if the r^3 term is positive?

$$\dot{r} = \mu r + r^3$$

$$\dot{\phi} = \omega + b r^2$$

Rewrite in Cartesian coordinates again:

$$\Rightarrow \dot{x} = \dot{r}\cos\phi - r\sin\phi \cdot \dot{\phi}$$

$$= (\mu r + r^3)\cos\phi - r\sin\phi(\omega + b r^2)$$

$$= \mu x + x r^2 - y(\omega + b r^2)$$

$$\boxed{\dot{x} = \mu x - \omega y + (x - b y)(x^2 + y^2)}$$

Similarly with $\dot{y}$:

$$\Rightarrow \dot{y} = \dot{r}\sin\phi + r\cos\phi \cdot \dot{\phi}$$

$$= (\mu r + r^3)\sin\phi + r\cos\phi(\omega + b r^2)$$

$$= \mu y + r^2 y + \omega x + b x r^2$$

$$\boxed{\dot{y} = \omega x + \mu y + (b x + y)(x^2 + y^2)}$$

The Jacobian is again, $\underline{J} = \begin{pmatrix} \mu & -w \\ w & +\mu \end{pmatrix}$

They are inherently non-linear { so, linear stability analysis cannot distinguish between a supercritical Hopf bifurcation from the subcritical case.

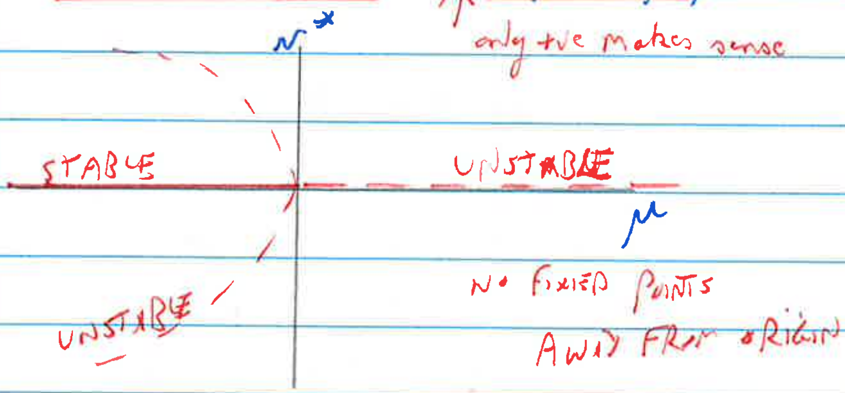
obviously, we again have a stable spiral at the origin.

Now what happens as μ increases through ϕ?
Where are fixed points:

$\dot{r} = r(\mu + r^2)$

$\therefore r^* = 0$ or $r^* = \pm\sqrt{|\mu|}$ if $\mu < 0$

only +ve makes sense



Note. $F(r,\mu) = \mu r + r^3$

$\Rightarrow F'(r,\mu) = \mu + 3r^2 \quad \therefore F'(0,\mu) = \mu < 0$ is stable

$F'(+\sqrt{|\mu|}, \mu) = 2|\mu| > 0$ is unstable.

A more realistic case has a stabilising term $-r^5$.

$$\dot{r} = \mu r + r^3 - r^5$$

Why not r^4?
To maintain symmetry.

$$\dot{\varphi} = \omega + b r^2$$

Again we rewrite into cartesian coordinates:

$$x = r\cos\varphi \qquad r^2 = x^2 + y^2$$
$$y = r\sin\varphi \qquad \tan\varphi = \frac{y}{x}$$

$$\Rightarrow \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi}$$

$$= (\mu r + r^3 - r^5)\cos\varphi - r\sin\varphi(\omega + b r^2)$$

$$= (\mu + r^2 - r^4)x - y(\omega + b r^2)$$

$$= \mu x - \omega y + (x - by) r^2 - x r^4$$

$$\boxed{\therefore \dot{x} = \mu x - \omega y + (x - by)(x^2+y^2) - x(x^2+y^2)^2}$$

and similarly with y:

$$\dot{y} = \dot{r}\sin\varphi + r\cos\varphi \cdot \dot{\varphi}$$

$\Rightarrow \quad \dot{y} = (\mu r + r^3 - r^5)\sin\varphi + r\cos\varphi(\omega + br^2)$

$= (\mu + r^2 - r^4)\, y \quad + x(\omega + br^2)$

$$\boxed{\dot{y} = \omega x + \mu y + (xb + y)(x^2+y^2) - y(x^2+y^2)^2}$$

Jacobian is again: $\underset{\approx}{J} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$

So, again a stable spiral at origin when $\mu < 0$.
What happens at $\mu > 0$?

Are there more fixed points / limit cycles?

$\dot{r} = \mu r + r^3 - r^5 \quad = \quad -r(r^4 - r^2 - \mu)$

Let $x = r^2 \Rightarrow x^2 - x - \mu = 0$ defines new fixed points.

$$\Rightarrow x = \frac{1 \pm \sqrt{1+4\mu}}{2}$$

$$\boxed{\text{so, } r^* = 0 \text{ or } (r_\pm^*)^2 = \frac{1 \pm \sqrt{1+4\mu}}{2}}$$

or $\quad \underline{\underline{\dot{r} = -r(r^2 - r_-^2)(r^2 - r_+^2)}}$

NB sign r^5 term is negative as it should be

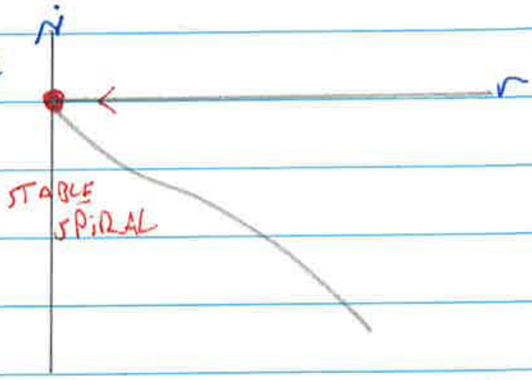
$$r_\pm^2 = \frac{1 \pm \sqrt{1+4\mu}}{2}$$

What does Graph ? look like?

$\mu < -1/4$ $r_\pm$ are complex; only $r=0$ is real

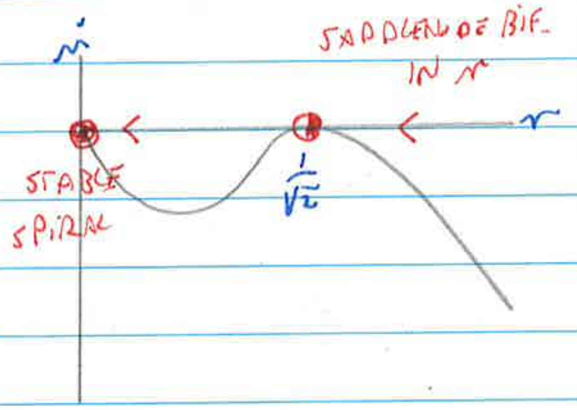
$F'(0) = \mu + 3r^2 - 5r^4 = \mu < 0$

$\therefore$ stable

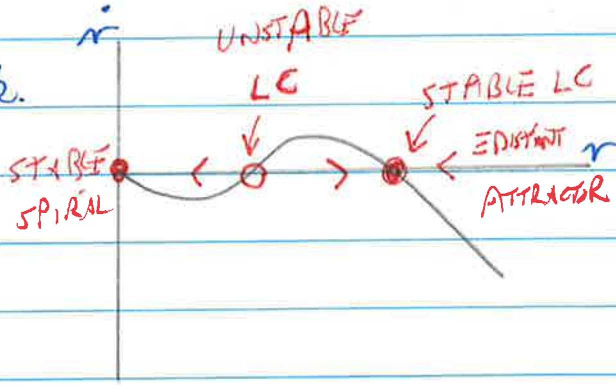


$\mu = -1/4$ $r_\pm = \pm 1/\sqrt{2}$, so $0, +1/\sqrt{2}$ are physically relevant

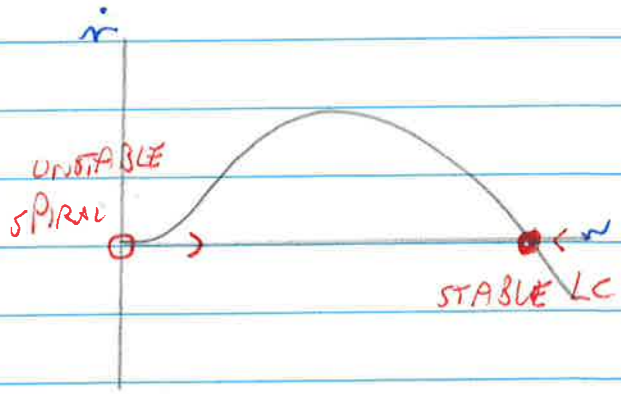
NB. $r_+ = -1/\sqrt{2}$ does not depend on μ, but new FP appears at $O(1)$ away from origin.



$-\frac{1}{4} < \mu < 0$, $r_\pm$ are real and distinct, and are a stable and unstable limit cycle.

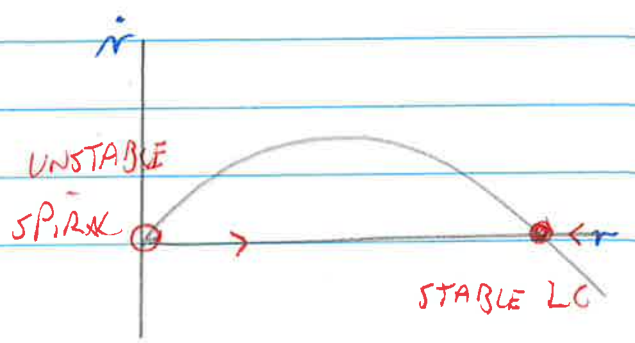


$\mu = 0$, $r_\pm = 0, 1$ origin is now an unstable spiral surrounded by a stable limit cycle at large r $\sim O(1)$ not $O(\sqrt{\mu})$



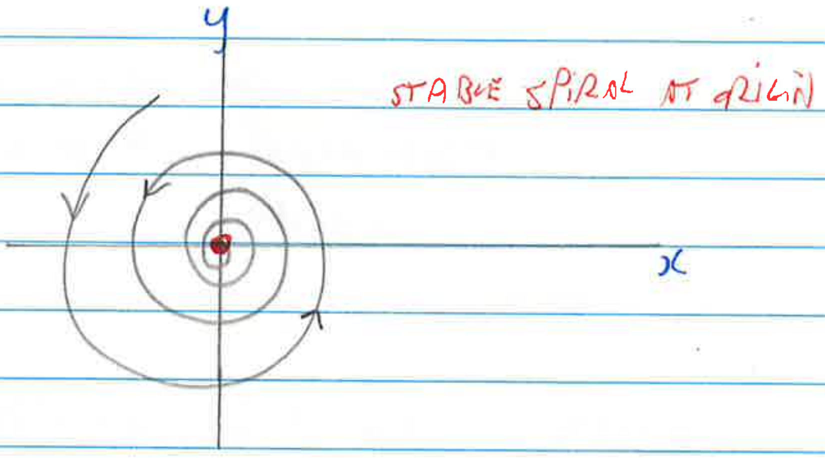
$\mu > 0$, $r_+ = \frac{1 \pm \sqrt{1+4\mu}}{2}$, r_- complex

stable limit cycle increases to larger radius.



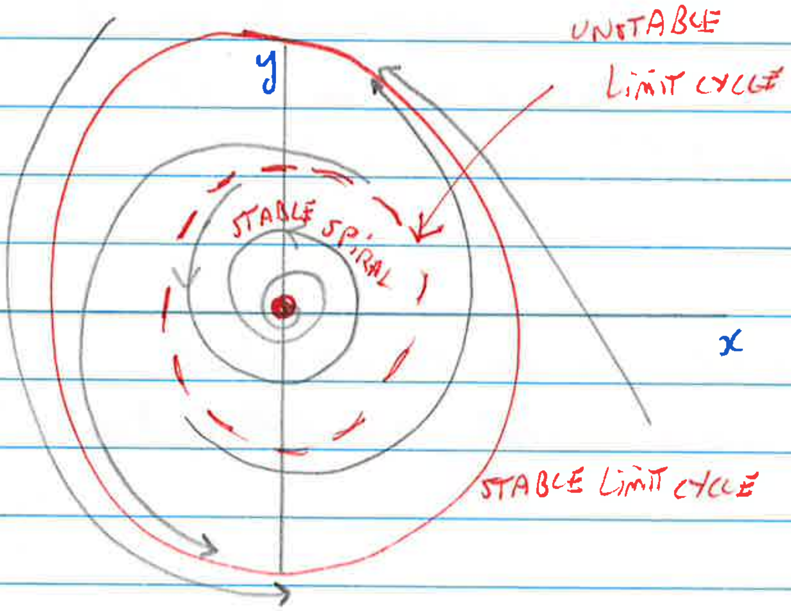
Phase Portraits now?

$\mu < -1/4$

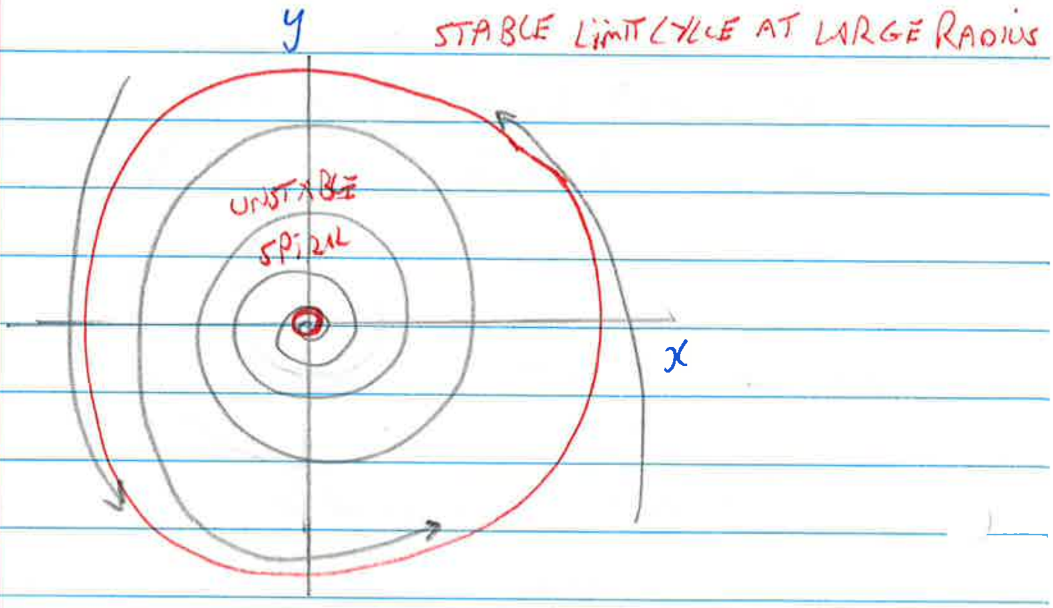


$-1/4 < \mu < 0$

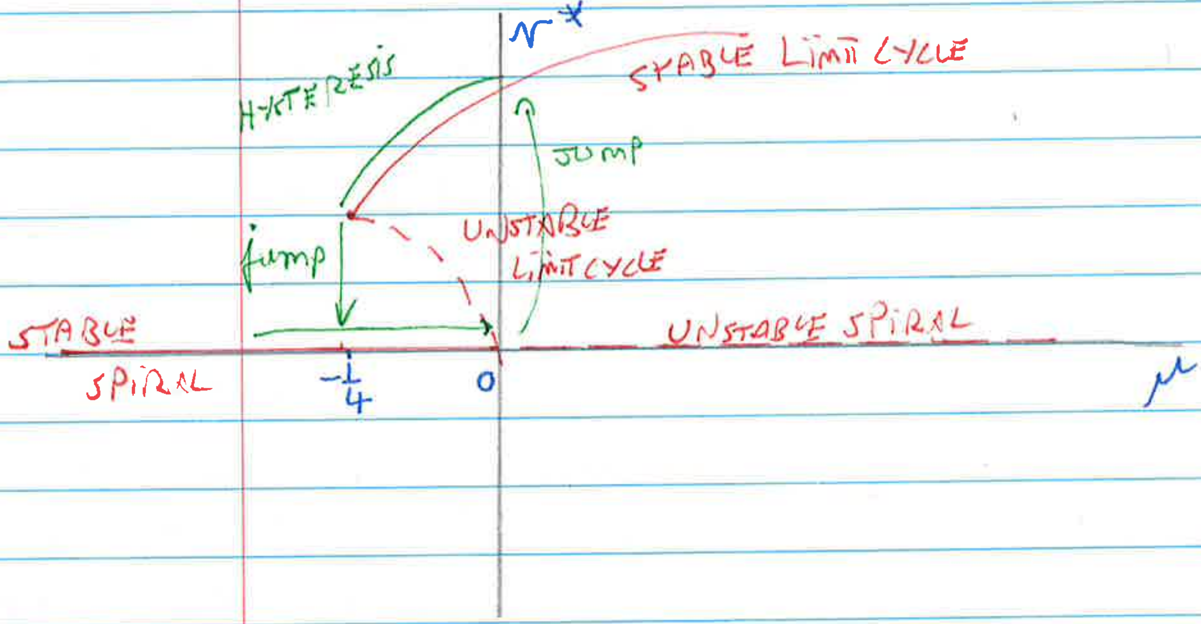
The unstable LC shrinks down to the origin and makes the stable spiral there unstable.

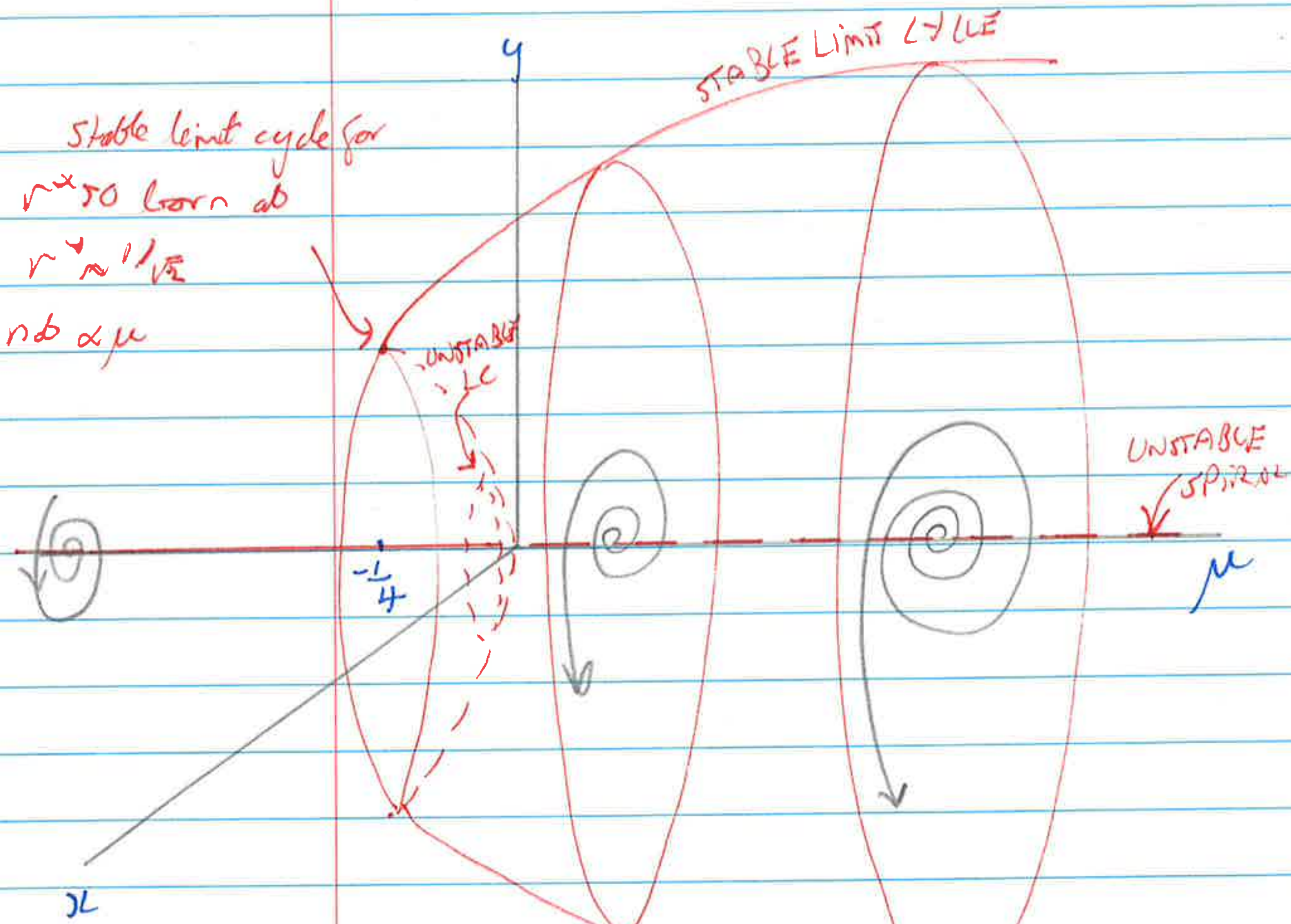


$\mu > 0$



What does the bifurcation diagram look like?





$\left\{ \begin{array}{l} \text{HARD} \\ \text{DISCONTINUOUS} \\ \text{DANGEROUS} \end{array} \right\}$ Transition

NB. Because the stable limit cycle is born at $r^* \sim O(1)$ not $O(\sqrt{\mu})$, it is far from the origin: so the system will jump from damped oscillations (i.e. stable spiral at $r^* = 0$) to a stable limit cycle a long way from the origin.

And, reversing μ to ϕ won't destroy these large amplitude oscillations. You have to reduce μ to $-1/4$ before the system returns to small oscillations. i.e. HYSTERESIS