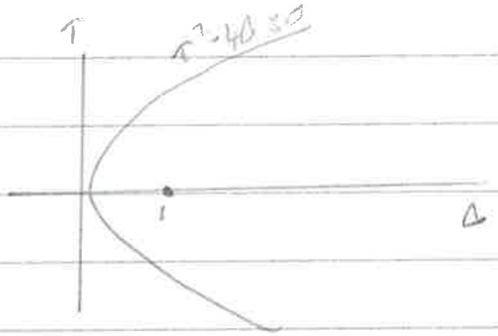
How can we plot trajectories of a centre if the eigenvalue & Eigenvector are complex?

24/10/24 Consider the system:

$$\dot{\underset{\sim}{x}} = \underset{\approx}{M}\,\underset{\sim}{x} \qquad \text{where } \underset{\approx}{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\operatorname{tr} \underset{\approx}{M} = 0$ $\quad \therefore$ Centre

$\det \underset{\approx}{M} = 1$



Eigenvalues

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0 \quad \therefore \lambda = \pm i$$

Eigenvectors

$\underline{\lambda = i}$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow -i v_1 - v_2 = 0$$

$$v_1 - i v_2 = 0$$

$$\therefore v_2 = -i v_1 \Rightarrow \underset{\sim}{v_1} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\underline{\lambda = -i}$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$\Rightarrow i v_1 - v_2 = 0$

$v_1 + i v_2 = 0$

$\therefore v_2 = i v_1 \quad \Rightarrow \underline{v_2} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

So, the general trajectory is: $\underline{x}(t) = c_1 e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix}$

where c_1, c_2 are complex numbers. From the initial condition:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ i(-c_1 + c_2) \end{pmatrix}$$

$\therefore x_0 = c_1 + c_2$

$y_0 = i(-c_1 + c_2)$

obviously, x_0, y_0 are real coordinates, so:

$c_1 + c_2$ must be real

$c_2 - c_1$ must be pure imaginary

Let $c_1 = a + ib$ where $a, b, c, d \in \mathbb{R}$
$c_2 = c + id$

$\Rightarrow c_1 + c_2 = a + c + i(b+d) \in \mathbb{R} \Rightarrow \underline{d = -b}$

$c_2 - c_1 = c - a - 2ib \in \mathbb{C} \Rightarrow \underline{c = a}$

$\therefore c_1 = a + ib$
$c_2 = a - ib$

So, we have: $x_0 = c_1 + c_2 = 2a \quad \therefore \underline{a = \frac{1}{2} x_0}$

$y_0 = i(-c_1 + c_2) = i \cdot -2ib = 2b \quad \therefore \underline{b = \frac{1}{2} y_0}$

and therefore: $c_1 = \frac{1}{2} x_0 + \frac{1}{2} y_0 i$

$c_2 = \frac{1}{2} x_0 - \frac{1}{2} y_0 i$

and the general trajectory is: $$\underline{x}(t) = \frac{1}{2}(x_0 + iy_0) e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1}{2}(x_0 - iy_0) e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

But obviously, we need $\underline{x}(t)$ to be real.

consider:

$$e^{it}\begin{pmatrix}1\\-i\end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix}1\\-i\end{pmatrix} = \begin{pmatrix}\cos t + i\sin t\\ -i\cos t + \sin t\end{pmatrix}$$

$$e^{-it}\begin{pmatrix}1\\i\end{pmatrix} = (\cos t - i\sin t)\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}\cos t - i\sin t\\ i\cos t + \sin t\end{pmatrix}$$

so, the x component is:

$$x(t) = \tfrac{1}{2}(x_0 + iy_0).(\cos t + i\sin t) + \tfrac{1}{2}(x_0 - iy_0)(\cos t - i\sin t)$$

$$= \frac{1}{2}\Big[x_0\cos t - y_0\sin t + i(\cancel{y_0\cos t} + \cancel{x_0\sin t})$$

$$+ x_0\cos t - y_0\sin t + i(\cancel{-y_0\cos t} - \cancel{x_0\sin t})\Big]$$

$$\underline{x(t) = x_0\cos t - y_0\sin t}$$

and the y component is:

$$\tfrac{1}{2}(x_0 + iy_0)(\cos t + i\sin t).-i + \tfrac{1}{2}(x_0 - iy_0)(\cos t - i\sin t).+i$$

$$= \frac{1}{2}\Big[-i\{x_0\cos t - y_0\sin t + i(y_0\cos t + x_0\sin t)\}$$

$$+ i.\{x_0\cos t - y_0\sin t - i(y_0\cos t + x_0\sin t)\}$$

$$= \frac{1}{2}\Big[-(x_0 \cos t \cancel{-} y_0 \sin t).i + y_0 \cos t + x_0 \sin t$$

$$+ (x_0 \cos t \cancel{-} y_0 \sin t).i + y_0 \cos t + x_0 \sin t \Big]$$

$$\therefore \underline{y(t) = x_0 \sin t + y_0 \cos t}$$

$\therefore$ General trajectory is:

$$\underline{\underline{x(t)}} = \begin{pmatrix} x_0 \cos t - y_0 \sin t \\ x_0 \sin t + y_0 \cos t \end{pmatrix} = \underline{\underline{\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}}$$

An anticlockwise rotation around the origin