Lecture 6: Ecological model

(CH. 4.2)

9/10/25

we've looked at nodes and saddlepoints, stars, and degenerate nodes.

What if the eigenvalues of $\underset{=}{M}$ are complex?

Given $\dot{x} = ax + by$, $\dot{y} = cx + dy$ $\Big\}$ $\underset{=}{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

characteristic eqn. is: $\det|\underset{=}{M} - \lambda \underset{=}{I}| = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - \underbrace{(a+d)}_{\tau}\lambda + \underbrace{ad - bc}_{\Delta} = 0$$

with solutions: $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$

Because $\underset{=}{M}$ is a Real matrix, λ_1, λ_2 must be either both real or complex conjugates.

If $\tau^2 < 4\Delta$, $\lambda_{1,2}$ are complex and we have a spiral

If $Re\,\lambda > 0 \Rightarrow$ unstable spiral
$\quad\quad\quad < 0 \Rightarrow$ stable "

Consider a model for two interacting populations:

$$\dot{N} = N\underbrace{(1-N}_{\text{LOGISTIC}}\underbrace{-p)}_{\text{COMPETITION}} \equiv F(N,p)$$

$$\dot{p} = \underbrace{-p}_{}\,(\tfrac{1}{2}\,\underbrace{-N}_{})\, \equiv G(N,p)$$

predators decline if no prey

predators increase by eating prey

This is a population model so $N, p \geqslant 0$.

Clearly X, Y axes are nullclines, and $(0,0)$ is an F.P.

Where are others?

Nullclines

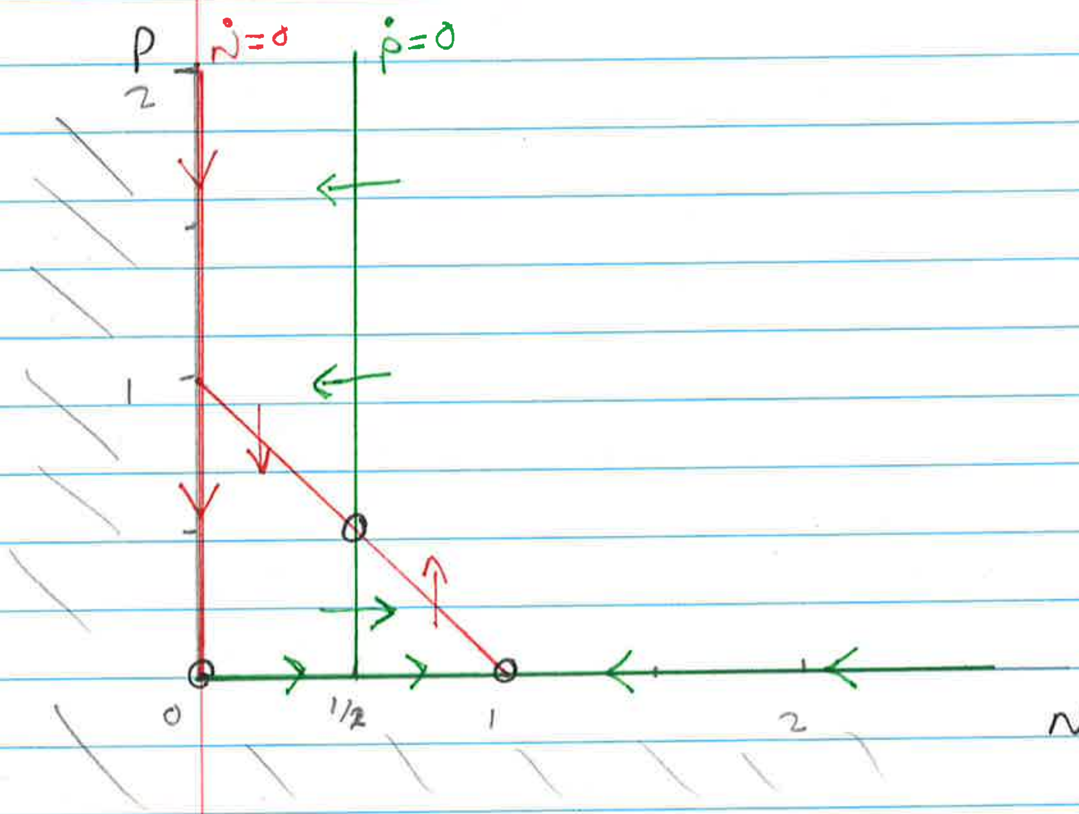
N nullcline: $\dot{N} = 0 \Rightarrow \underline{N = 0}$

or $1 - N - p = 0$

$\Rightarrow \underline{p = 1 - N}$

p nullcline: $\dot{p} = 0 \Rightarrow \underline{p = 0}$

or $\underline{N = 1/2}$



We can easily see the FP at the intersections of nullclines

How does the vector field point on the nullclines?

N nullcline: Along $N=0$ what is $\dot{P}(N,P)$?

$$\dot{P} = -P\left(\tfrac{1}{2}-N\right) = -\tfrac{1}{2}P < 0 \quad \therefore \text{Points down}$$

and along $P=1-N$, what is it?

$$\dot{P} = -(1-N)\left(\tfrac{1}{2}-N\right)$$

and we see: $N < 1/2 \Rightarrow \dot{P} < 0$

$1/2 < N < 1 \Rightarrow \dot{P} > 0$

$1 < N \Rightarrow \dot{P} < 0$ again, but this is outside the first quadrant, so we ignore it.

P nullcline: Along $P=0$, what is $\dot{N}(N,P)$?

$$\dot{N} = N(1-N-P) = N(1-N) \quad \begin{array}{l} >0 \text{ for } N<1 \\ <0 \text{ for } N>1. \end{array}$$

Along $N=\frac{1}{2}$, what is $\dot{N}$?

$$\dot{N} = \tfrac{1}{2}\left(\tfrac{1}{2}-P\right) \quad \begin{array}{l} <0 \text{ for } P>1/2 \\ >0 \text{ for } P<1/2 \end{array}$$

NB Because the axes are nullclines here, trajectories cannot cross them nor leave them if they start on them.

Finding the Fixed points and classifying them

The FPs are seen to be: $(0,0)$, $(1,0)$, $(\frac{1}{2},\frac{1}{2})$

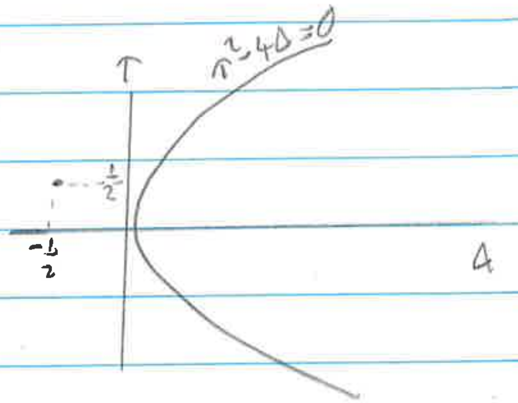
What is the Jacobian?

$$\underline{\underline{J}} = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial P} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial P} \end{pmatrix} = \begin{pmatrix} 1-P-2N & -N \\ P & N-\frac{1}{2} \end{pmatrix}$$

A) $(0,0)$ $\underline{\underline{J}}_A = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

$\therefore \tau = 1/2$

$\Delta = -1/2$ which is < 0.

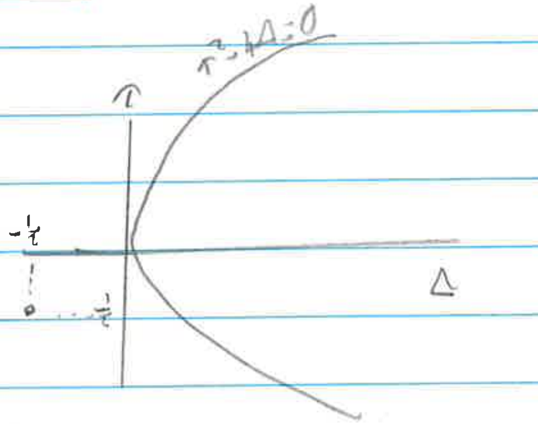


$\therefore$ A saddlepoint at $(0,0)$

B) $(1,0)$ $\underline{\underline{J}}B = \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}$

$\therefore \tau = -1/2$

$\Delta = -1/2$

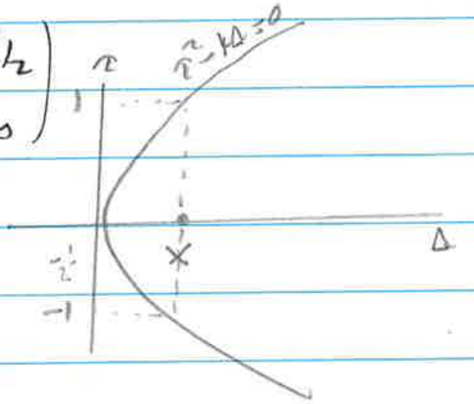


$\therefore$ A saddlepoint also at $(1,0)$

c) $(\frac{1}{2}, 1/2)$ $\underline{\underline{J}}_c = \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{pmatrix}$

$\therefore \tau = -1/2$

$\Delta = 1/4$



$\therefore$ A stable spiral at $(1/2, 1/2)$

Now we have the fixed points, what are the stable/unstable manifolds of the saddlepoints, and the eigenvalues/eigenvectors of the spiral?

How do we even draw trajectories if λ_i $\underline{v}_i$ are complex? (See moodle today for complex trajectories

A) $(0,0)$ $\underset{\sim}{J}_A = \begin{vmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{vmatrix}$ Saddlepoint

Eigenvalues: $\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\frac{1}{2}-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(-\frac{1}{2}-\lambda) = 0$

$\therefore \lambda = -\frac{1}{2}, 1$

Eigenvectors

$\lambda = -\frac{1}{2}$ $\begin{pmatrix} 1-(-\frac{1}{2}) & 0 \\ 0 & -\frac{1}{2}-(-\frac{1}{2}) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\therefore \frac{3}{2}v_1 = 0$ $\therefore \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = 1$ $\begin{pmatrix} 0 & 0 \\ 0 & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\therefore -\frac{3}{2}v_2 = 0$ $\therefore \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore$ General trajectory is:

$$\underset{\sim}{x}(t) = c_1 e^{-\frac{1}{2}t}\begin{pmatrix}0\\1\end{pmatrix} + c_2 e^{t}\begin{pmatrix}1\\0\end{pmatrix}$$

STABLE MANIFOLD UNSTABLE MANIFOLD

B) $(1,0)$ $\underset{=}{J}_B = \begin{pmatrix}-1 & -1\\ 0 & 1/2\end{pmatrix}$ Saddlepoint

Eigenvalues: $\begin{vmatrix}-1-\lambda & -1\\ 0 & \frac{1}{2}-\lambda\end{vmatrix} = 0$

$\Rightarrow (-1-\lambda)(\frac{1}{2}-\lambda) = 0$

$\therefore \lambda = -1, 1/2$

Eigenvectors

$\lambda = -1, \begin{pmatrix}0 & -1\\ 0 & 3/2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = 0$

$\therefore \frac{3}{2}v_2 = 0 \quad \therefore \underset{\sim}{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$\lambda = 1/2, \begin{pmatrix}-3/2 & -1\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = 0$

$\Rightarrow -\frac{3}{2}v_1 - v_2 = 0$

$\therefore v_2 = -\frac{3}{2}v_1 \quad \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$

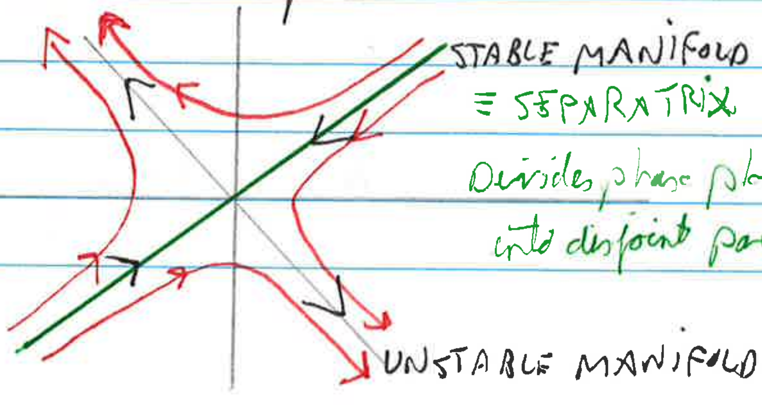
$\therefore$ General trajectory:

$$\underline{x}(t) = \underbrace{c_1 e^{\frac{1}{2}t} \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}}_{\text{UNSTABLE MANIFOLD}} + \underbrace{c_2 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{STABLE MANIFOLD}}$$

from Strogatz, p130

The stable manifold of a saddlepoint is the set of initial points $\underline{x}_0$ that approach $\underline{x}^*$ as $t \to +\infty$; the unstable manifold is the set of initial points that approach $\underline{x}^*$ as $t \to -\infty$.

A typical trajectory asymptotically approaches the UNSTABLE manifold as $t \to +\infty$, and the STABLE manifold as $t \to -\infty$.



c) $(1/2, 1/2)$ $\underline{\underline{J}}_c = \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ Stable Spiral

Eigenvalues: $\begin{vmatrix} -1/2 - \lambda & -1/2 \\ 1/2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (1/2 + \lambda).\lambda + 1/4 = 0$

$\Rightarrow \lambda^2 + 1/2 \lambda + 1/4 = 0$

$\therefore \lambda = \dfrac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}}{2} = \dfrac{-1/2 \pm \frac{\sqrt{3}}{2} i}{2} = -1/4 \pm \sqrt{3}/4\, i$

$\mathrm{Re}\,\lambda < 0 \Rightarrow$ STABLE

Eigenvectors:

$\lambda = -1/4 + \sqrt{3}/4\, i$

$$\begin{pmatrix} -\frac{1}{2} - \left(-\frac{1}{4} + \frac{\sqrt{3}}{4} i\right) & -\frac{1}{2} \\ \frac{1}{2} & -\left(-\frac{1}{4} + \frac{\sqrt{3}}{4}\right) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{4} - \frac{\sqrt{3}}{4} i & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} - \sqrt{3}/4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$\Rightarrow -\left(\frac{1}{4} + \frac{\sqrt{3}}{4} i\right) v_1 - \frac{1}{2} v_2 = 0$

$\Rightarrow v_2 = -\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = -\frac{1}{2}(1 + \sqrt{3}\, i).v_1$

$$\therefore \underline{v}_1 = \begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix}$$

$$\lambda = -1/4 - \sqrt{3}/4 i$$

$$\begin{pmatrix} -\frac{1}{2} - \left(-\frac{1}{4} - \sqrt{3}/4 i\right) & -1/2 \\ 1/2 & + \left(1/4 + \sqrt{3}/4 i\right) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{4} + \sqrt{3}/4 i & -1/2 \\ 1/2 & 1/4 + \sqrt{3}/4 i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \left(-\frac{1}{4} + \sqrt{3}/4 i\right) v_1 - 1/2 \, v_2 = 0 \qquad \Rightarrow v_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) v_1$$

$$1/2 \, v_1 + \left(1/4 + \sqrt{3}/4 i\right) v_2 = 0$$

$$\therefore \underline{v}_2 = \begin{pmatrix} 1 \\ -\frac{1}{2} + \sqrt{3}/2 i \end{pmatrix}$$

$\therefore$ General solution is:

$$\underline{x}(t) = c_1 e^{\left(-\frac{1}{4} + \frac{\sqrt{3}}{4} i\right) t} \begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix} + c_2 e^{\left(-\frac{1}{4} - \sqrt{3}/4 i\right) t} \begin{pmatrix} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i \end{pmatrix}$$

See moodle for why this produces a REAL trajectory for any c_1, c_2, which can be complex, but (x_0, y_0) must be real.

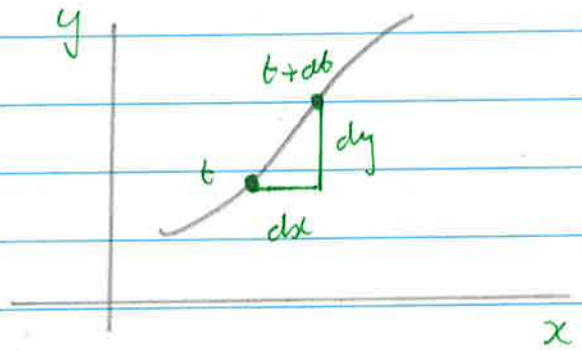
Can we draw the trajectories in the phase portrait yet?

IMPORTANT! We need to know what happens for large N, P, i.e. far from the fixed points.

Start with $\dot{N} = N(1-N-P) \to -N(N+P)$ for $N,P \gg 1$

and $\dot{P} = -P(\frac{1}{2}-N) \to NP$ for $N,P \gg 1$.

Mathematical Aside



Suppose $(x(t), y(t))$ defines a curve in XY plane.

What is $\frac{dy(t)}{dx}$?

$$\frac{dy}{dx} = \frac{y(t+dt)-y(t)}{x(t+dt)-x(t)} \sim \frac{y(t)+\dot{y}\,dt+\dots-y(t)}{x(t)+\dot{x}\,dt+\dots-x(t)}$$

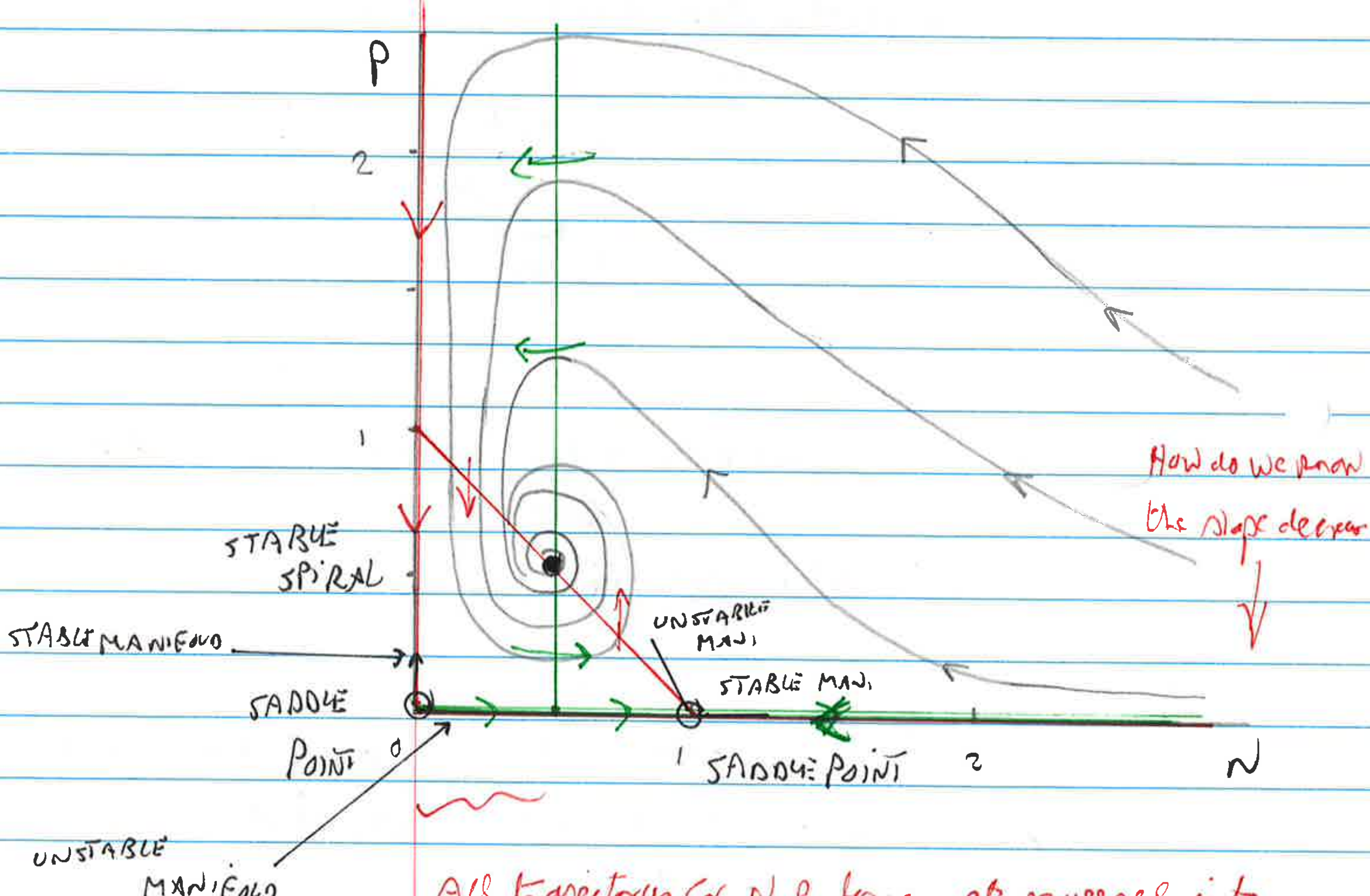
$$\boxed{\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}}$$

so, $\dfrac{dP}{dN} = \dfrac{\dot{P}}{\dot{N}} \sim \dfrac{NP}{-N(N+P)} \sim \dfrac{P}{-(N+P)} < 0$

i.e. Trajectories point down to the right as $N, P \to \infty$

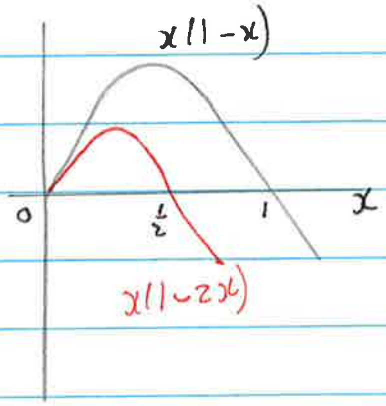
Note also that when $N = 1/2$, $\dot{P} = 0$, so $\dfrac{dP}{dN} = 0$

And when $N < 1/2$, and $P \to \infty$ (up Y axis)
then $dP/dN > 0$ i.e. points upwards to the right.



All trajectories for N, P large get squeezed into $N = (0, 1/2)$!

CONCLUSION: populations stabilise at $(1/2, 1/2)$ as $t \to \infty$.



What happens if we change the coefficients?

$$\dot{N} = N(1 - 2N - P)$$

smaller carrying capacity for prey

$$\dot{P} = -P(2 - N)$$

fast loss of predators if no prey (or few)

Where are the nullclines now?

$\dot{N} = 0$ $\Rightarrow N = 0$ or $P = 1 - 2N$

and what is $\dot{P}$ along the curves for $\dot{N} = 0$?

$N = 0$: $\dot{P} = -2P < 0$

$P = 1 - 2N$: $\dot{P} = -(1-2N)(2-N)$

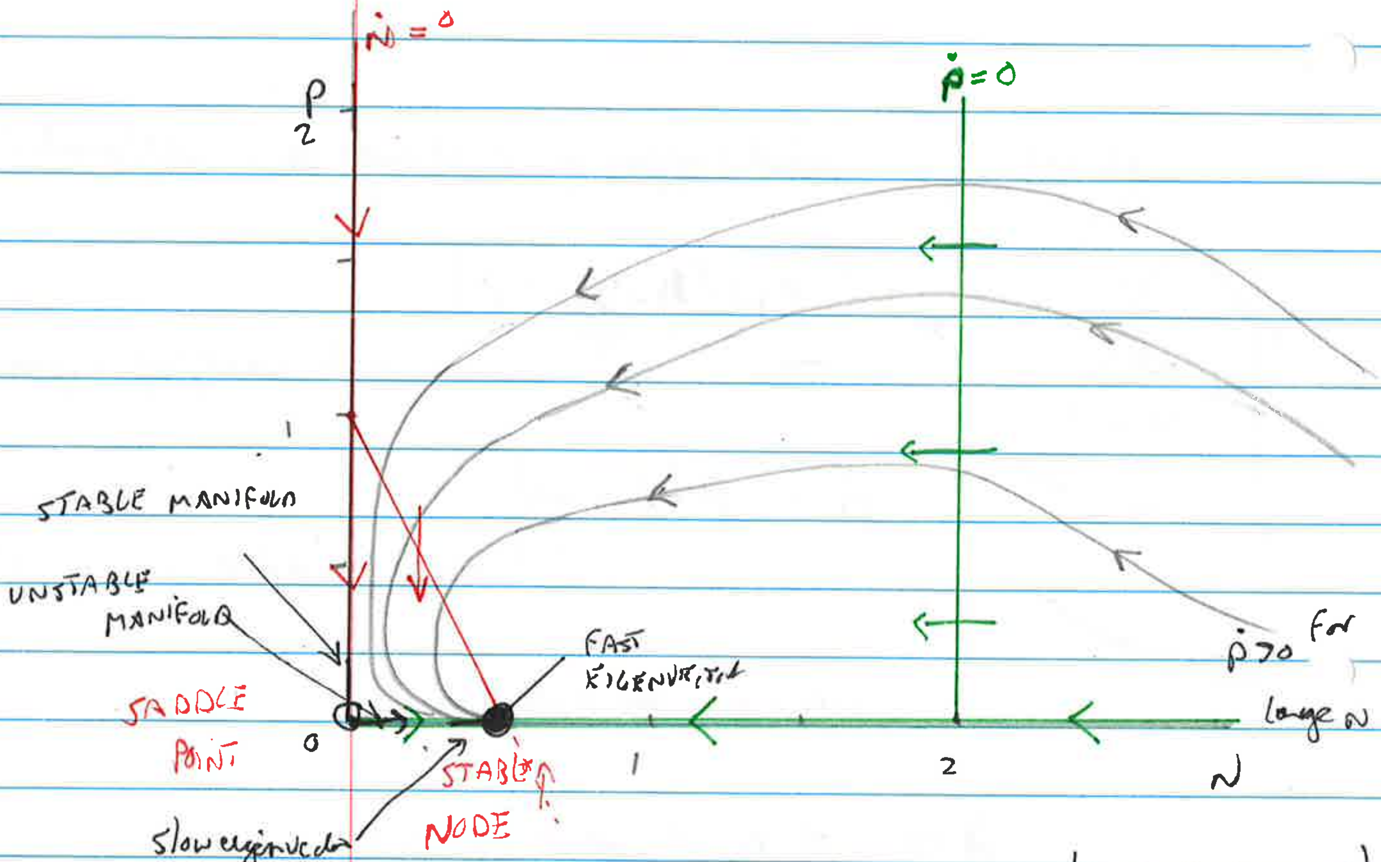
$= (2N-1)(2-N)$ which is > 0 for $\frac{1}{2} < N < 2$
and < 0 for $N < 1/2$

$\dot{P} = 0$ $\Rightarrow P = 0$ or $N = 2$

and what is $\dot{N}$ along those curves?

$P = 0$: $\dot{N} = N(1-2N)$ > 0 for $N < 1/2$, otherwise < 0

$N = 2$: $\dot{N} = 2(-3 - P) < 0$ for all P.

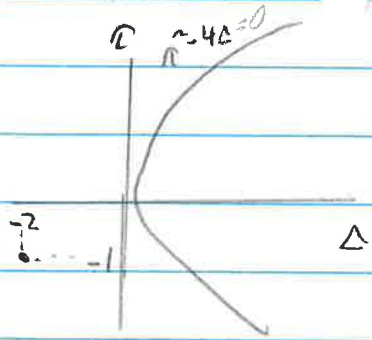


only two fixed points now! $\underline{\underline{J}} = \begin{pmatrix} 1-p-4N & -N \\ p & N-2 \end{pmatrix}$

$\dot p = 0$ has shifted to larger N, and $\dot N = 0$ to $N = \frac{1}{2}$, so no longer intersecting.

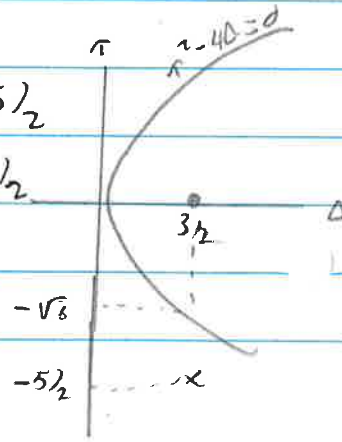
Fixed points now? clearly still (0,0) and now $(\frac{1}{2}, 0)$

A) (0,0) $\underline{\underline{J_A}} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ ∴ $\tau = -1$, $\Delta = -2$



∴ saddlepoint still at (0,0)

B) $(\frac{1}{2}, 0)$ $\underline{\underline{J_B}} = \begin{pmatrix} -1 & -\frac{1}{2} \\ 0 & -3/2 \end{pmatrix}$ ∴ $\tau = -5/2$, $\Delta = 3/2$



∴ stable node at $(1/2, 0)$

i.e. saddlepoint has changed to a stable node!

What happens to trajectories as $N, p \to \infty$ now?

$$\frac{dp}{dN} = \frac{\dot{p}}{\dot{N}} = \frac{-p(2-N)}{N(1-2N-p)} \to \frac{N \cdot p}{-N(2N+p)} = \frac{-p}{2N+p}$$

similar to the original case. So, trajectories point down and right as $N, p \to \infty$.

Eigenvalues, eigenvectors of stable node at $(\frac{1}{2}, 0)$

$$\begin{vmatrix} -1-\lambda & -\frac{1}{2} \\ 0 & -\frac{3}{2}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1+\lambda)(\frac{3}{2}+\lambda) = 0$

$\therefore \lambda = -1, -\frac{3}{2}$

$\lambda = -1 \quad \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_2 = 0 \quad \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -\frac{3}{2} \quad \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \frac{1}{2} v_1 - \frac{1}{2} v_2 = 0$

$\therefore v_2 = v_1 \quad \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

slow eigenvector

FAST EIGENVECTOR

$$\therefore \underline{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-\frac{3}{2}t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

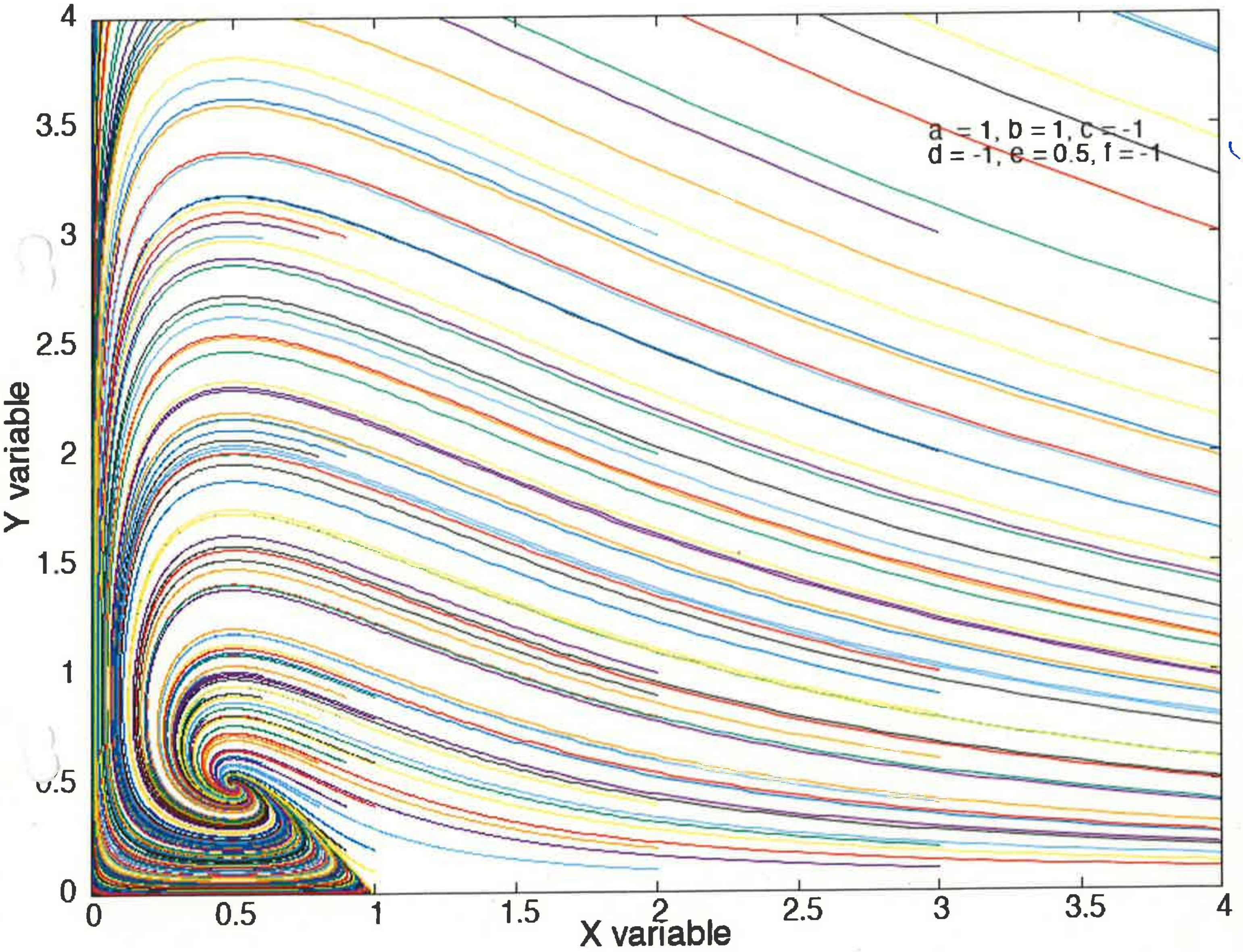
LECTURE 6

ORIGINAL MODEL

$\dot{N} = N(1-N-P)$

$\dot{P} = -P(\frac{1}{2}-N)$

Stable spiral



LECTURE 6

$\dot{N} = N(1-2N-P)$

$\dot{P} = -P(2-N)$

$(\frac{1}{2}, 0)$ is now a stable node