Lecture 5: Non-Linear 2D Systems

(CH. 4.1)

7/10/24 Start with the general 2D dynamical system:

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and suppose that (x^*, y^*) is a fixed point:

i.e. $f(x^*, y^*) = 0$
$g(x^*, y^*) = 0$

We expand f and g about (x^*, y^*) in a Taylor series:

Let $u = x - x^*$ with u, v small
$v = y - y^*$

$$\Rightarrow \dot{u} = \dot{x} = f(x, y) = f(x^* + u, y^* + v)$$

and $\dot{v} = \dot{y} = g(x, y) = g(x^* + u, y^* + v)$

$$\Rightarrow \dot{u} = \underbrace{\cancel{f(x^*, y^*)}}_{0} + u \underbrace{\left(\frac{\partial f}{\partial x}\right)_{x^*, y^*}}_{f_x} + v \underbrace{\left(\frac{\partial f}{\partial y}\right)_{x^*, y^*}}_{f_y} + \cdots$$

$$\text{and } \dot{v} = \underbrace{\cancel{g(x^*, y^*)}}_{0} + u \underbrace{\left(\frac{\partial g}{\partial x}\right)_{x^*, y^*}}_{g_x} + v \underbrace{\left(\frac{\partial g}{\partial y}\right)_{x^*, y^*}}_{g_y} + \cdots$$

$$\therefore \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

and $\underline{J} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ = JACOBIAN MATRIX

NOTES

1). $\underline{\underline{J}} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ is the Jacobian around the point (x^*, y^*)

You must evaluate $\underline{\underline{J}}$ around each fixed point to find their behaviour.

2) This only works for saddlepoints, nodes, and spirals.

The edge cases: centres, stars, and degenerate nodes are $O(u^2, v^2)$, and you need to examine higher-order terms in the Taylor series.

Model of two interacting populations
≡ Lotka–Volterra model
≡ Predator–Prey "
≡ Rabbits – Foxes "

Let x = # rabbits
y = # sheep

and assume rabbits breed faster than sheep, and they compete to eat grass.

Bigger for rabbits

$$\dot{x} = \underbrace{x(3 - x}\, \underbrace{-\,y)}$$

competition

$$\dot{y} = \underbrace{y(2 - y}\, - x)$$

Logistic growth

NB $x, y \geq 0$ as they are populations!

Where are the fixed points?

$(0,0)$ is clearly one. cp $\dot{N} = N.R(N)$ from lecture 2.

If $x = 0$, $y = 2$ is another: $(0, 2)$

If $y = 0$, $x = 3$ " " : $(3, 0)$

Are there any more? Use the nullclines to find out.

Find the nullclines

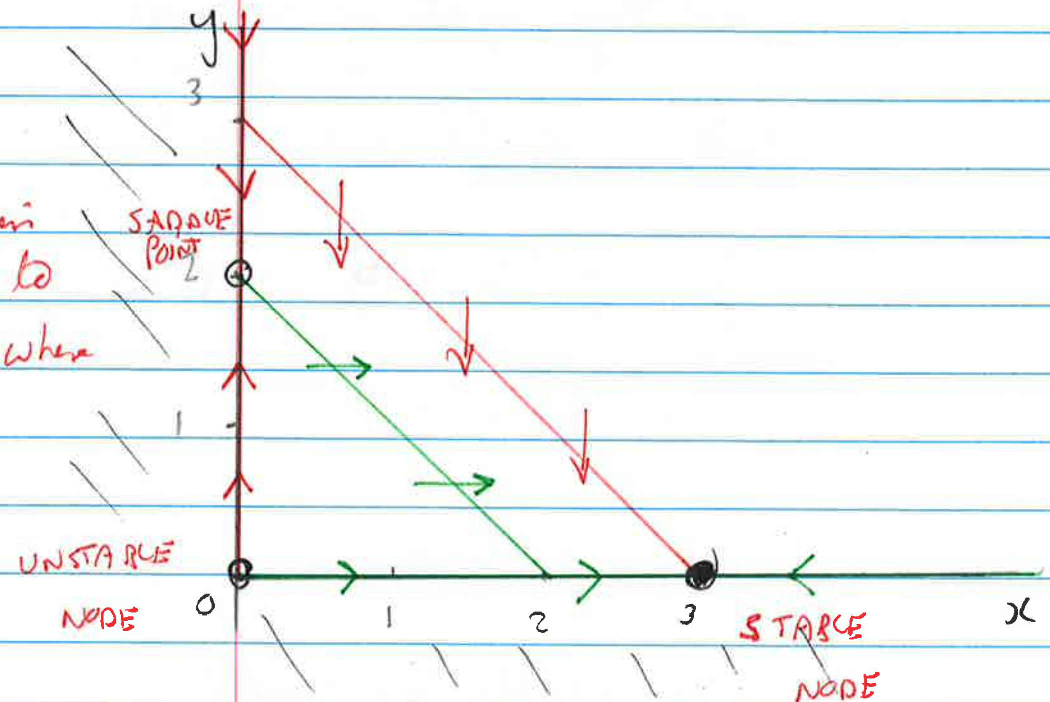
$\dot{x} = 0 \implies x.(3-x-y) = 0$

$\implies x = 0$ or $y = 3 - x$ NB TWO curves!

$\dot{y} = 0 \implies y.(2-y-x) = 0$

$\implies y = 0$ or $y = 2 - x$

HINT
Draw nullclines in different colours to see more easily where they cross



Fixed points are where the nullclines intersect (not necessarily at the origin now!)

We see that $(0,0)$, $(3,0)$, $(0,2)$ are only F.Ps

What is the direction of the vector field along the nullclines?

X nullcline 1) $\dot{x} = 0$; $x = 0$ $\Rightarrow$ $\dot{y}$ (along $x=0$) $= y(2-y)$

> 0 for $y < 2$
< 0 for $y > 2$

2) $y = 3 - x$ $\Rightarrow$ $\dot{y} = (3-x)(2-(3-x)-x)$

$= (3-x)(-1) = x-3$
> 0 for $x > 3$
< 0 for $x < 3$

Y nullcline 1) $\dot{y} = 0$, $y = 0$ $\Rightarrow$ $\dot{x}$ (along $y=0$) $= x.(3-x)$

> 0 for $x < 3$
< 0 for $x > 3$

2) $y = 2-x$ $\Rightarrow$ $\dot{x} = x(3-x-(2-x))$

$= x.(1)$ > 0 for $x > 0$

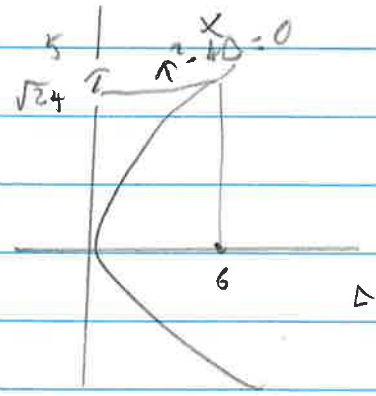
Now add arrows to the nullclines in the phase portrait.

Next analyse each fixed point using the Jacobian at the FP.

$$\underline{\underline{J}} = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3-y-2x & -x \\ -y & 2-x-2y \end{pmatrix}$$

1) $(0,0)$

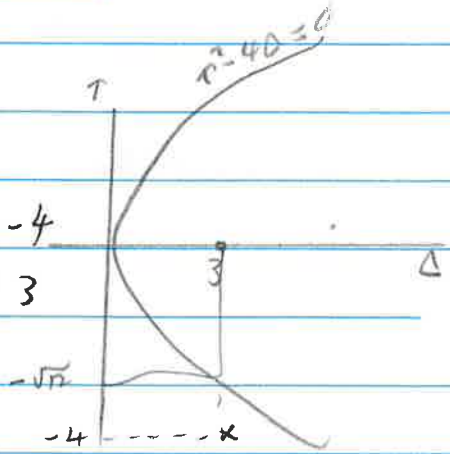
$$\underline{\underline{J}}(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad \therefore \tau = 5, \quad \Delta = 6$$



$\therefore$ unstable node at $(0,0)$

2) $(3,0)$

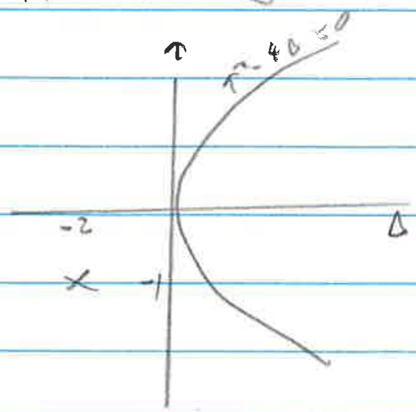
$$\underline{\underline{J}}(3,0) = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \quad \therefore \tau = -4, \quad \Delta = 3$$



$\therefore$ stable node at $(3,0)$

3) $(0,2)$

$$\underline{\underline{J}}(0,2) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \quad \therefore \tau = -1, \quad \Delta = -2$$



$\therefore$ saddlepoint at $(0,2)$

(Label/colour FPs in the phase portrait).

Find eigenvalues/eigenvectors for each F.P.

1) $(0,0)$ $\begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(2-\lambda) = 0 \quad \therefore \lambda = 2, 3$

Eigenvectors:

$\lambda = 2 \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow v_1 = 0 \quad \therefore \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = 3 \quad \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\Rightarrow -v_2 = 0 \quad \therefore v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore \underline{x}(t) = c_1 e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Slow eigenvector   Fast eigenvector

2) $(3,0)$ $\begin{vmatrix} -3-\lambda & -3 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (-3-\lambda)(-1-\lambda) = 0$

$\Rightarrow (1+\lambda)(3+\lambda) = 0 \quad \therefore \lambda = -1, -3$

Eigenvector:

$$\lambda = -1 \quad \begin{pmatrix} -2 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow -2v_1 - 3v_2 = 0 \quad \therefore v_2 = -\tfrac{2}{3}v_1 \quad \therefore \underline{v}_1 = \begin{pmatrix} 1 \\ -2/3 \end{pmatrix}$$

$$\lambda = -3 \quad \begin{pmatrix} 0 & -3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow -3v_2 = 0 \quad \therefore \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\therefore \underline{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -2/3 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Slow eigenvector Fast eigenvector

3) $(0,2)$ $\begin{vmatrix} 1-\lambda & 0 \\ -2 & -2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)(-2-\lambda) = 0$$

$$\therefore \lambda = 1, -2$$

Eigenvectors:

$$\lambda = 1 \quad \begin{pmatrix} 0 & 0 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

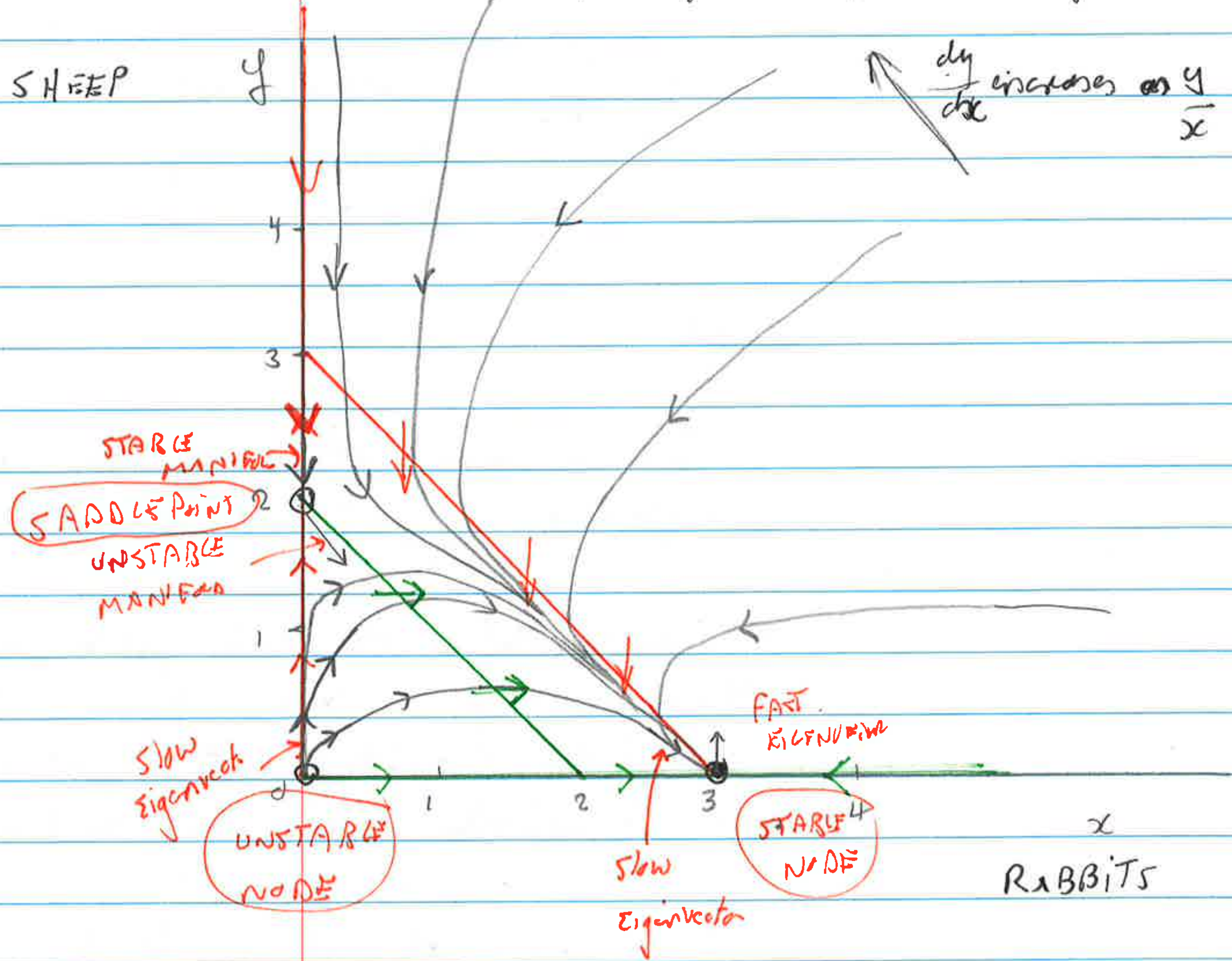
$$\Rightarrow -2v_1 - 3v_2 = 0 \quad \therefore v_2 = -2/3\, v_1$$

$$\lambda = -2 \quad \begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow 3v_1 = 0 \;\therefore\; \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore \underline{x}(t) = \underbrace{c_1 e^{t} \begin{pmatrix} 1 \\ -2/3 \end{pmatrix}}_{\text{UNSTABLE MANIFOLD}} + \underbrace{c_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{STABLE MANIFOLD}}$$

Let's draw the phase portrait from this information.



conc. sheep go extinct due to pressure from rabbits breeding faster

What are $\dot{x}, \dot{y}$ for large x, y?

$$\dot{x} = x(3 - x - y)$$

$$\dot{y} = y(2 - y - x)$$

$\Rightarrow$ As $x, y \to +\infty$

$$\frac{\dot{y}}{\dot{x}} = \frac{y(2 - y - x)}{x(3 - x - y)} \to \frac{y}{x}\frac{(-\cancel{x} - y)}{(-\cancel{x} - y)} = \frac{y}{x} > 0$$

and $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$ so, both $\dot{x}, \dot{y}$ are negative but $\frac{dy}{dx}$ is positive

NOTES

1. For a non-linear problem, eigenvectors and manifolds are only straight lines close to the FP

2. When calculating the vector field far from the origin (e.g. $x, y \to \infty$), dy/dx is the slope of the trajectories with respect to x axis, not the direction of time along the trajectory.

3 The only stable node is at $(3,0)$ so the rabbits dominate and sheep go extinct!

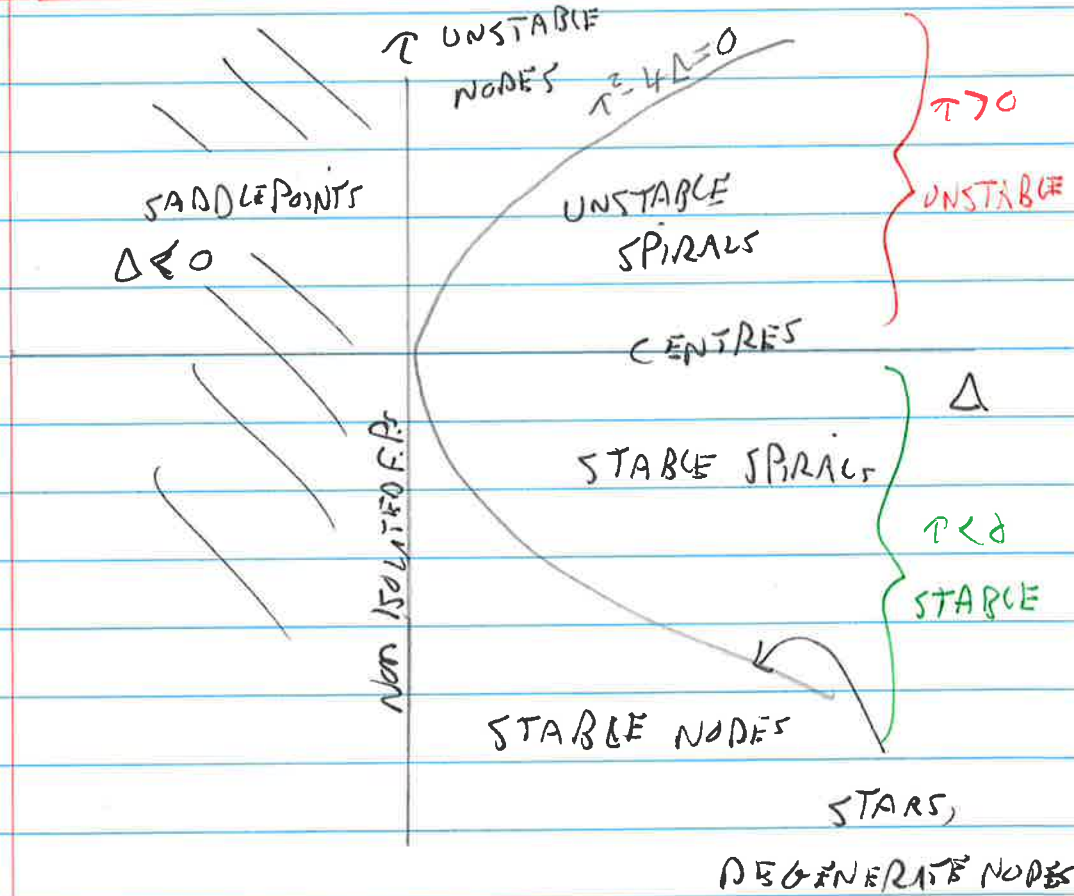
2D Linear Recipe (Side blackboard)

$$\left.\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}\right\} \quad \underline{\underline{M}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\tau = \mathrm{Tr}\, \underline{\underline{M}} = a + d$$

$$\Delta = \det.\, \underline{\underline{M}} = ad - bc$$

Then: $\underline{x}(t) = C_1 e^{\lambda_1 t} \underline{v_1} + C_2 e^{\lambda_2 t} \underline{v_2}$



You find the type and stability of the fixed point just from τ, Δ.

Recipe for solving any 2D linear system

1 Given $\dot{x} = ax + b$, $\dot{y} = cx + d$ $\Big\}$ $\underline{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

2 Find $\tau = \mathrm{Tr}\,\underline{M} = a + b$

$\Delta = \det \underline{M} = ad - bc$

x axis   y axis

Draw the τ, Δ plot and find point (Δ, τ)

and read off the type and stability of the F.P.

(Stop here if all you need is F.P.)

3 Find eigenvalues and eigenvectors of $\underline{M}$.

4 Find the nullclines: x nullcline is $\dot{x} = 0$

y nullcline is $\dot{y} = 0$.

And find the direction of the vector field along the nullclines.

5 Draw the nullclines, eigenvectors and some representative trajectories on the phase portrait.

(Stop here if you only need a qualitative solution.)

6 Write the general solution as:

$$\underline{x}(t) = c_1 e^{\lambda_1 t}\,\underline{v}_1 + c_2 e^{\lambda_2 t}\,\underline{v}_2$$

and find c_1, c_2 from the initial point (x_0, y_0)

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \underline{v}_1 + c_2 \underline{v}_2$$