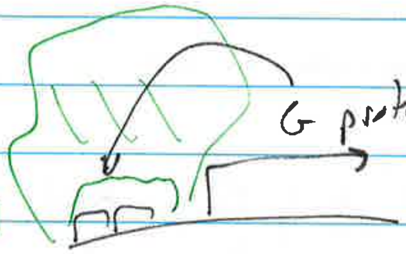
Hysteresis (Lecture 3)



If [G] changes continuously, how does this give rise to a switch?

$$\dot{G} = \underbrace{S_0}_{\text{Constitutive production}} - \underbrace{k_1 G}_{\text{Degradation}} + k_2 \underbrace{\frac{G^2}{K^2+G^2}}_{\text{Saturated +ve Feedback}} \qquad S_0, k_1, k_2, K$$

Degradation: k_1 = Rate $= 1/\text{Lifetime}$

Saturated +ve Feedback: K = Half-"conc", k_2 = magnitude of effect

Constitutive production

Non-Dimensionalising

- start with the most complicated term:

$$\Rightarrow \frac{G^2}{K^2+G^2}$$

Try $g = \dfrac{G}{K}$

$$\Rightarrow K\dot{g} = S_0 - k_1 K g + k_2 \frac{g^2}{1+g^2}$$

Devide by k_2

$$\Rightarrow \frac{K}{k_2}\dot{g} = \overset{s}{\frac{50}{k_2}} - \overset{r}{\frac{k_1 K}{k_2}} g + \frac{g^2}{1+g^2}$$

Absorb K/k_2 into time scale

$$\tau = \frac{k_2 t}{K}$$

$$\Rightarrow \frac{d}{dt} = \frac{d\tau}{dt}\cdot\frac{d}{d\tau} = \frac{k_2}{K}\frac{d}{d\tau}$$

$$\Rightarrow \frac{K}{k_2}\frac{dg}{dt} = \frac{K}{k_2}\cdot\frac{k_2}{K}\frac{dg}{d\tau} = \frac{dg}{d\tau}$$

$$\therefore \dot{g} = s - rg + \frac{g^2}{1+g^2}$$

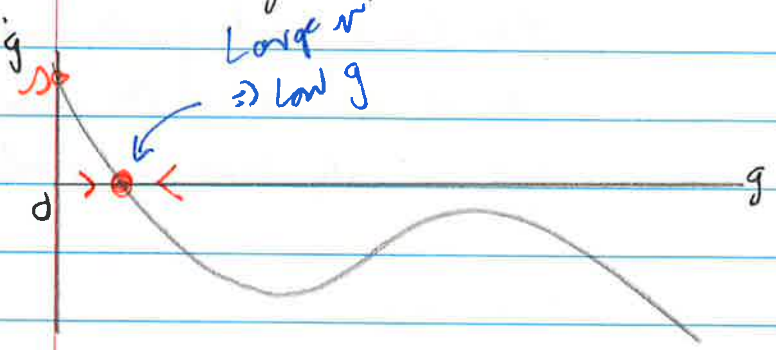
$\equiv \frac{d}{d\tau}$ now

Hysteresis

- Comes from the fixed points appearing / disappearing at different values of g when increasing or decreasing a parameter

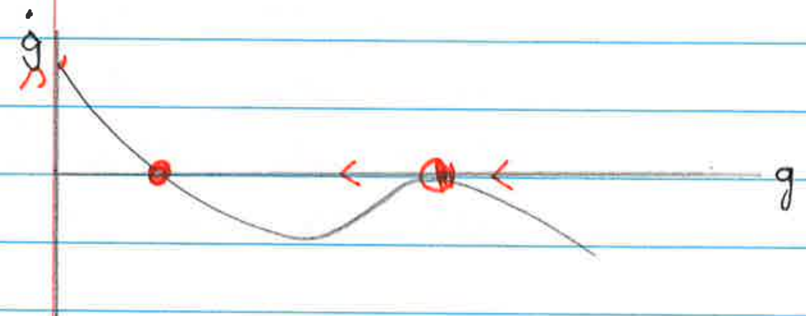
$$\dot{g} = s - rg + \frac{g^2}{1+g^2}$$

NB could also be $\frac{g^3}{1+g^3}$, $\frac{g^4}{1+g^4}$, etc

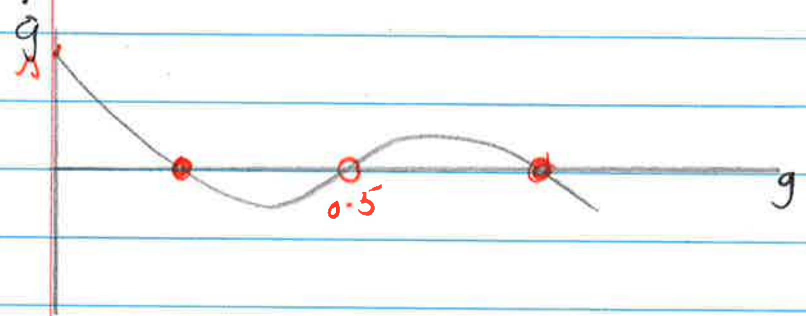


$s = 0.05$
$r = 0.7$

saddlenode bifurcation

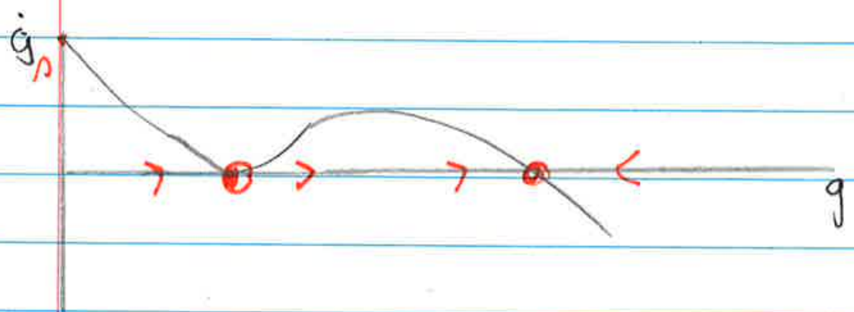


$s = 0.05$
$r = 0.5528$

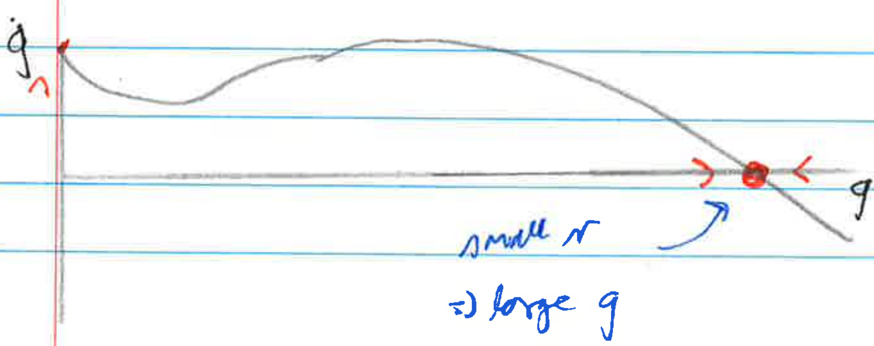


$s = 0.05$
$r = 0.5$

saddlenode bifurcation at a different value of r when it increases compared to decreasing



$s = 0.05$
$r = 0.4435$



$s = 0.05$
$r = 0.3$

Try graph of $\dot{g} = s - rg + \frac{g}{1+g}$

The fixed point just moves to larger g as r decreases; the equation is a quadratic and can only have 2 roots at most.

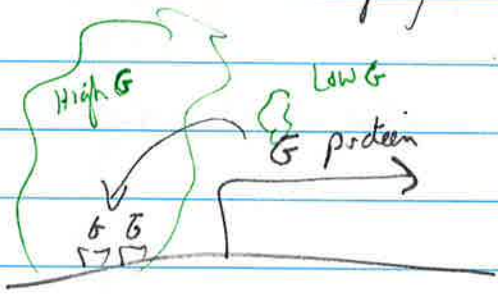
Conclusion

Hysteresis requires at least a cubic relation of $\dot{g}$ on g, so there are 3 fixed points, with an unstable one in the middle of two stable ones

Side blackboard

Recap of Lecture 3

What are we trying to do?



A. continuous change in expression?

or a switch?

$$\frac{dG}{dt} = s_0 - k_1 G + k_2 \frac{G^2}{K^2+G^2}$$

↓ Non-dimensionalise

$$\dot{g} = s - r g + \frac{g^2}{1+g^2}$$

source, degradation, positive feedback

To do:—
1) Find the fixed points graphically
2) Plot Graph 1 for values of s, r
3) How do fixed points depend on s, r?
4) Can we represent this dependence in a diagram?
5) Why is it a bistable switch, and what is hysteresis?
6) Draw trajectories (Graph 2)

Main Board

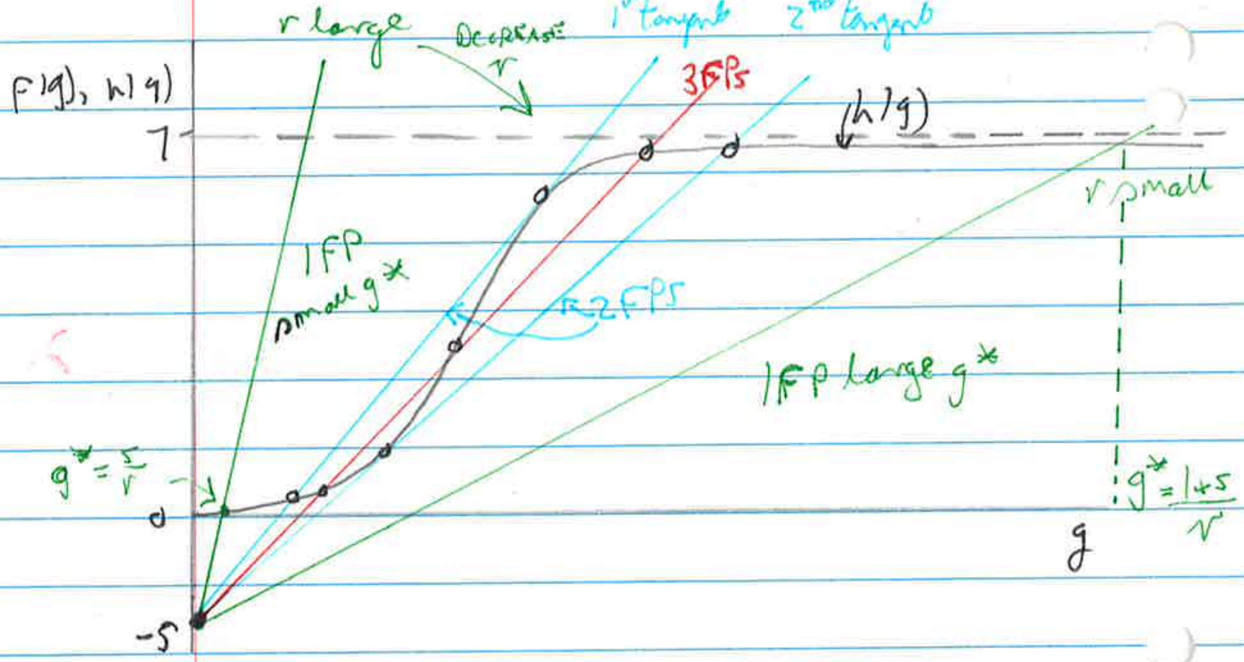
1) Find the fixed points

$$\dot{g} = s - rg + \frac{g^2}{1+g^2} = -f(g) + h(g)$$

where $f(g) = rg - s$

$$h(g) = \frac{g^2}{1+g^2}$$

Fixed points g^* are when $f(g^*) = h(g^*)$.



We can get the fixed points in the limits:

r very small (low degradation) g^* is large $\Rightarrow h(g^*) \sim 1$

Given $\dot{g} = 0$

$$\Rightarrow s - rg^* + 1 = 0$$

$$\therefore g^* = \frac{1+s}{r}$$

High expression because degradation is low

Similarly, you can see that for r large, $g^* \simeq 0$,
so $g^2/(1+g^2) \sim 0$

$$\Rightarrow \quad s - r g^* + O(0) = 0$$

$$\therefore \; g^* = \frac{s}{r}$$ i.e. v. low expression as degradation is fast

Now imagine, reducing r from a high value. At some point, $f(g)$ will become tangent to $h(g)$. When this happens we have

$$f(g) = h(g)$$ Equal positions
and $$f'(g) = h'(g)$$ " slopes

Before we go on, notice that there are 3 FPs between the two tangent conditions. Why?

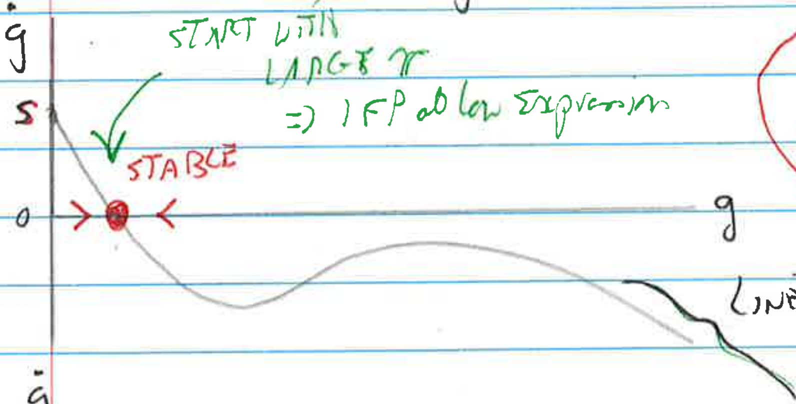
Draw Graph 1 for several values of r.

NOTE

Tangents appear at different values of r when r is decreasing and increasing!
This is the key to bistability and is what defines hysteresis.

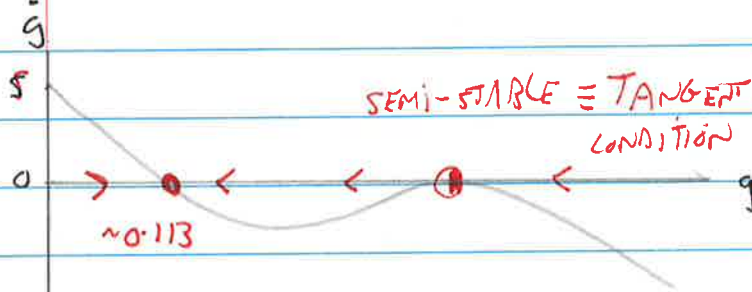
2) $\dot{g} = s - rg + \frac{g^2}{1+g^2}$

(A) 1 F.P. Here



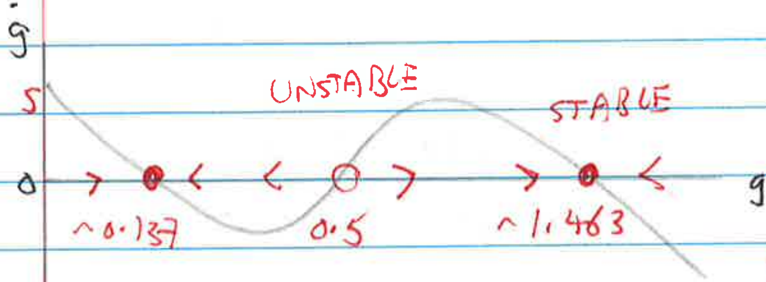
$s = 0.05$ CONST.
$r = 0.7$

(B) 2 F.P.s



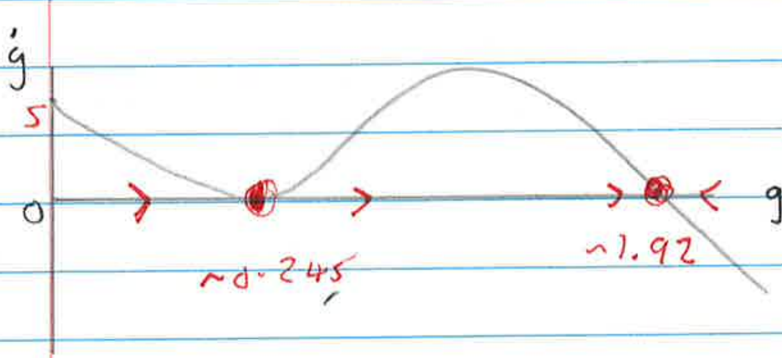
$s = 0.05$
$r \simeq 0.5528$

(C) 3 F.P.s



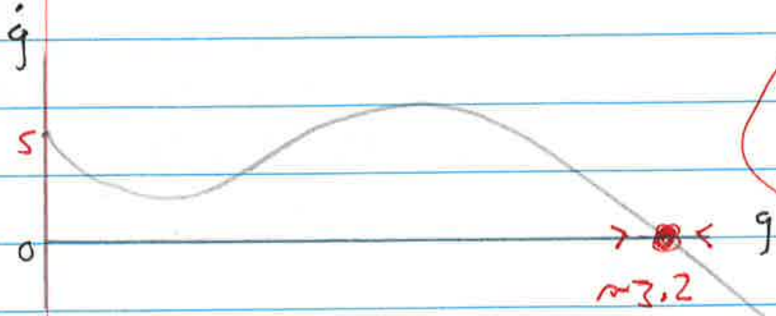
$s = 0.05$
$r = 0.5$

(D) 2 F.P.s



$s = 0.05$
$r = 0.4435$

(E) 1 F.P.



$s = 0.05$
$r = 0.3$

3) How do the FPs depend on s, r ?

For large and small r, there is only one FP. Where do two appear? This is the tangent condition. We must solve the two simultaneous equations:

$$rg - s = \frac{g^2}{1+g^2} \qquad (1)$$

and
$$\frac{d}{dg}\left(rg - s\right) = \frac{d}{dg}\left(\frac{g^2}{1+g^2}\right)$$

$$\Rightarrow r = \frac{(1+g^2).2g - g^2.2g}{(1+g^2)^2} = \frac{2g}{(1+g^2)^2} \qquad (2)$$

If we put r from (2) into (1), we get:

$$s = \frac{g^2(1-g^2)}{(1+g^2)^2}, \qquad r = \frac{2g}{(1+g^2)^2}$$

These define the location of the fixed point(s) g^* implicitly for any s, r combination that has a tangent!

Obviously if $g^* = 0$ we have $r = 0$ and $s = 0$.

and if $g^* = 1$, $r = 1/2$, $s = 0$

And there's a maximum value of s, r for which these equations have a solution.

$$\frac{dr}{dg} = \frac{(1+g^2)^2 . 2 - 2g \cdot 2(1+g^2) . 2g}{(1+g^2)^4} = 0$$

$$\Rightarrow \quad 2(1+g^2)^{\cancel{2}} - \cancel{8}g^2(1+g^2) = 0$$

$$\Rightarrow \quad 1+g^2 - 4g^2 = 0$$

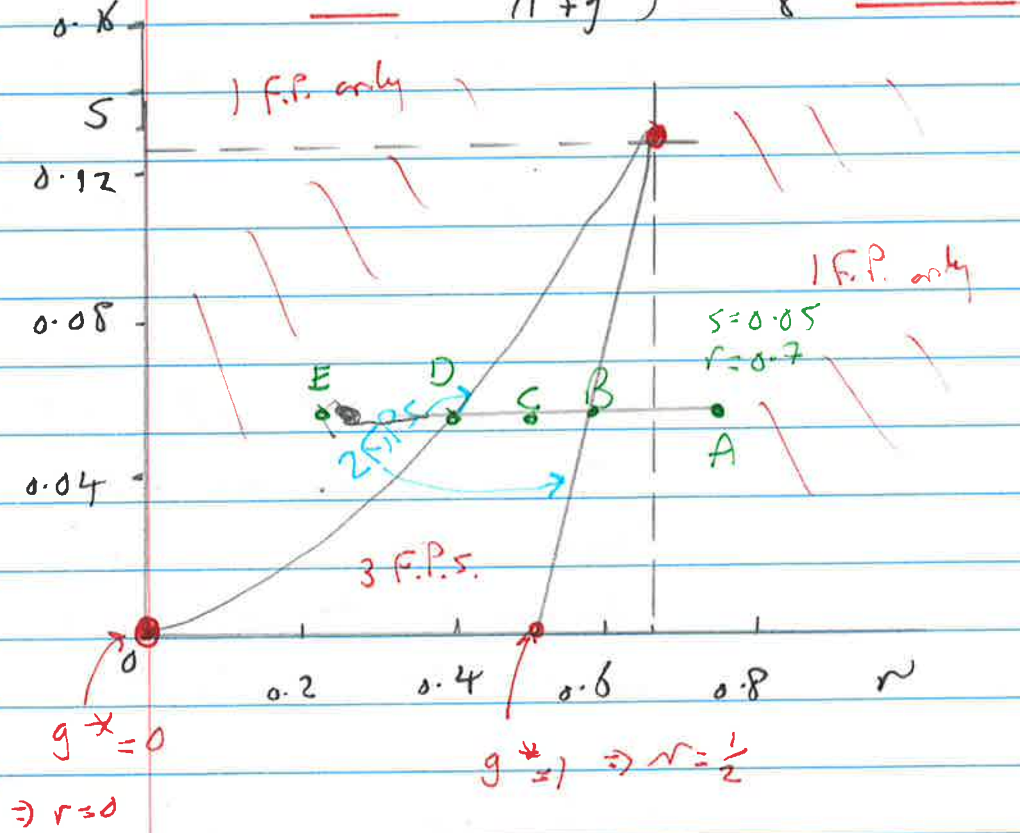
$$\therefore 3g^2 = 1 \quad \therefore g^* = \frac{1}{\sqrt{3}}$$

and so $r_{max} = \dfrac{2g^*}{(1+g^{*2})^2} = \dfrac{9}{8\sqrt{3}} \sim 0.65$

and also $s_{max} = \dfrac{g^{*2}(1-g^{*2})}{(1+g^{*2})^2} = \dfrac{1}{8} = 0.125$

4) Diagram

NB The plot only shows how many FPs there are, not what value of g^* occurs at them — except for $s=0$

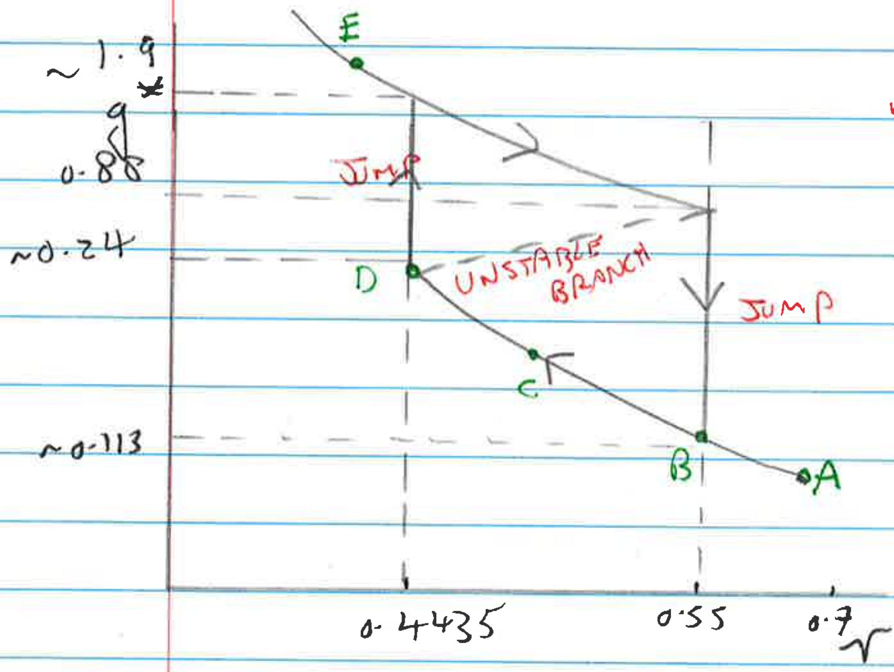


You can show that $s(r) \sim r^2$ for $r \sim 0$

$s(r) \sim r$ for $r \sim 0.5$

5) Why is it a bistable switch?

Consider how the fixed points change as we reduce r from, e.g. 0.7 to 0.3 (points A - B - ... E-A)



NB $S = 0.05$ here

Notice that there 3 FPs in the order:

STABLE - UNSTABLE - STABLE

The unstable one keeps the system at one of the stable ones until it disappears AND the unstable F.P. appears and disappears at different values of r when it is increasing or decreasing.

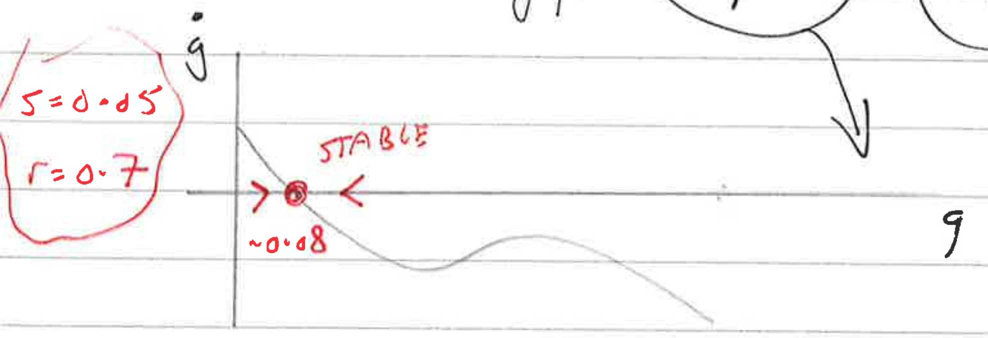
CONC. If we decrease r from a large value, the system stays at the small g^* fixed point (low expression) until $r \sim 0.4435$, when it jumps to the high expression fixed point.

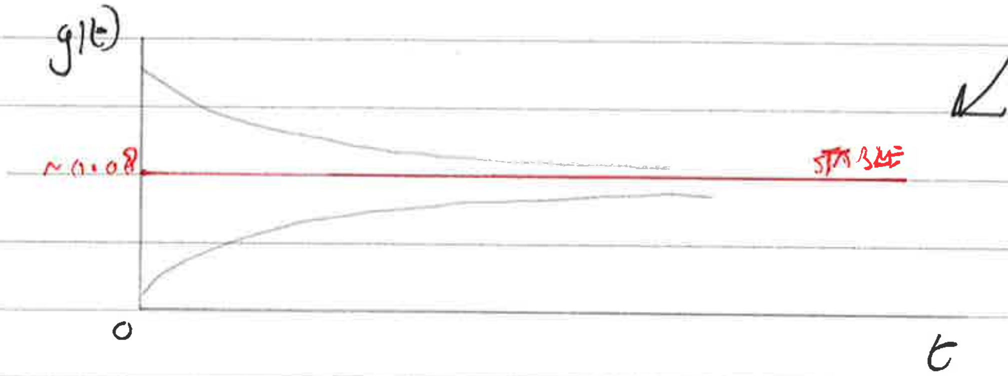
But when raising r, it switches at $r \sim 0.5528$

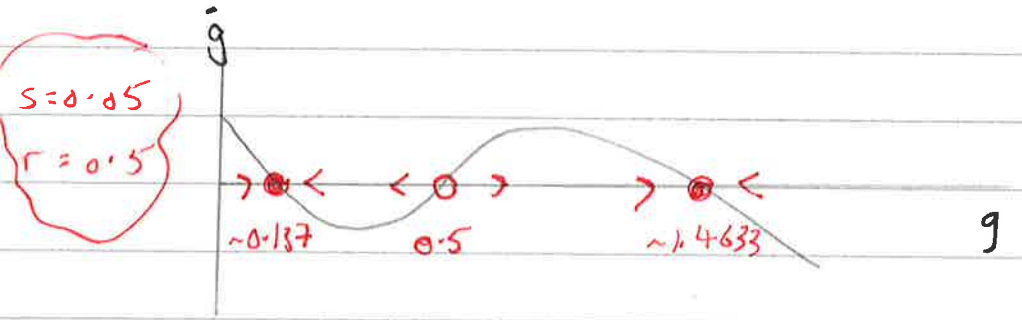
$\therefore$ HYSTERESIS!

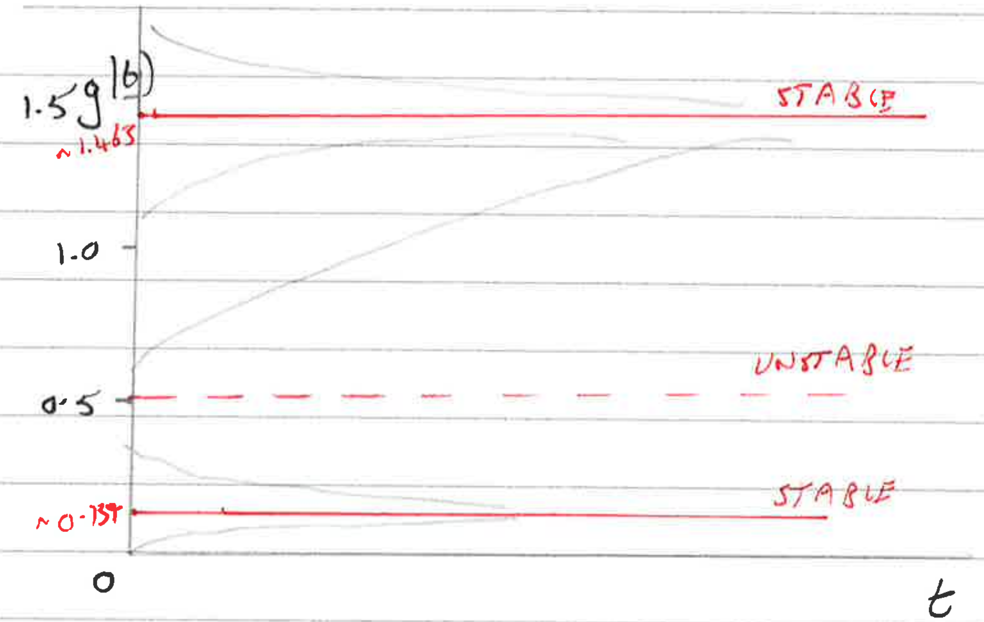
Trajectories for $s = 0.05$, $r = 0.7, 0.5, 0.3$

Going from Graph 7 to Graph 2









$s = 0.05$
$r = 0.3$