Dynamical Systems Review

How do complex systems change over time?

1D: $\dot{x} = f(x, r)$

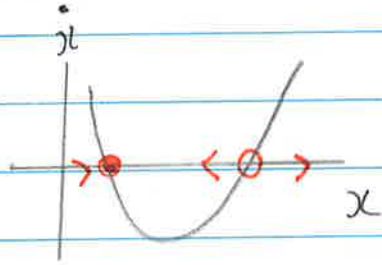
AUTONOMOUS ≡ No explicit time dependence
(Why should a chemical reaction / rabbits be different on Monday or Wednesday?)

↓

Find Fixed Points: why? These are where a system ends up over time

↓

Stability of F.P.s
GRAPH 1



Tells you which F.P. governs long-time behaviour

$f'(x^*) < 0$ STABLE
$f'(x^*) > 0$ UN "

↓

Trajectories are monotonic

↓

Can fixed points change?
BIFURCATIONS

BIFURCATIONS IN 1D

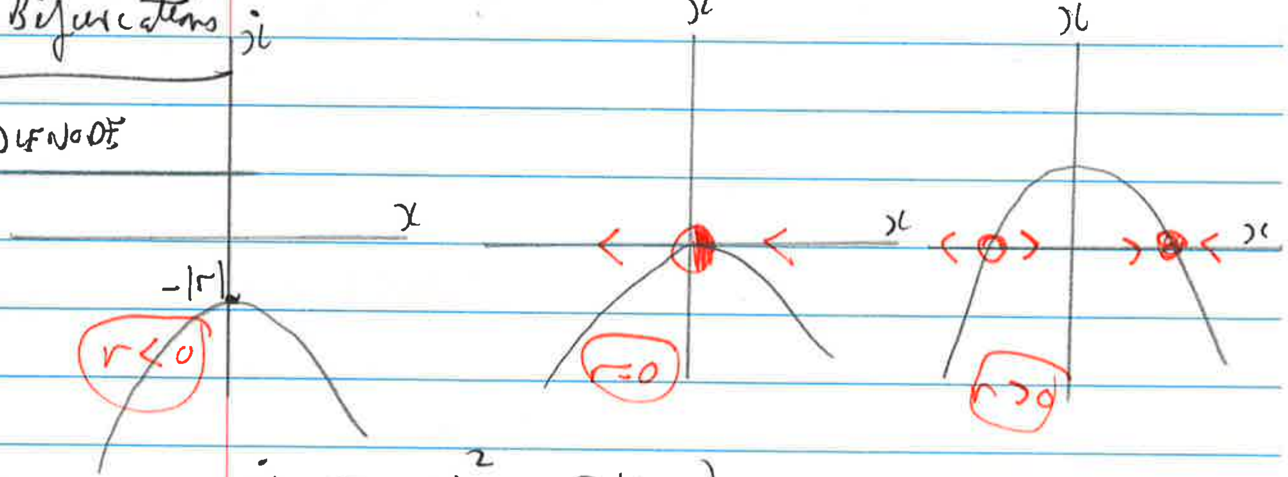
Def^n: A qualitative (topological) change in the dynamics of a system as a parameter is changed

What can fixed points do? Appear/Disappear change stability

Can they do it any way? NO

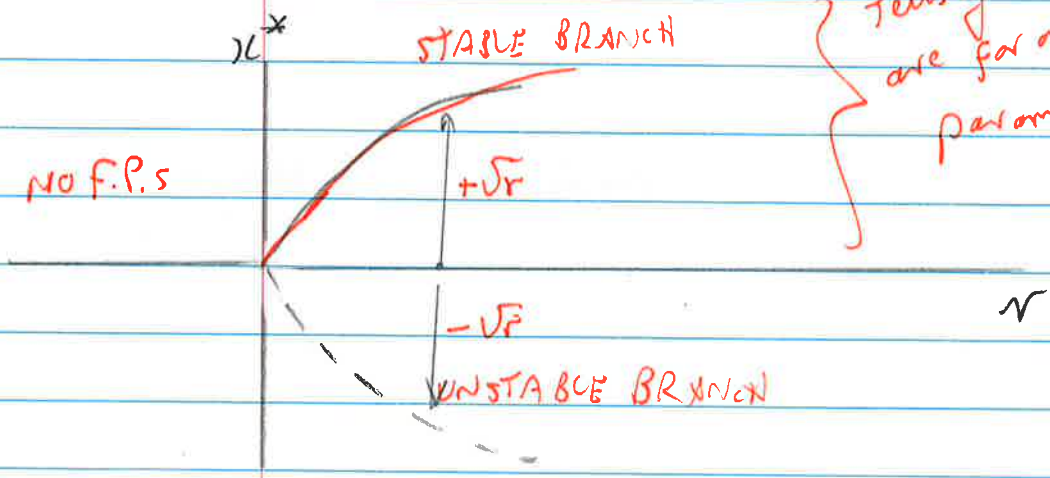
1D Bifurcations

A) SADDLENODE



$$\dot{x} = r - x^2 \equiv f(x, r)$$

$$\Rightarrow \dot{x} = 0 \Rightarrow x^* = \pm\sqrt{r} \text{ if } r > 0$$



Tells you where F.Ps are for any value of parameter r

Conclusion

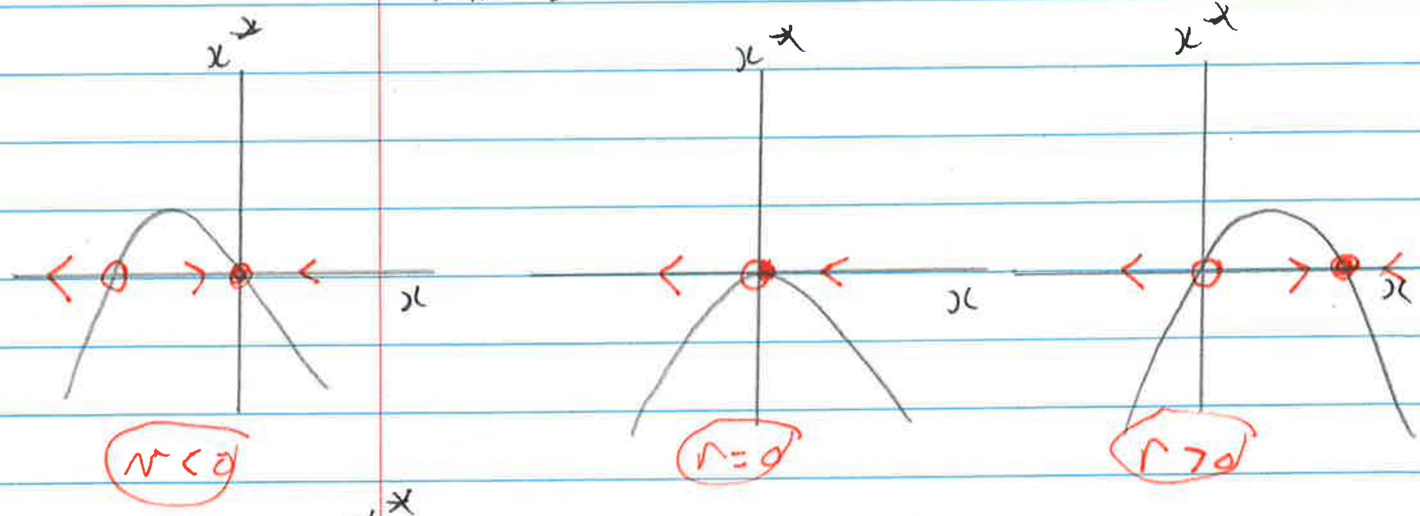
The Bifurcation diagram tells you where a system will end up as time goes to ∞.

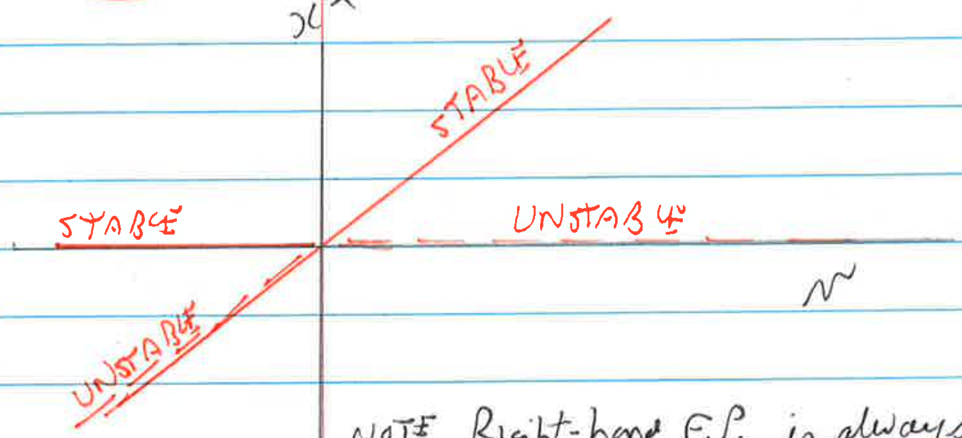
It is a summary of all possible Graph 1's for the values of the parameter.

Adjacent branches must have opposite stability because adjacent Fixed points in Graph 1 must have opposite stability.

B) TRANSCRITICAL : $\dot{x} = rx - x^2$

F.P.s $x^* = 0$ or r





NOTE. Right-hand F.P. is always stable: origin F.P. changes stability.

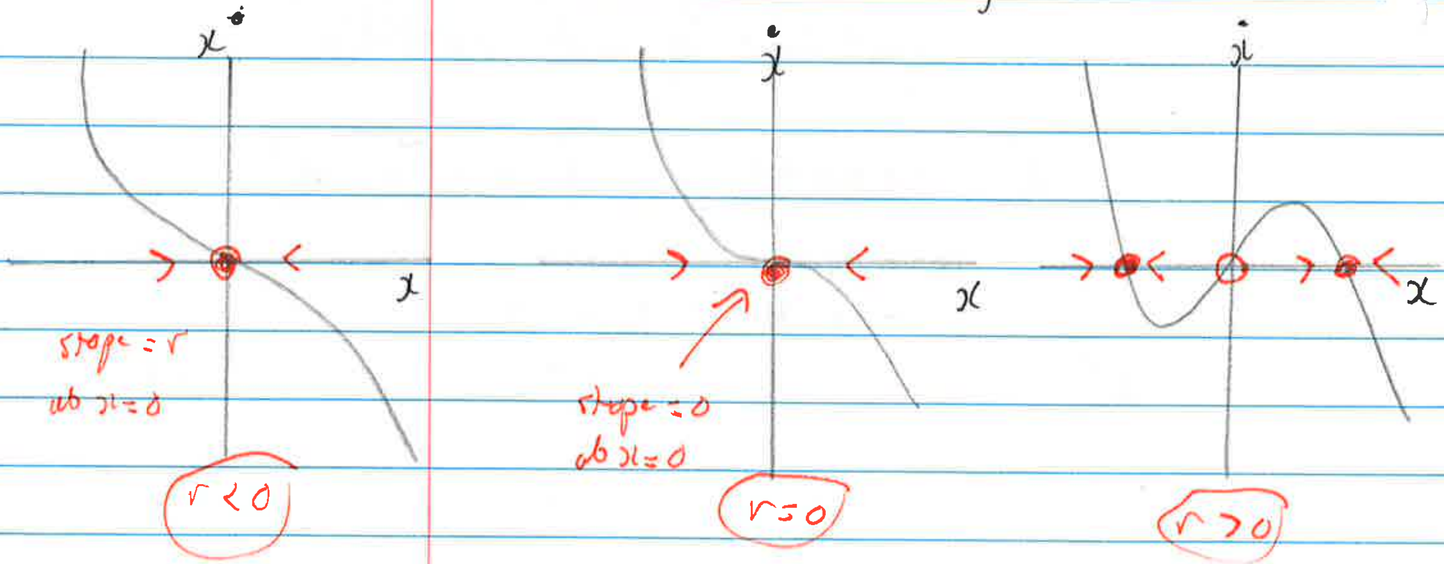
c) Pitchfork Bifurcation

- in systems with a symmetry, e.g. $x \to -x$

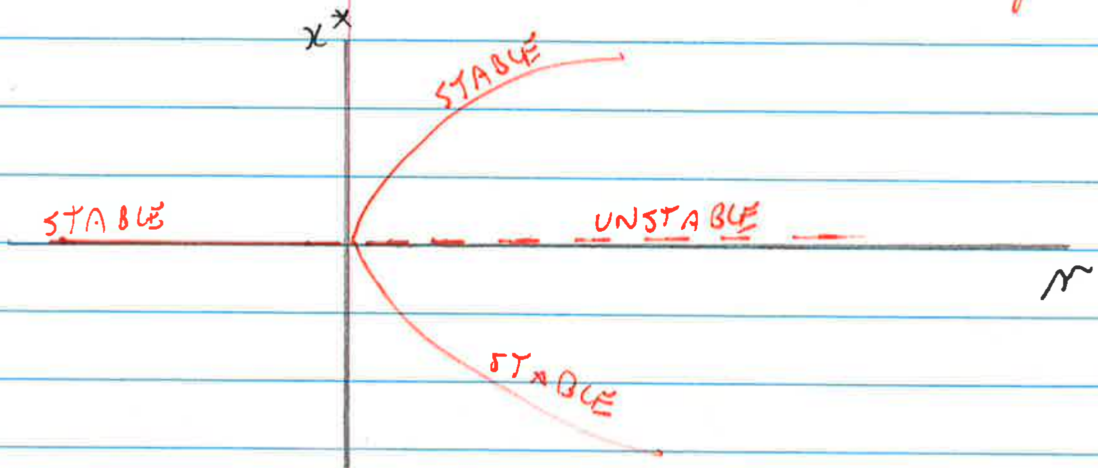
supercritical: $\dot{x} = rx - x^3$

$\dot{x} = 0 \Rightarrow x(r - x^2) = 0$

$\therefore x^* = 0$ or $\pm\sqrt{r}$ if $r > 0$



Note that 2 new F.P.s emerge from $x^* = 0$ when $r > 0$, and move apart: A saddlenode bifurcation. AND $x^* = 0$ changes stability



Example 1) Suppose $\dot{x} = x - rx^3$?

$\dot{x} = 0 \Rightarrow x(1 - rx^2) = 0$

$$\therefore x^* = 0 \text{ or } x^* = \frac{\pm 1}{\sqrt{r}}$$

$\therefore$ Same as usual supercritical pitchfork.

Ex 2) $\dot{x} = rx + x^2 - x^3$

$\dot{x} = 0 \Rightarrow x(r + x - x^2) = 0$

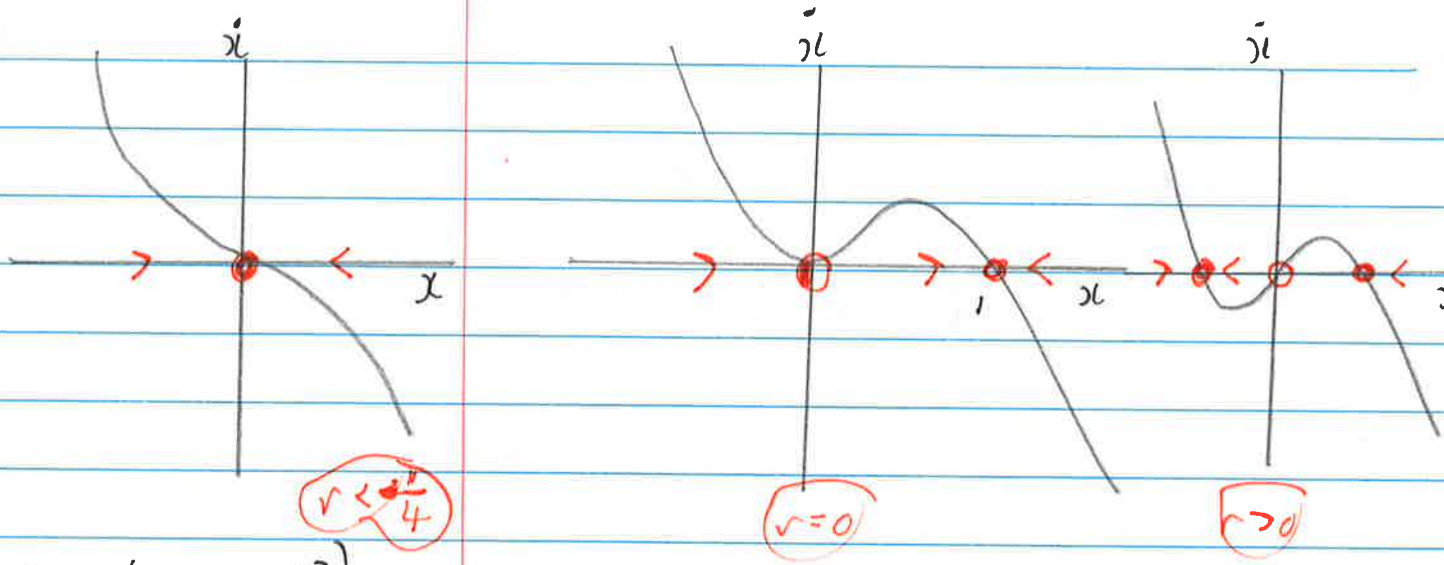
$\therefore x^* = 0$ or $x^2 - x - r = 0$

$$\therefore x^* = \frac{1 \pm \sqrt{1 + 4r}}{2}$$

so if $r > -1/4$ this has two real roots.

Let's take $r = 2 \Rightarrow x^* = 0, \frac{1 \pm 3}{2} = 2, -1$

$\therefore -1, 0, 2$



$\dot{x} = x(r + x - x^2)$

Let's do $r=0$ first as it's easy:

$\dot{x} = x(x - x^2) = x^2(1 - x)$

$\therefore x^* = 0, 1$ and we have a $-x^3$ term, so $x \ll 0 \Rightarrow \dot{x} < 0$ and $x \gg 0$ has $\dot{x} \sim -x^3 < 0$

$r < 0$ Two cases:

$-1/4 < r < 0$ x^* still has two roots

$r < -1/4$ no real roots from quadratic

$\lim_{x \to -\infty} \dot{x} = -x^3 > 0$

$r > 0$ 3 real roots, one of which is always 0.

Stability: $f'(x) = r + 2x - 3x^2$

$\therefore f'(0) = r$

$f'(1) = r - 1$ assumed

How to draw the bifurcation diagram?

Try $r = 2$ then $x^* = -1, 0, 2$

$r = 0$ then $x^* = x(x - x^2) = x^2(1-x)$
$= 0, 1$

$r = -1/4$ then $x^* = x(-\frac{1}{4} + x - x^2)$

$\therefore x^* = 0, 1/2$

What is their stability?

$f'(0, -\frac{1}{4}) = -\frac{1}{4}$ $\therefore$ Stable

$f'(\frac{1}{2}, -\frac{1}{4}) = -\frac{1}{4} + 2.\frac{1}{2} - 3.\frac{1}{4} = 0$

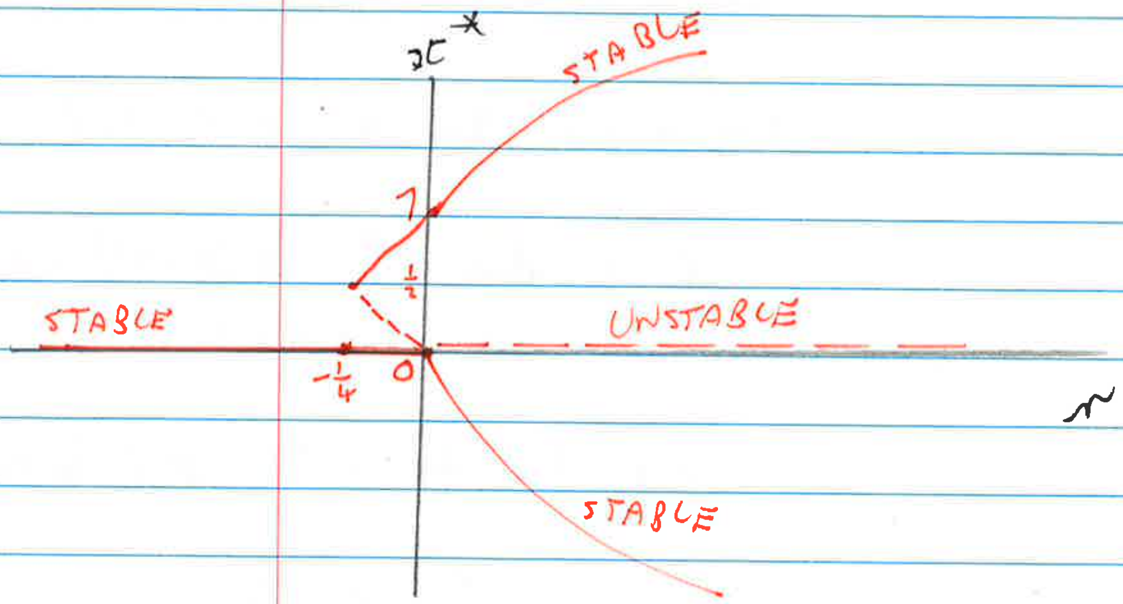
Let $r \sim 0^-$ i.e. $-\varepsilon$

$$\Rightarrow x^* = \frac{1 \pm (1-4\varepsilon)^{\frac{1}{2}}}{2} = \frac{1 \pm (1 - 2\varepsilon + \cdots)}{2}$$

$= 1 - \varepsilon, \quad \varepsilon$

$$f'(1-\varepsilon, -\varepsilon) = -\varepsilon + 2(1-\varepsilon) - 3(1-\varepsilon)^2$$
$$= 2 - 3\varepsilon - 3(1 - 2\varepsilon + \varepsilon^2)$$
$$= -1 + 3\varepsilon + O(\varepsilon^2)$$
< 0 $\therefore$ Stable

$$f'(\varepsilon, -\varepsilon) = -\varepsilon + 2\varepsilon - 3\varepsilon^2 = \varepsilon - 3\varepsilon^2$$
$$= \varepsilon(1 - 3\varepsilon) > 0 \quad \therefore \text{ unstable}$$



x^*
STABLE
1
$\frac{1}{2}$
STABLE
UNSTABLE
$-\frac{1}{4}$
0
r
STABLE

Graded Exercise 2: Drawing Trajectories

$$\dot{x} = x(2 - x - y)$$

$$\dot{y} = y(4x - x^2 - 2)$$

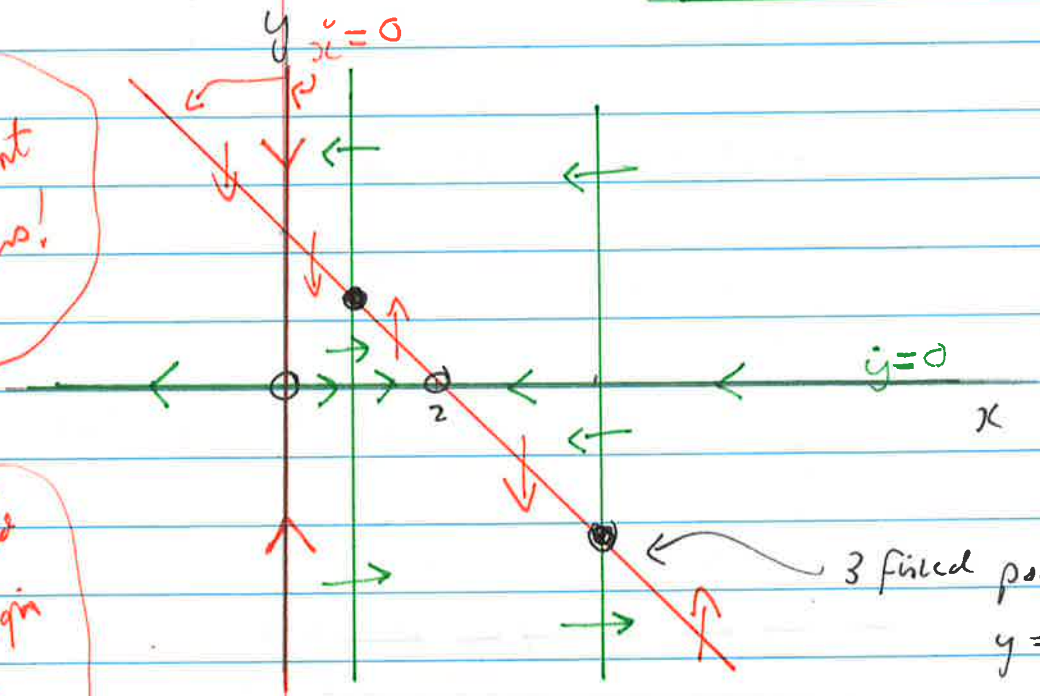
Nullclines:

$\dot{x} = 0 \Rightarrow x = 0$ or $y = 2 - x$

$\dot{y} = 0 \Rightarrow y = 0$ or $4x - x^2 - 2 = 0$

$\Rightarrow x^* = 2 \pm \sqrt{2} \sim 3.414, 0.586$

Use different colours!



3 fixed points lie on this line $y = 2 - x$

NB Vector field changes sign across a fixed point

When $a \to 4$ the 3 FPs merge along $y = 2 - x$ in a saddle-node bifurcation

Direction of the vector field on null clines

We know $\dot{x} = x(2-x-y)$

$$\dot{y} = y(4x - x^2 - 2)$$

we need to find $\dot{y}$ along the curves $\dot{x} = 0$
ie along $x = 0$ and $y = 2 - x$.

substituting $x = 0$ into $\dot{y}$ gives:

$\dot{y} = -2y \quad \therefore \dot{y} < 0$ for $y > 0$ and vice versa

and substituting $y = 2 - x$ into $\dot{y}$ gives:

$\dot{y} = (2-x)(4x - x^2 - 2)$

But we know the roots are $0.586, 3.414$

$\Rightarrow \dot{y} = (x-2)(x^2 - 4x + 2)$

$= (x-2)(x-0.586)(x-3.414)$

$\therefore \dot{y} > 0$ for $x > 3.414$
$\dot{y} < 0$ for $2 < x < 3.414$
$\dot{y} > 0$ for $0.586 < x < 2$
$\dot{y} < 0$ for $x < 0.586$

Next we need $\dot{x}$ along $\dot{y}=0$

Substituting $y=0$ into $\dot{x}$ gives:

$\dot{x} = x(2-x)$ <0 for $2<x$
>0 for $0<x<2$
<0 for $x<0$

and also substituting $x = 2 \pm \sqrt{2}$

$\dot{x} = (2 \pm \sqrt{2})(2-(2 \pm \sqrt{2}) - y)$

$= (2 \pm \sqrt{2})(\mp \sqrt{2} - y)$

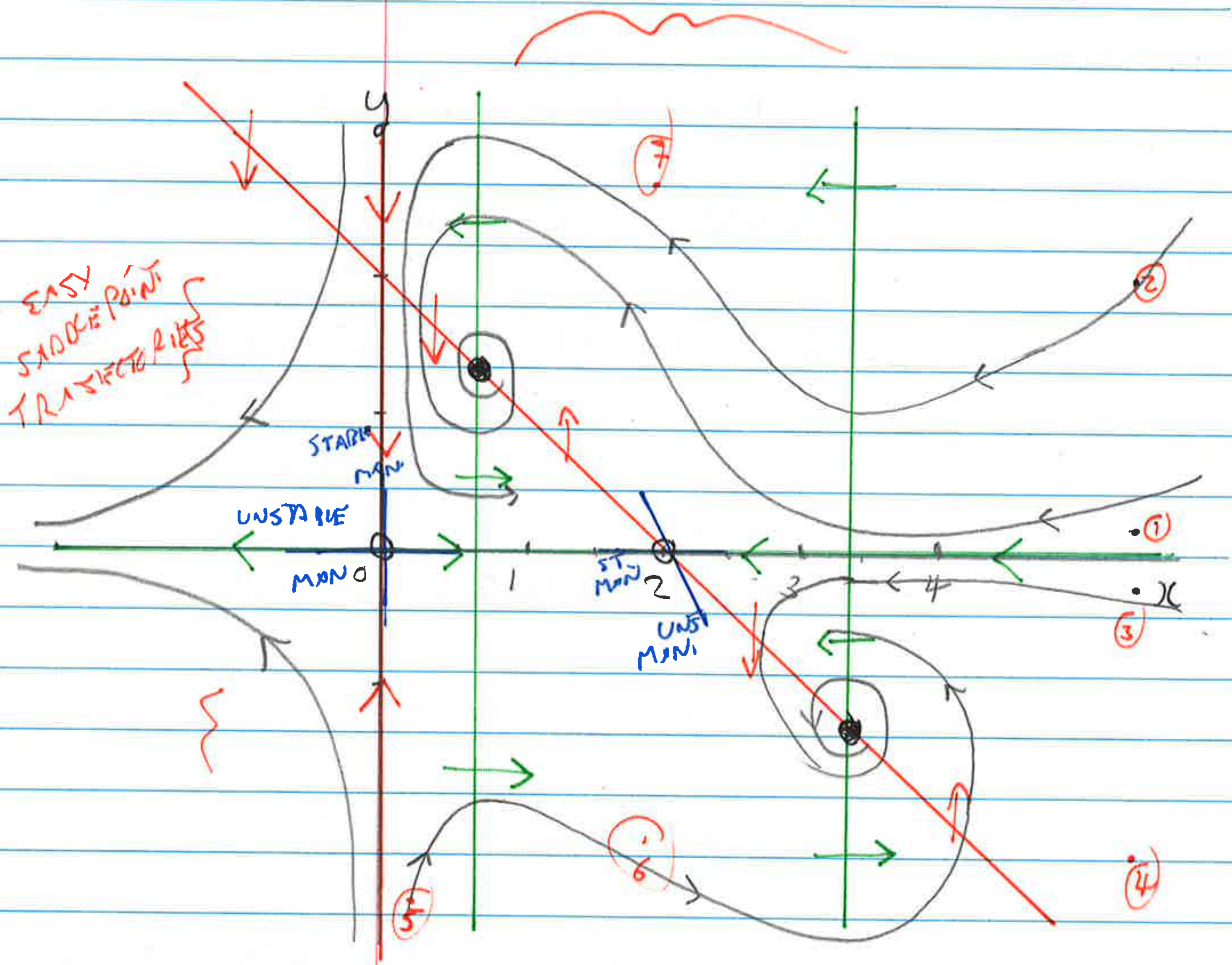
$= -(2+\sqrt{2})(\sqrt{2}+y)$ <0 for $y > -\sqrt{2}$
>0 for $y < -\sqrt{2}$

or $\underbrace{(2-\sqrt{2})}_{>0}(\sqrt{2}-y)$
which is >0 for $y<\sqrt{2}$
<0 for $y>\sqrt{2}$

Fixed points

$(0,0)$, $(2,0)$, $(2-\sqrt{2}, \sqrt{2})$, $(2+\sqrt{2}, -\sqrt{2})$

Note slope must be ϕ on here!



How does the vector field flow?

(Need saddlepoint eigenvectors ...)

① Large x, small $y > 0$

$$\dot{x} = x(2 - x - y)$$
$$\dot{y} = y(4x - x^2 - 2)$$

$$\Rightarrow \quad \left.\begin{aligned} \dot{x} &\longrightarrow -x^2 \\ \dot{y} &\longrightarrow -x^2 y \end{aligned}\right\} \quad \frac{\dot{y}}{\dot{x}} \equiv \frac{dy}{dx} = y \sim 0 \text{ and +ve}$$

i.e. slopes up to right

② Try large x and large y

$\Rightarrow \dot{x} \to -x(x+y) < 0$ but large now

$\dot{y} \to -x^2 y < 0$ also large

and $\dfrac{\dot{y}}{\dot{x}} = \dfrac{xy}{x+y}$ ie also large

③ Large x, small $y < 0$

As for ① $\dot{x} \to -x^2 < 0$

but now $\dot{y} \to -|y| \cdot -x^2 = +|y| x^2 > 0$

$\therefore \dfrac{\dot{y}}{\dot{x}} = -|y| \sim 0^-$ i.e. slopes ~~up~~ down to the right

④ Large x, large $y < 0$ (similar to ② with $y < 0$)

$\dot{x} \to -x(x - |y|)$

$\dot{y} \to x^2 |y| > 0$

not so easy, try somewhere else.

5 small $x>0$, large $y<0$

$\dot{x} \to x(2+|y|) \sim x|y| > 0$

$\dot{y} \to -|y| \cdot -2 = 2|y| > 0$

$\therefore \frac{\dot{y}}{\dot{x}} = \frac{2}{x}$ is large and > 0.

6 Substitute $x=2$, $y=-3$ say

$\dot{x} = 2(2-2+3) = 6$

$\dot{y} = -3(8-4-2) = -6$

$\therefore \frac{\dot{y}}{\dot{x}} = -1$ i.e. sloping down to right

7 Substitute $x=2$, $y=4$

$\dot{x} = 2(2-2-4) = -8$

$\dot{y} = 4(8-4-2) = 8$

$\therefore \frac{\dot{y}}{\dot{x}} = -1$ i.e. sloping down to right.

Why are fixed points in a non-linear 2D system controlled by the same matrix as for a linear system

$$\dot{x} = f(x,y) \qquad \text{cp} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underline{\underline{A}} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\dot{y} = g(x,y)$$

(refer to Graded Exercise phase portrait)

Because trajectories are smooth, differentiable unique curves, we can Taylor expand the non-linear equations near any point.

And because $f(x^*, y^*) = 0$, the dynamics is determined by f' i.e. first order terms, which gives the Jacobian.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + O\left(f_{xx}^{2}, f_{yy}^{2} \ldots\right)$$

once you do this, the equations only apply VERY close to (x^*, y^*) not anywhere else, e.g. $(0,0)$ of the full non-linear system

If you go further away higher-order terms are not ignorable any more.

Linearised 2D System

24/1/23 consider $\dot{x} = f(x,y)$
$\dot{y} = g(x,y)$

and let x^*, y^* be a fixed point:

$$f(x^*, y^*) = 0$$
$$g(x^*, y^*) = 0$$

Let $u = x - x^*$
$v = y - y^*$

Then $\dot{u} = \dot{x} = f(x,y) = f(x^* + u, y^* + v)$

$\dot{v} = \dot{y} = g(x,y) = g(x^* + u, y^* + v)$

Expand $\dot{u}, \dot{v}$ in Taylor series:

$$\dot{u} = \cancelto{0}{f(x^*, y^*)} + \left(\frac{\partial f}{\partial x}\right)_{x^* y^*} u + \left(\frac{\partial f}{\partial y}\right)_{x^* y^*} v + \dots$$

$$\dot{v} = \cancelto{0}{g(x^*, y^*)} + \left(\frac{\partial g}{\partial x}\right)_{x^* y^*} u + \left(\frac{\partial g}{\partial y}\right)_{x^* y^*} v + \dots$$

$\therefore \dot{u} = f_x u + f_y v$ or $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}}_{\text{Jacobian at the fixed point } x^*, y^*} \begin{pmatrix} u \\ v \end{pmatrix}$

$\dot{v} = g_x u + g_y v$

NB only works for saddles, nodes and spirals (centres, stars deg. or $O(u^2)$)

Strogatz Ex. 6.3.2

$$\dot{x} = -y + ax(x^2+y^2) = f(x,y)$$

$$\dot{y} = x + ay(x^2+y^2) = g(x,y)$$

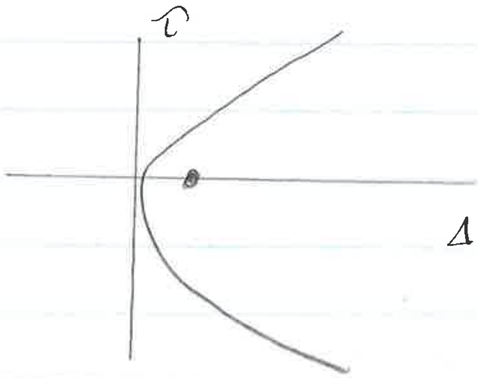
Let $u = x - x^*$
$v = y - y^*$

$$\Rightarrow \dot{u} = f_x u + f_y v = (3ax^2 + ay^2)u + (-1 + 2axy)v$$

$$\dot{v} = g_x u + g_y v = (1 + 2axy)u + (ax^2 + 3ay^2)v$$

so about the point $(0,0)$, the Jacobian is:

$$\underline{\underline{J}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



cp. Fig. 5.2.8

$$\left.\begin{aligned} \tau &= \mathrm{Tr}\,\underline{\underline{J}} = 0 \\ \Delta &= \det \underline{\underline{J}} = 1 \end{aligned}\right\} \Rightarrow \text{centre}$$

But is it really a centre?

To analyse the non-linear case, we transform to polar coordinates:

$x = r\cos\theta$ $\qquad x^2 + y^2 = r^2$
$y = r\sin\theta$

Recap of Lecture 4: Linear 2D systems

Given $\dot{x} = ax + by$

$\dot{y} = cx + dy$

Let $\underline{\underline{M}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so $\underline{\dot{X}} = \underline{\underline{M}}\,\underline{X}$, $\underline{X} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

Find $\tau = \text{Trace}\,\underline{\underline{M}} = a + d = \lambda_1 + \lambda_2$

$\Delta = \det.\,\underline{\underline{M}} = ad - bc = \lambda_1 . \lambda_2$

where λ_i, $i = 1, 2$ are eigenvalues of $\underline{\underline{M}}$, i.e. solutions of:

$\det | \underline{\underline{M}} - \lambda \underline{\underline{I}} | = 0$

Then solve: $(\underline{\underline{M}} - \lambda_i \underline{\underline{I}})\underline{v_i} = 0$

to get eigenvector $\underline{v_i}$

The general solution, i.e. trajectory, is:

$$\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{v_1} + c_2 e^{\lambda_2 t} \underline{v_2}$$

where c_1, c_2 are fixed by the initial condition:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1 \underline{v_1} + c_2 \underline{v_2}$$

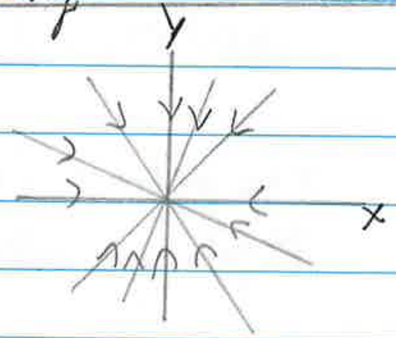
Q. Why are the eigenvectors related to the time evolution of the system?

What do they have to do with nodes, stars, saddlepoints?

It comes because: $\dot{\underline{x}} = \underline{M}\,\underline{x}$ is AUTONOMOUS i.e. trajectories cannot intersect.

Suppose we have a stable FP at origin, how can the system approach it?

1) From anywhere ⟹ STAR node



Try $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

All vectors are eigenvectors

2) Two eigenvectors

We can write any vector as a linear combination of any two, non-colinear, vectors. So choose $\underline{v_1}, \underline{v_2}$:

Then $\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{v_1} + c_2 e^{\lambda_2 t} \underline{v_2}$

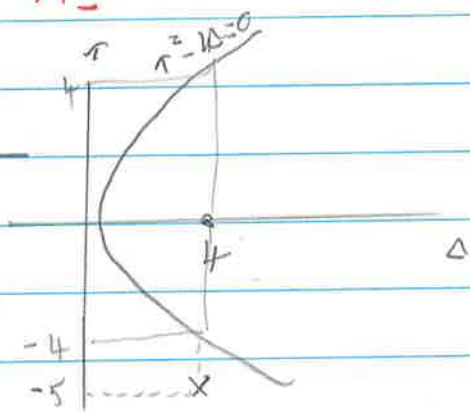
Why can we do this?

$$\underline{\underline{M}}\,\underline{x} = c_1 e^{\lambda_1 t} \underbrace{\underline{\underline{M}}\,\underline{v_1}}_{\lambda_1 \underline{v_1}} + c_2 e^{\lambda_2 t} \underbrace{\underline{\underline{M}}\,\underline{v_2}}_{\lambda_2 \underline{v_2}} = \frac{d\underline{x}}{dt}$$

Try $\underline{\underline{M}} = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}$

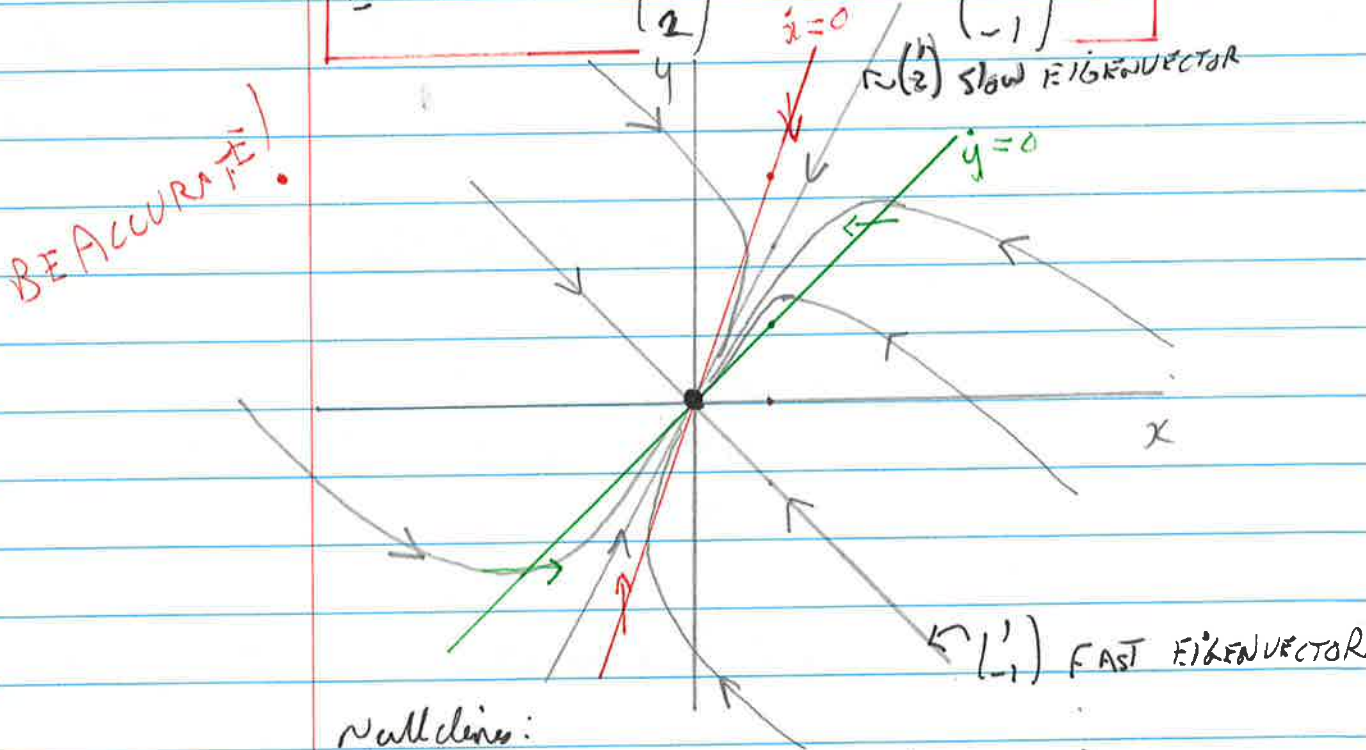
then $\tau = -5$

$\Delta = 4$



∴ stable node

$$\boxed{\underline{x}(t) = c_1 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-4t}\begin{pmatrix}1\\-1\end{pmatrix}}$$



Nullclines:

$\dot{x} = 0 \Rightarrow y = 3x$, and $\dot{y}$ (along $\dot{x} = 0$) $= -4x$

$\dot{y} = 0 \Rightarrow y = x$, and $\dot{x}$ (along $\dot{y} = 0$) $= -2x$

$\dot{x} = -3x + y$

$\dot{y} = 2x - 2y$

If we start on an eigenvector, we stay on it, and if we start anywhere else, we head towards the slow eigenvector as $t \to \infty$ (if stable, or $t \to -\infty$, if unstable).

They only scale along the eigenvector.

Because eigenvectors don't change direction (when acted on by $\underline{M}$, all systems end up approaching the slow one.

3) What if there is only one eigenvector?

$\dot{x} = x$
$\dot{y} = -x + y$

Try $\underline{M} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, one find: $\lambda = 1$, $\underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore \underline{x}(t) = c_1 e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ i.e. unstable as $\lambda > 0$. A degenerate node.

How do we interpret this?

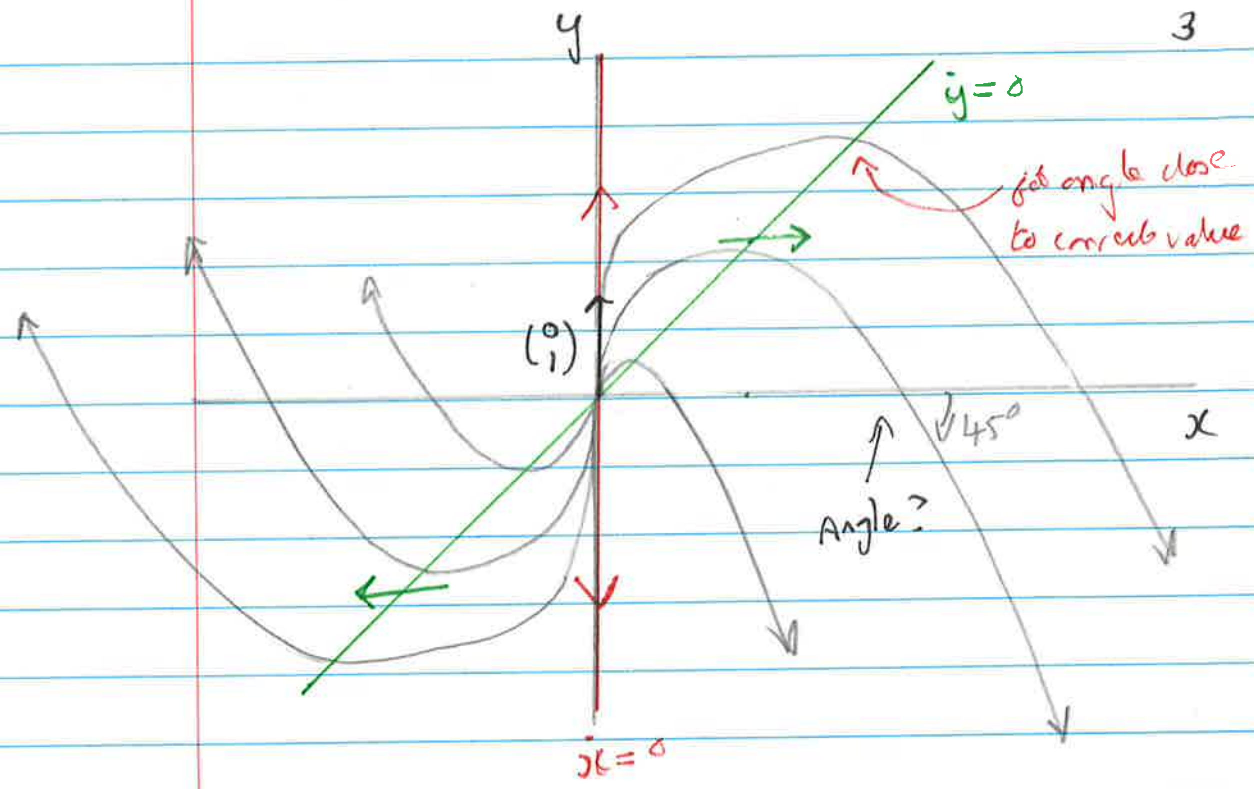
Nullclines:

$\dot{x} = 0 \Rightarrow x = 0$ i.e. y axis

$\dot{y} = 0 \Rightarrow y = x$

and $\dot{y}(\text{along } x = 0) = y > 0$ for $y > 0$.

and $\dot{x}(\text{along } \dot{y} = 0) = x > 0$ for $x > 0$.



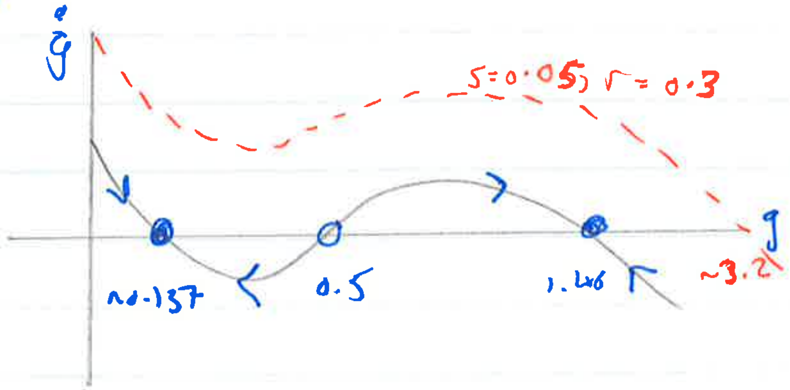
when $y=0$: $\dfrac{\dot{y}}{\dot{x}} = \dfrac{-x}{x} = -1$ i.e. 45° to x axis

and $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}}$ use this to find orientation of trajectories away from origin or fixed points

2D Linear Vs Non-Linear Systems

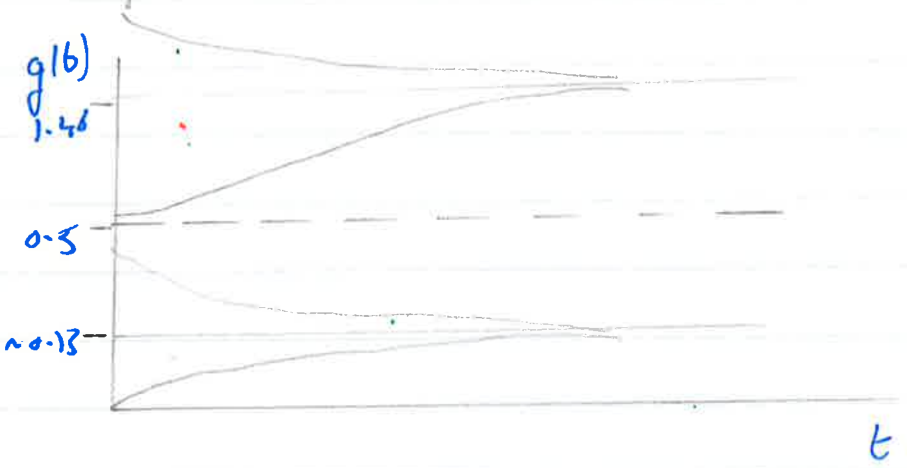
Recall 1D: $\dot{g} = s - rg + \frac{g^2}{1+g^2}$

Graph 1 $s = 0.05, r = 0.5$

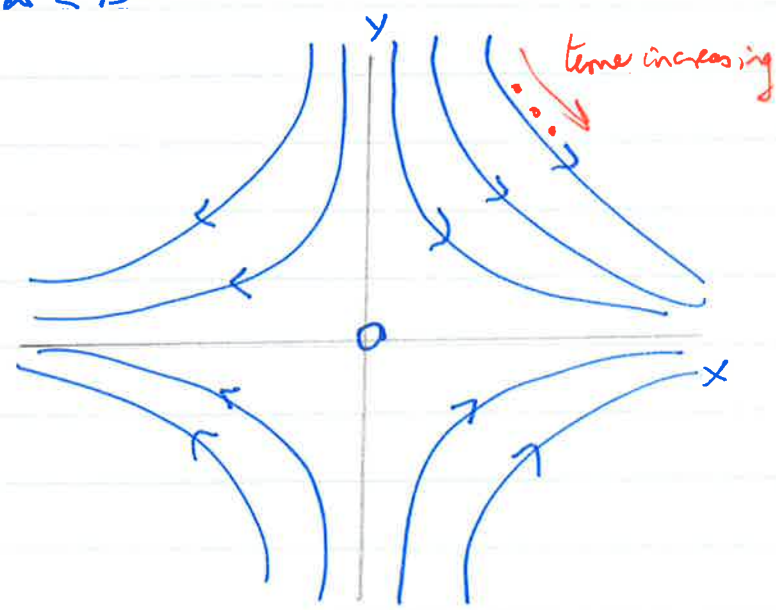


when the unstable F.P vanishes trajectories jump off up to ...?

Graph 2



But in 2D:



where has time gone?

$M = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

or $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$