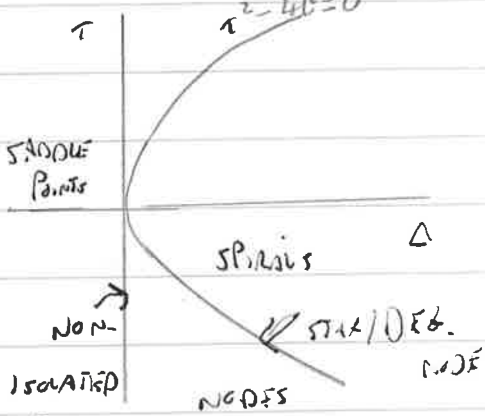
# How to distinguish a star node from a degenerate node.

Recall from the $\tau$-$\Delta$ plot, star nodes and degenerate nodes occur on the curve $\tau^2 - 4\Delta = 0$




Consider the identity matrix:

$$\underline{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

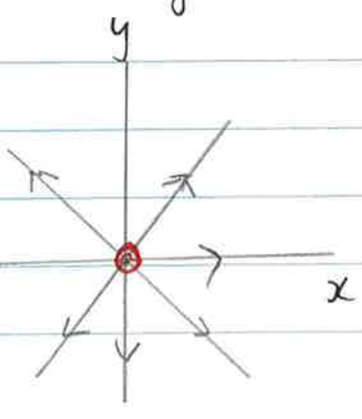
i.e. $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

we have: $$\left.\begin{aligned} \tau &= 2 \\ \Delta &= 1 \end{aligned}\right\} \Rightarrow \tau^2 - 4\Delta = 0$$

So, the fixed point is either a star or a degenerate node. Which is it?

Eigenvalue: $$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1 \text{ only one eigenvalue}$$

Eigenvector: $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \therefore \text{Any vector is an eigenvector}$$




So, the origin is a star node, and it is unstable because $\tau \equiv \lambda > 0$.

solution is: $\underline{x}(t) = e^{t} \underline{x}_0$

Now try $\underline{M} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

We have now: $\tau = 2$ so again it is either a star or
$\Delta = 1$ a degenerate node as
$\tau^2 - 4\Delta = 0.$

Eigenvalues: $\begin{vmatrix} 1-\lambda & 0 \\ -1 & 1-\lambda \end{vmatrix} = 0 \quad \therefore \lambda = +1$ again

Eigenvectors: $\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow -v_1 = 0 \quad \therefore \underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$v_2$ is any number

The general solution is then: $\underline{x}(t) = c e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

But this cannot be the full solution:

$\dot{x} = x$
$\dot{y} = -x + y$

which means $x(t) = x_0 e^{t}$ but our solution above has NO $x$ component!

Let us find the y component:

$$\dot{y} = -x + y = -x_0 e^{t} + y$$

$$\Rightarrow \dot{y} - y = -x_0 e^{t}$$

we solve this with an integrating factor:

$$IF = e^{-\int dt} = e^{-t}$$

$$\Rightarrow e^{-t}(\dot{y} - y) = -x_0 e^{t} \cdot e^{-t} = -x_0$$

$$\Rightarrow \frac{d}{dt}\left(y e^{-t}\right) = -x_0$$

$$\therefore y e^{-t} = -x_0 t + A \qquad A: \text{constant of integration}$$

$$\therefore y(t) = (A - x_0 t) e^{t}$$

We find A from the initial condition:

$$y(0) = y_0 = A$$

$$\therefore y(t) = (y_0 - x_0 t) e^{t}$$

our full solution is: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{t} \\ (y_0 - x_0 t) e^{t} \end{pmatrix}$

So, $x(t)$ just increases exponentially from $x_0$, but $y(t)$ changes as $y_0 - x_0 t$, and changes sign if $y = x_0 t$

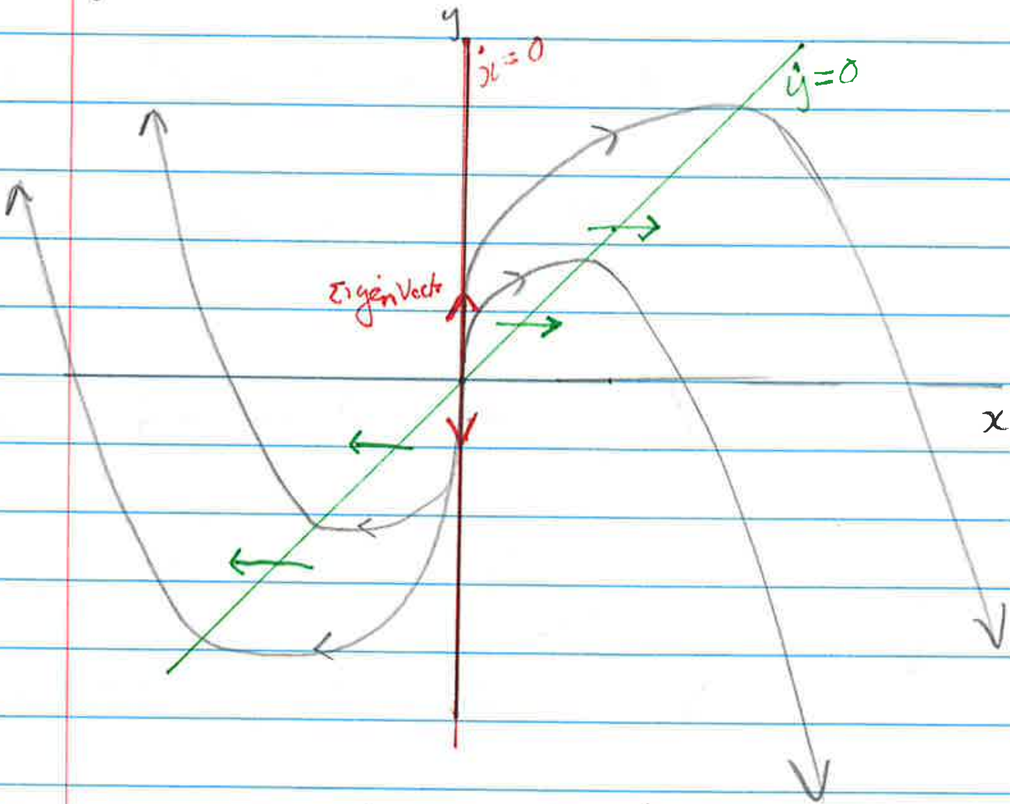
Where are the nullclines?

$\dot{x} = 0 \Rightarrow x = 0$ i.e. $Y$ axis is $x$ nullcline

and along $\dot{x} = 0$, $\dot{y} = y$.

$\dot{y} = 0 \Rightarrow y = x$ i.e. $45°$ to $x$ axis, and $\dot{x} = x$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 e^t \\ (y_0 - x_0 t)e^t \end{pmatrix}$$




For a degenerate node, the trajectories approach (leave) the node parallel to the only eigen vector, AND become parallel to it again as $t \to \infty$.