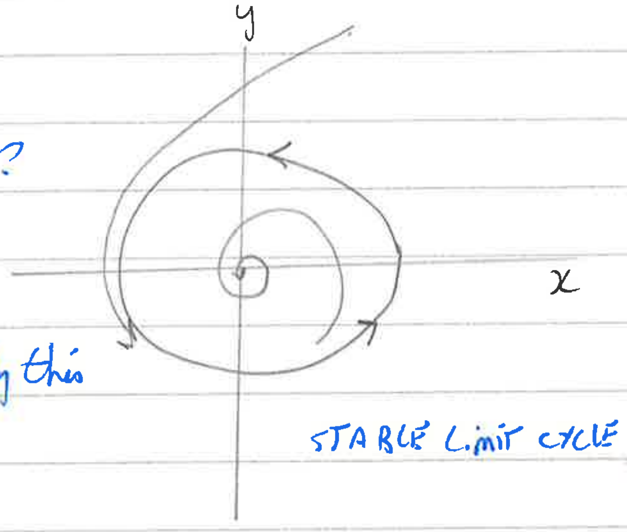
Forced Phase Oscillators

16/1/25

Q How does a stable limit cycle respond to an external stimulus?



- Its shape can change
- Its period can change // Let's study this

consider $\dot{\theta} = \omega + f(\theta, \alpha)$

external stimulus

$f(\theta, \alpha)$ must be periodic in $(\theta - \alpha)$ why? so that $\dot{\theta}$ is single-valued

i.e. $f(\theta, \alpha) = f(\theta + 2n\pi, \alpha)$

e.g. $\dot{\theta} = \omega + K \sin(\alpha - \theta)$ K = coupling constant, α = external phase angle

and assume: $\dot{\alpha} = \Omega$

NOTE

1) If $\theta = \alpha$ $\dot{\theta} = \omega$ and the oscillator is in phase with the external stimulus.

2) If $\theta > \alpha$, $\sin(\alpha - \theta) < 0$, and θ "slows down"

3) If $\theta \ngtr \alpha$, $\sin(\alpha - \theta) > 0$, and θ "speeds up."

Let $\phi = \alpha - \theta$

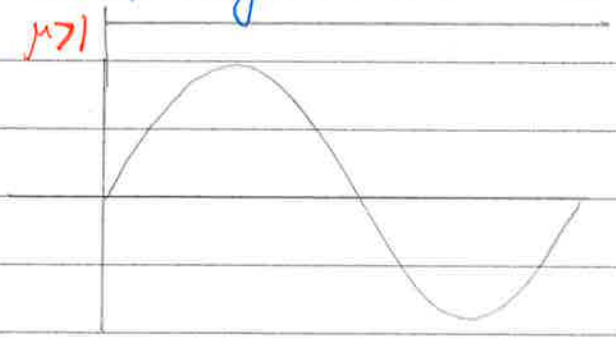
$\Rightarrow \dot{\phi} = \dot{\alpha} - \dot{\theta}$

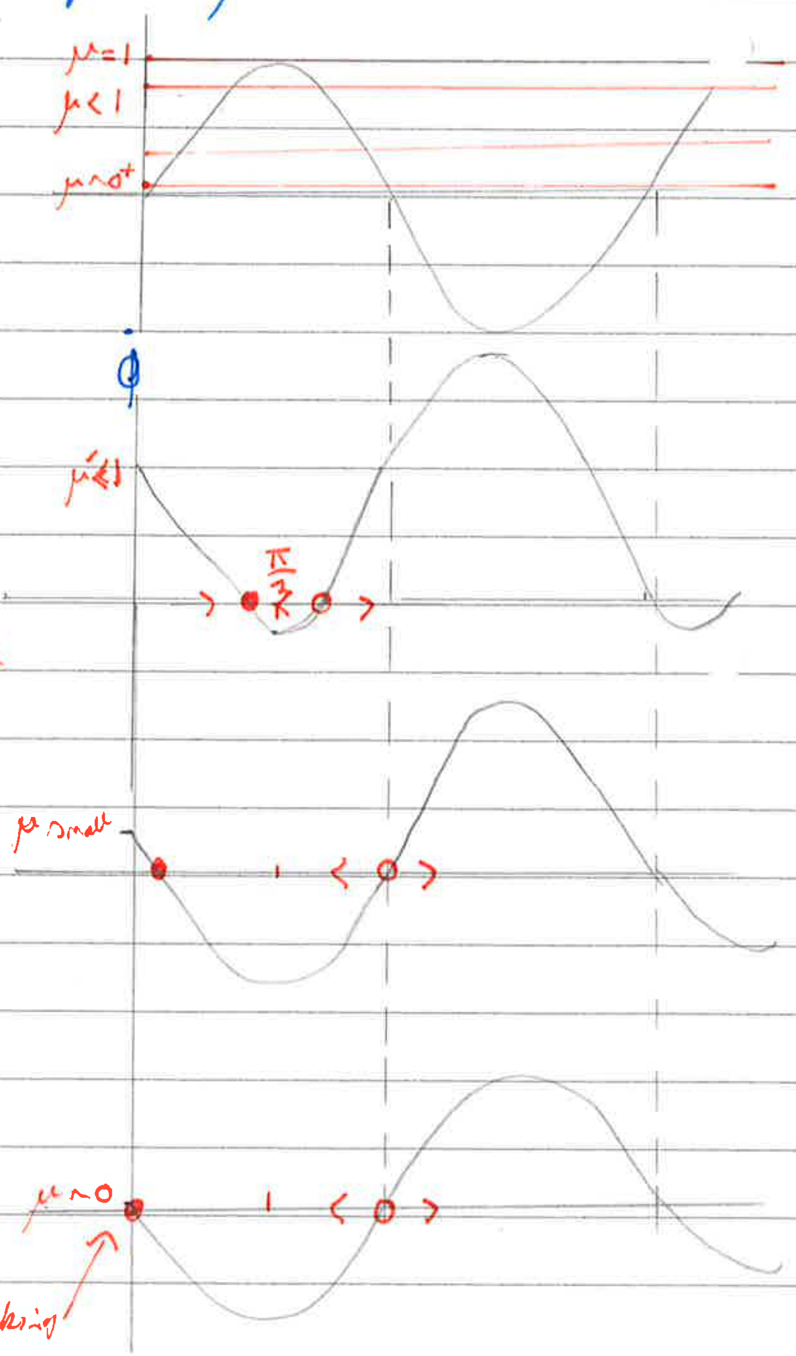
$\Rightarrow \quad \dot{\phi} = \Omega - \omega - K\sin(\phi)$

Let $\tau = Kt$

$\Rightarrow \quad \dfrac{d\phi}{d\tau} = \dfrac{\Omega - \omega}{K} - \sin\phi \equiv \mu - \sin\phi$

How does ϕ vary with time? It depends on μ.





no fixed points $\Rightarrow$ no entrainment

ϕ just drifts longer and longer

(but periodic!)

The stable fixed point moves from $\pi/2$ (exactly at $\mu = 1$) towards ϕ as μ decreases towards ϕ.

The FP occurs at : $\dot{\phi} = 0$

$\Rightarrow \quad \sin\phi = \mu$

$\therefore \alpha - \theta = \sin^{-1}\mu$

$\therefore \theta = \alpha - \sin^{-1}\mu$

Total locking

$\theta = \alpha$

what is the phase shift precisely at the bifurcation point?

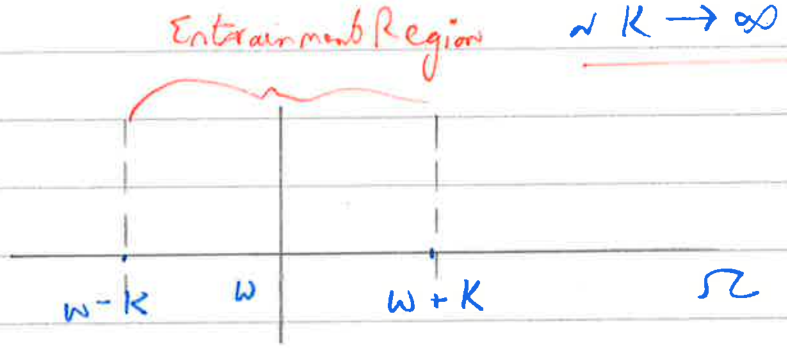
The bifurcation occurs at $\mu = 1$ as then $\dot{\varphi} = 1 - \sin\varphi^* = 0$

$$\Rightarrow \varphi^* = \pi/2$$

and the critical value of K is: $K_c = \Omega - \omega$ $\quad$ $\left(\text{when } \mu = \frac{\Omega - \omega}{K} \to 1 \right)$

There are two ways of inducing entrainment: Ω is close to ω

or $K \to \infty$

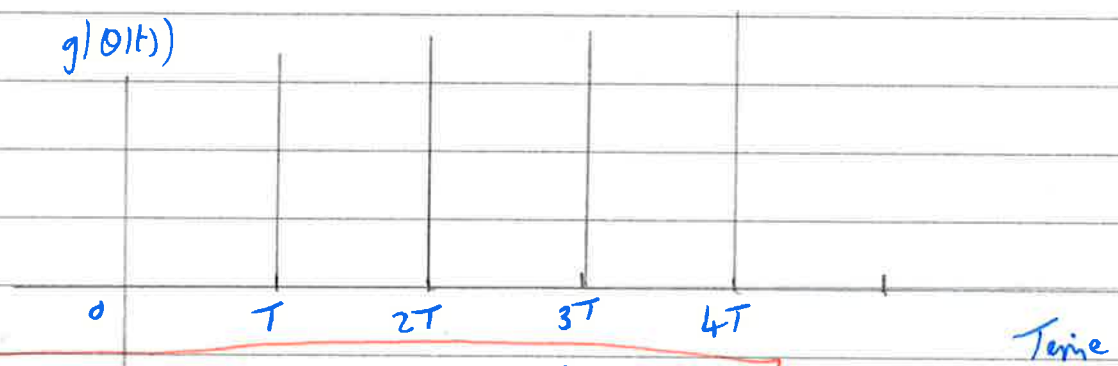


and we only have entrainment for $|\Omega - \omega| < K$

Pulsed coupling

$$\dot\theta = \omega + \sum_n \delta(t-nT)\, g(\theta(t))$$

$T =$ period of external oscillator



$$\boxed{\text{i.e., } \dot\theta = \omega + \delta(t-T)\, g(\theta(t)) + \delta(t-2T)\, g(\theta(t)) + \dots}$$

only one δ-function acts at any one time pulse

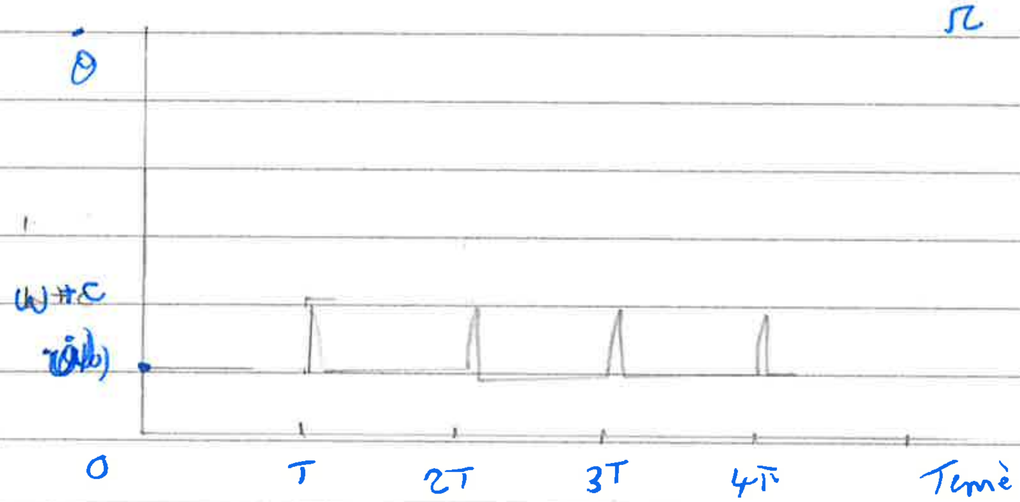
$\delta(x-x_0) = 0$ if $x \neq x_0$

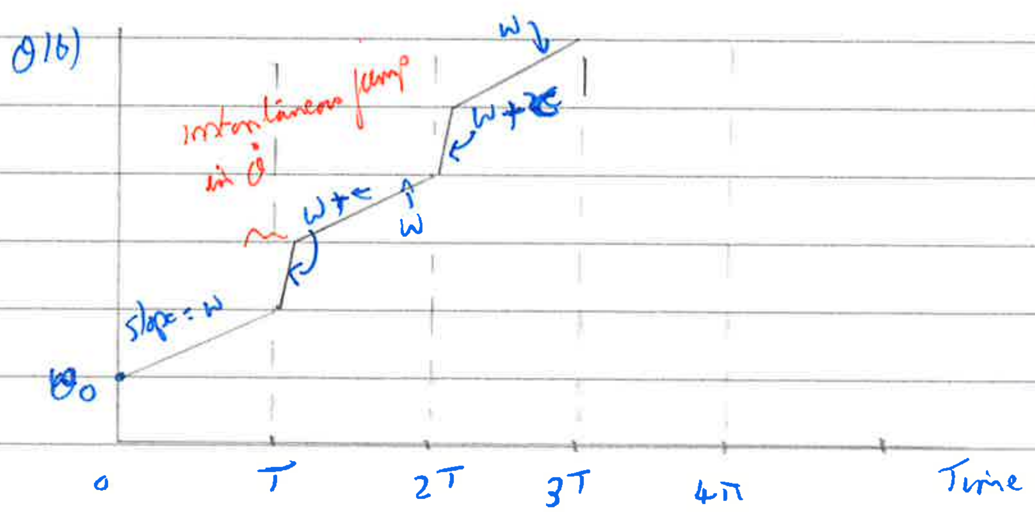
$= \infty$ if $x = x_0$

$\int_\Omega \delta(x-x_0)\,dx = 1$ if Ω includes x_0

Suppose $g(\theta) = c =$ constant.

Each time t passes through nT, $\dot\theta$ is incremented by c instantaneously

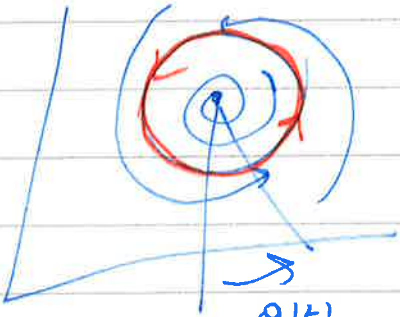




NB $\theta(t) = \omega t$ when $t \neq nT$

LIMIT CYCLE → OSCILLATOR
(isolated, closed trajectory ⇒ periodic)

$$\theta(t+T) = \theta(t) \quad T = \text{period}$$
$$W = \frac{2\pi}{T}$$



motion on the LC: $\dot{\theta} = W$

What could happen if we perturb an LC? – it moves (Boring)
changes shape
change period

perturbed LC: $\dot{\theta} = W + F(\theta)$

must be periodic – why?

IF coupled to an external oscillator Ω

$$\dot{\theta} = W + K\sin(\alpha - \theta)$$
$$\dot{\alpha} = \Omega$$

CONTINUOUS COUPLING must be periodic

$\alpha(t) = \Omega t$

NB Know Trig identities!
$\sin(A \pm B) =$
$\cos(A \pm B) =$

Leads to entrainment if $K \uparrow$ or $\Omega \approx W$

A fixed point now becomes a fixed phase difference $\phi^* = \cancel{\alpha} - \theta$ = CONSTANT
When does $\phi(t) \to \phi^* = \text{CONSTANT}$

$$\dot{\phi} = \dot{\alpha} - \dot{\theta} = \cdots \quad \mu - \sin\phi \qquad \mu = \frac{\Omega - W}{K}$$

If $\mu > 1 \Rightarrow$ No entrainment
$\mu < 1 \Rightarrow$ Entrainment

$\mu_c = 1$ = critical coupling parameter at which entrainment first appears as μ increases.

DISCRETE COUPLING $\dot{\theta} = w + \sum_{n=1}^{\infty} \delta(t - nT)\, g(\theta(t))$

$$\theta_{n+1} - \theta_n = wT + g(\theta(T_n))$$

map or a discrete dynamical system

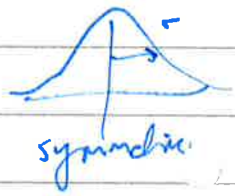
Kuramoto model

couple N oscillator to each other instead of a single external source

$$\dot{\theta}_i = w_i + F(\theta_i, \theta_j)$$

$$= w_i + \sum_{j=1}^{N} \frac{K}{N} \sin(\theta_j - \theta_i)$$

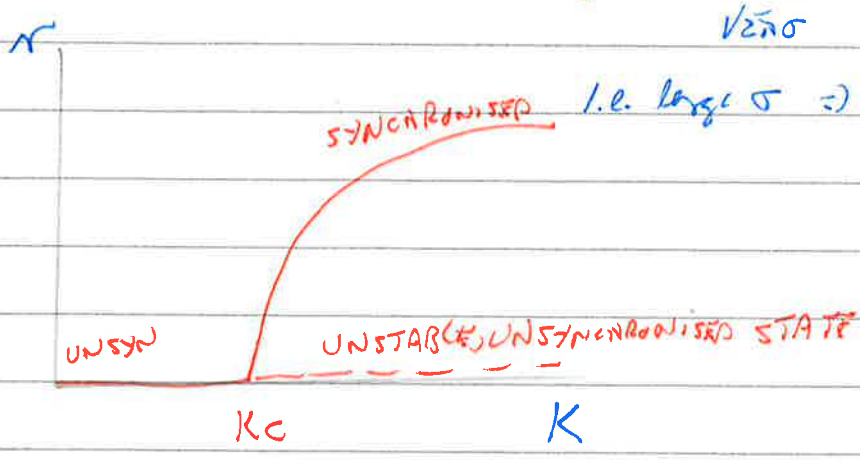
With solution: $r = rK \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, g(Kr\sin\theta)\, d\theta$ where $g(w) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(w-w_0)^2}{2\sigma^2}}$



and $K_c = \frac{2}{\pi g(0)}$, which depends on dispersion of the frequency

$$g(0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-w_0^2/2\sigma^2}$$

i.e. large σ ⇒ small $g(0)$ ⇒ large K_c



$r = 0$ ⇒ No synchronisation

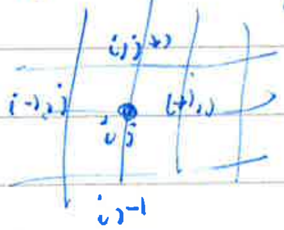
$r < 1$ ⇒ partial "

$r = 1$ ⇒ complete "

What other kinds of coupling could there be?

- all to all $\sum_{j \neq i} F(\theta_i, \theta_j)$
- some to some $\sum_{j \in \{i \pm 1\}} F(\theta_i, \theta_j)$ e.g. Nearest Neighbour
- distance dependent

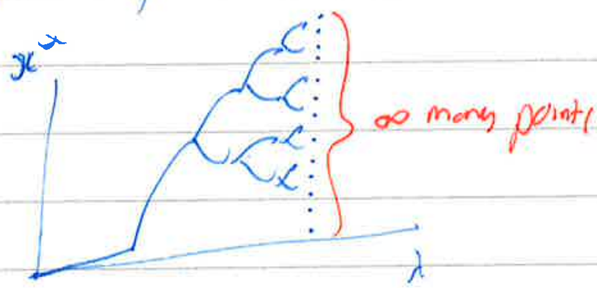
$$f_{ij} \sim \frac{K}{|\underline{x}_i - \underline{x}_j|} \sin(\theta_j - \theta_i)$$



3) CHAOS → 1D DISCRETE (poincare map)

$$x_{n+1} = F(x_n, \lambda) = \lambda x_n (1 - x_n)$$

Stability: $|F'(x^*, \lambda)| < 1$



Many fixed points → ∞ many as λ increases

How are they distributed?

Cantor set

			$j=$	$N(j)$	$\varepsilon(j)$
0	$\frac{1}{3}$	$\frac{2}{3}$ 1	0	1	1
			1	2	$1/3$
0 $\frac{1}{9}$			2	4	$1/9$
	ε		3	8	$1/3^3$

$$D = \lim_{j \to \infty} \frac{\ln N(\varepsilon_j)}{\ln(1/\varepsilon_j)} = \lim_{j \to \infty} \frac{\ln 2^j}{\ln(1/(1/3)^j)} = \frac{j \ln 2}{j \ln 3} = \frac{\ln 2}{\ln 3} < 1$$

How much is removed? = sum of the gaps

$$\lim_{N\to\infty} \sum_{j=1}^{N} 2^{j-1} \left(\tfrac{1}{3}\right)^{j} = \frac{1}{2} \sum_{j=1}^{\infty} \left(\tfrac{2}{3}\right)^{j} = \frac{1}{2}\left[\sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^{j} - 1 \right]$$

But $\displaystyle\sum_{j=0}^{\infty} x^{j} = \frac{1}{1-x}$ for $|x| < 1$

$\therefore$ Gaps $= \dfrac{1}{2}\left[\dfrac{1}{1-\frac{2}{3}} - 1 \right] = 1$!

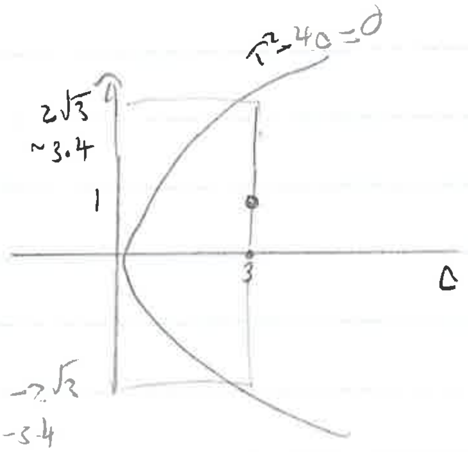
We have infinitely many points with no size in the Cantor set

2D Non-Linear Systems

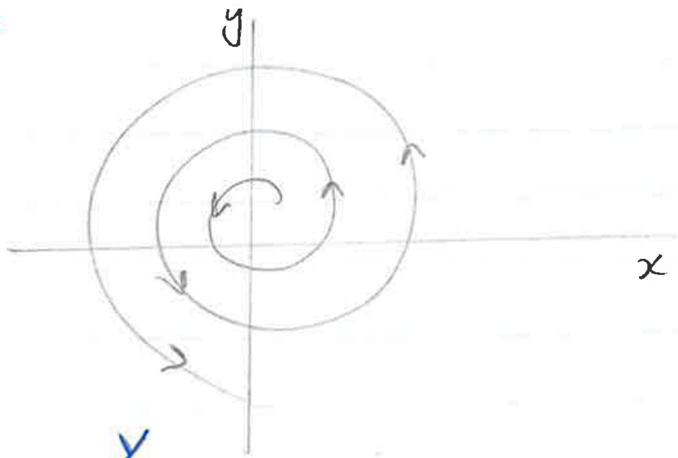
12/10/23 | Keep graph of T, Δ on the screen.

$$\dot{x} = x - y$$
$$\dot{y} = 3x$$

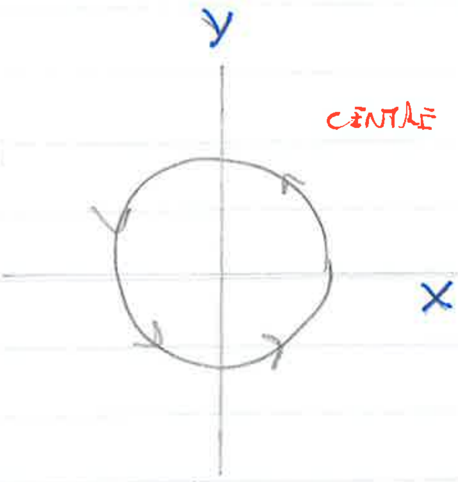
What is this? $\underline{M} = \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix} \Rightarrow \operatorname{Tr} \underline{M} = 1$, $\Delta = 3$



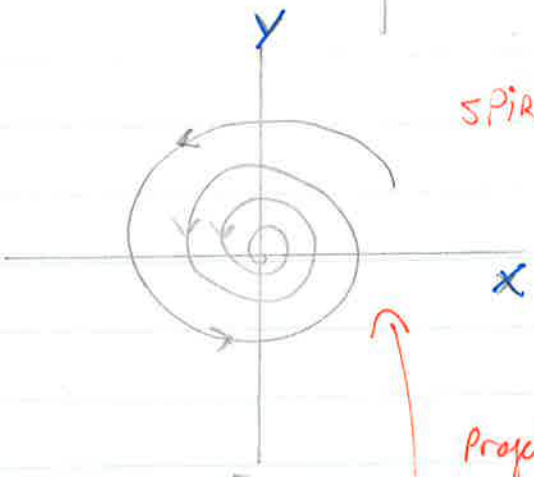
$\therefore$ It is an unstable spiral.



Look at $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 10 & 0 \end{pmatrix}$

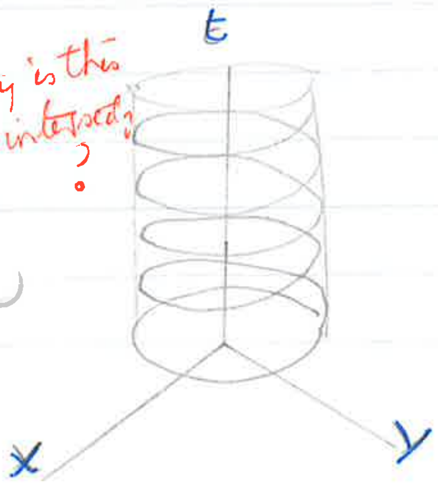


CENTRE

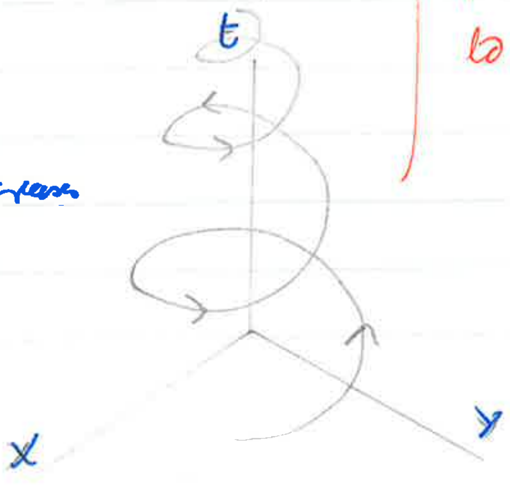


SPIRAL

why is this not interesting?



Time increases



Project onto xy plane to get phase portrait