Lecture 4: Linear 2D Dynamical systems

(CH. 3)

The general 2D linear system is:

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

we can write this as a matrix equation:

$$\dot{\underline{x}} = \underline{\underline{M}}\,\underline{x}, \quad \underline{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \underline{\underline{M}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Q. Is this the most general case? What about

$$\dot{\underline{x}} = \underline{\underline{M}}\,\underline{x} + \underline{c}, \quad \underline{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

You can always shift this to the origin by changing variables.

clearly $\underline{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a solution. Are there others?

If so, they must satisfy

$$ax + by = 0$$
$$cx + dy = 0$$

Solve for x, y: From second eqn: $y = -\frac{c}{d}x$

so $ax + \frac{b}{d}\cdot -cx = 0$ $\therefore$ $(ad - bc)x = 0$

Since we want $x \neq 0$, $\det \underline{M} = 0$.

Fixed points of linear system are : $\underline{x} = 0$ or $\det \underline{M} = 0$

What does $\det \underline{M} = 0$ look like?

Trivial case : $\underline{M} = 0 \Rightarrow x, y =$ constant

Non-Trivial case : $\underline{M} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$

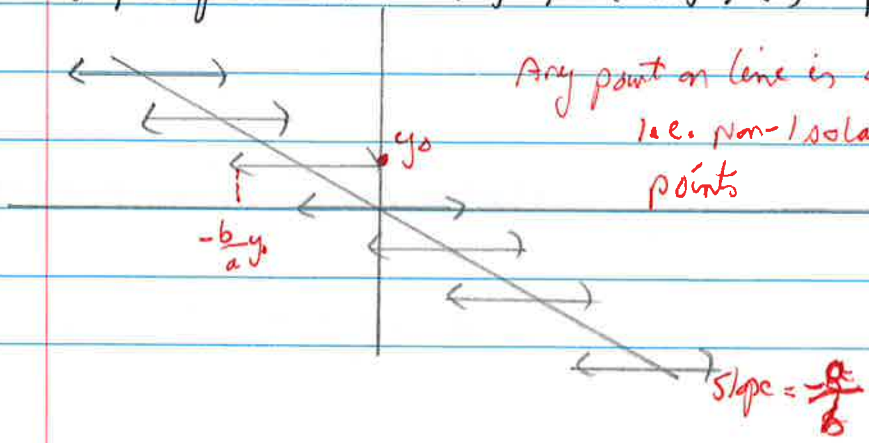
$$\Rightarrow \dot{x} = ax + by$$
$$\dot{y} = 0$$

$\therefore y(t) = y_0 \equiv$ constant

$\Rightarrow \dot{x} = ax + by_0 = 0$ at fixed point x^*.

$\therefore x^* = -b/a \cdot y_0$

So fixed point is : $(x^*, y^*) = (-\frac{b}{a} y_0, y_0)$ for any y_0



Any point on line is a fixed point
i.e. Non-Isolated Fixed points

Shifting a 2D Fixed point to the origin

28/6/24 Suppose $\underset{\sim}{\dot{x}} = \underset{=}{A}\,\underset{\sim}{x} + \underset{\sim}{c}$ $\qquad \underset{\sim}{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ = constant

Can we transform this so that $\underset{\sim}{\dot{x}} = \underset{=}{A}\,\underset{\sim}{x}$?

Let $u = x + \alpha$ and find α, β to make c_1, c_2 disappear.
$v = y + \beta$

Clearly $\dot{u} = \dot{x}$ and let $\underset{=}{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$\dot{v} = \dot{y}$

$$\Rightarrow \dot{u} = ax + by + c_1 = a(u-\alpha) + b(v-\beta) + c_1$$

$$\dot{v} = cx + dy + c_2 = c(u-\alpha) + d(v-\beta) + c_2$$

$$\Rightarrow \dot{u} = au + bv - a\alpha - b\beta + c_1$$

$$\dot{v} = cu + dv - c\alpha - d\beta + c_2$$

So, we want to find α, β so that: $-a\alpha - b\beta + c_1 = 0$

$$-c\alpha - d\beta + c_2 = 0$$

Solve for α: $\alpha = \dfrac{-b\beta + c_1}{a}$ and $\alpha = \dfrac{-d\beta + c_2}{c}$

$$\Rightarrow (-b\beta + c_1) \cdot c = (-d\beta + c_2) \cdot a$$

$$\Rightarrow (ad - bc)\beta = ac_2 - cc_1$$

$$\boxed{\therefore \beta = \frac{(ac_2 - cc_1)}{(ad - bc)}} \quad \text{or} \quad \beta = \frac{ac_2 - cc_1}{\det \underline{\underline{A}}}$$

$$\therefore \alpha = \frac{-b\beta + c_1}{a} = \frac{-b}{a}\left(\frac{ac_2 - cc_1}{(ad - bc)}\right) + \frac{c_1}{a}$$

$$= \frac{-abc_2 + b\cancel{c}c_1 + c_1(ad - \cancel{b}c)}{a(ad - bc)}$$

$$= \frac{-\cancel{a}bc_2 + \cancel{a}dc_1}{\cancel{a}(ad - bc)} = \frac{dc_1 - bc_2}{ad - bc}$$

$$\boxed{\therefore \alpha = \frac{dc_1 - bc_2}{ad - bc}} \quad \text{or} \quad \alpha = \frac{dc_1 - bc_2}{\det \underline{\underline{A}}}$$

e.g. $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad \underline{c} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$ $\quad$ so $a = 1,\ b = 2,\ c = 3,\ d = 4,\ c_1 = 5,\ c_2 = 10$

$\det \underline{\underline{A}} = -2$ and $\alpha = \dfrac{4.5 - 2.10}{-2} = 0$

$$\beta = \frac{1.10 - 3.5}{-2} = 2.5$$

$$\therefore u = x \qquad v = y + 2.5$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y - 2.5 \end{pmatrix} + \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} x + 2y - 5 + 5 \\ 3x + 4y - 10 + 10 \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{QED}$$

what if we are not on the line?

$$\begin{rcases} \dot{x} = ax + by \\ \dot{y} = 0 \end{rcases} \dot{x} = ax + by_0$$

Solve: $$\int_{x_0}^{x} \frac{dx}{ax+by_0} = \int_0^t dt$$

$$\Rightarrow \frac{1}{a}\Big[\ln(ax+by_0)\Big]_{x_0}^{x} = t$$

$$\Rightarrow \ln\left(\frac{ax+by_0}{ax_0+by_0}\right) = at$$

$$\therefore \frac{ax+by_0}{ax_0+by_0} = e^{at}$$

$$\Rightarrow ax+by_0 = (ax_0+by_0)e^{at}$$

$$\Rightarrow ax - ax_0e^{at} = -by_0 + by_0e^{at}$$

$$\Rightarrow x - x_0e^{at} = \frac{b}{a}\left(e^{at}-1\right)$$

$$\therefore x(t) = x_0e^{at} + \frac{b}{a}y_0\left(e^{at}-1\right)$$

x increases exponentially from x_0 ($a>0$, or goes to $-\frac{b}{a}y_0$ if $a<0$)

note that any value of y_0 corresponds to an x^* via $x^* = \frac{-b}{a} y_0$

$$\Rightarrow x(t) = x_0 e^{at} + \frac{b}{a} y_0 (e^{at} - 1)$$

$$= x_0 e^{at} - x^* e^{at} + x^*$$

$$= (x_0 - x^*) e^{at} + x^*$$

i.e. $x(t)$ moves away from x^* exponentially proportional to its initial distance (or towards it if $a < 0$).

General case

See Strogatz Example 5.1.2

$$\underset{=}{M} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \quad \text{i.e.} \quad \begin{aligned} \dot{x} &= a x \\ \dot{y} &= -y \end{aligned}$$

These are uncoupled, so its' easier, but they illustrate different types of fixed point.

Solutions are :

$$x(t) = x_0 e^{at} \qquad a = \text{constant}$$
$$y(t) = y_0 e^{-t}$$

Solutions depend on the constant a. Examine them for various values of a.

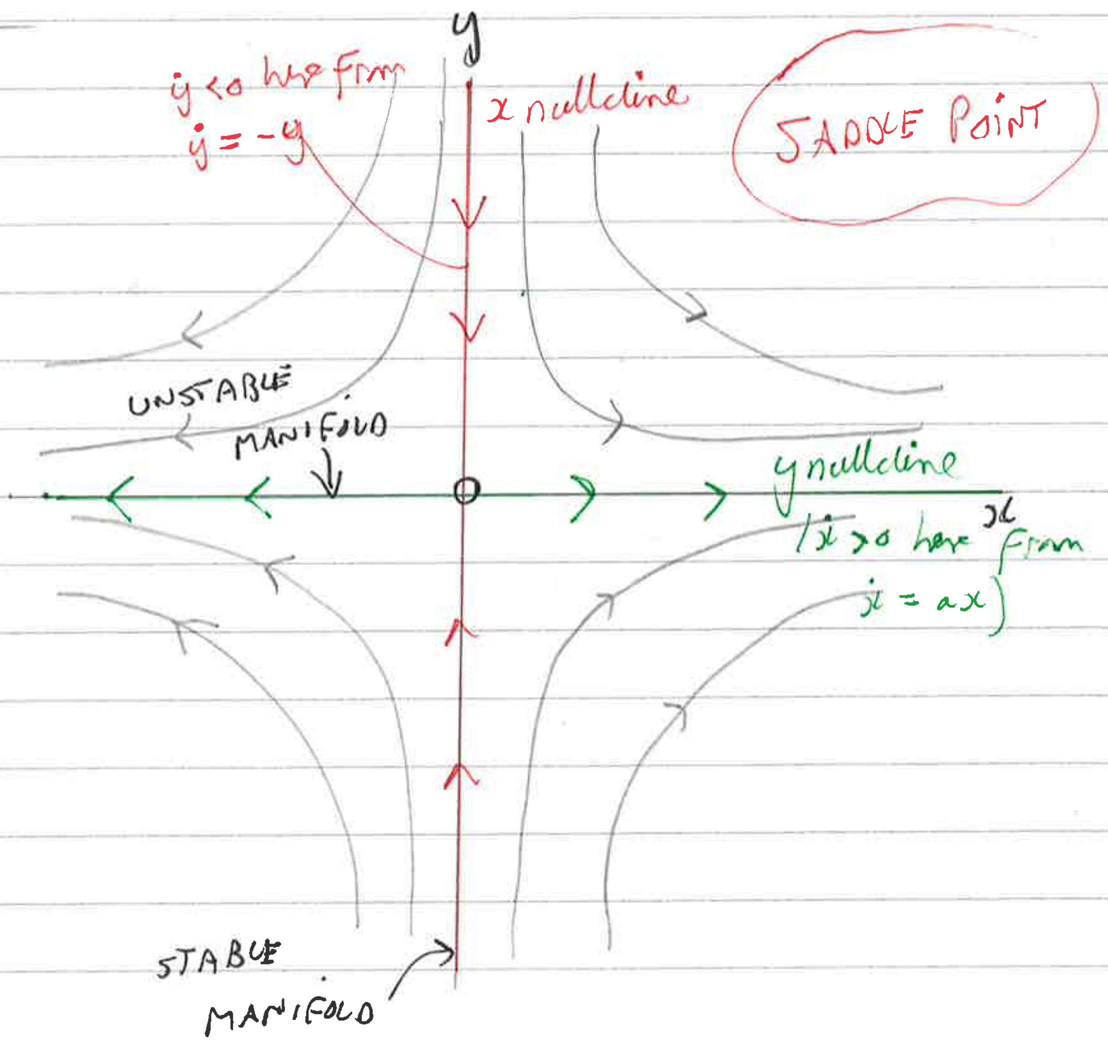
Case 1 : $a > 0$

$$\underset{=}{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \dot{x} = x$$
$$\dot{y} = -y$$

$$\Rightarrow x(t) = x_0 e^{at}$$
$$y(t) = y_0 e^{-t}$$

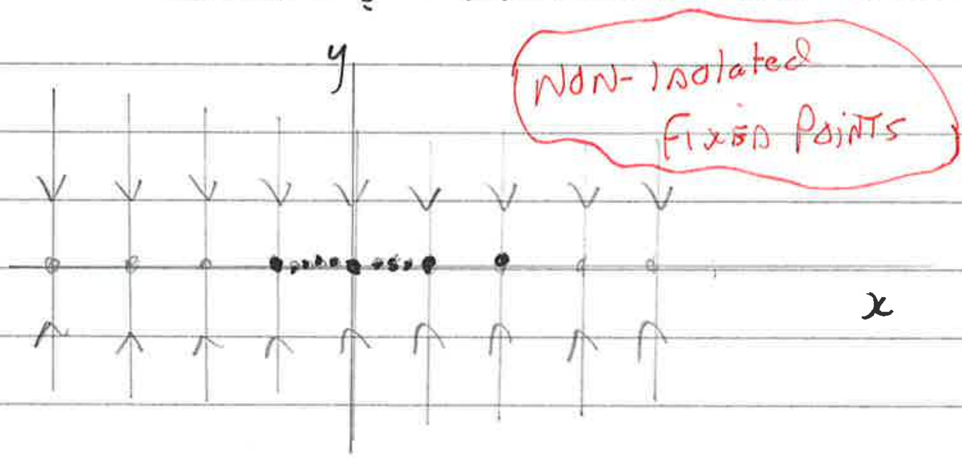
Fixed point is $(0,0)$



Case 2 : $a = 0$

$$\underline{\underline{M}} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

NB $\det \underline{\underline{M}} = 0$, so we have a non-isolated FP



$\Rightarrow \dot{x} = 0$

$\dot{y} = -y$

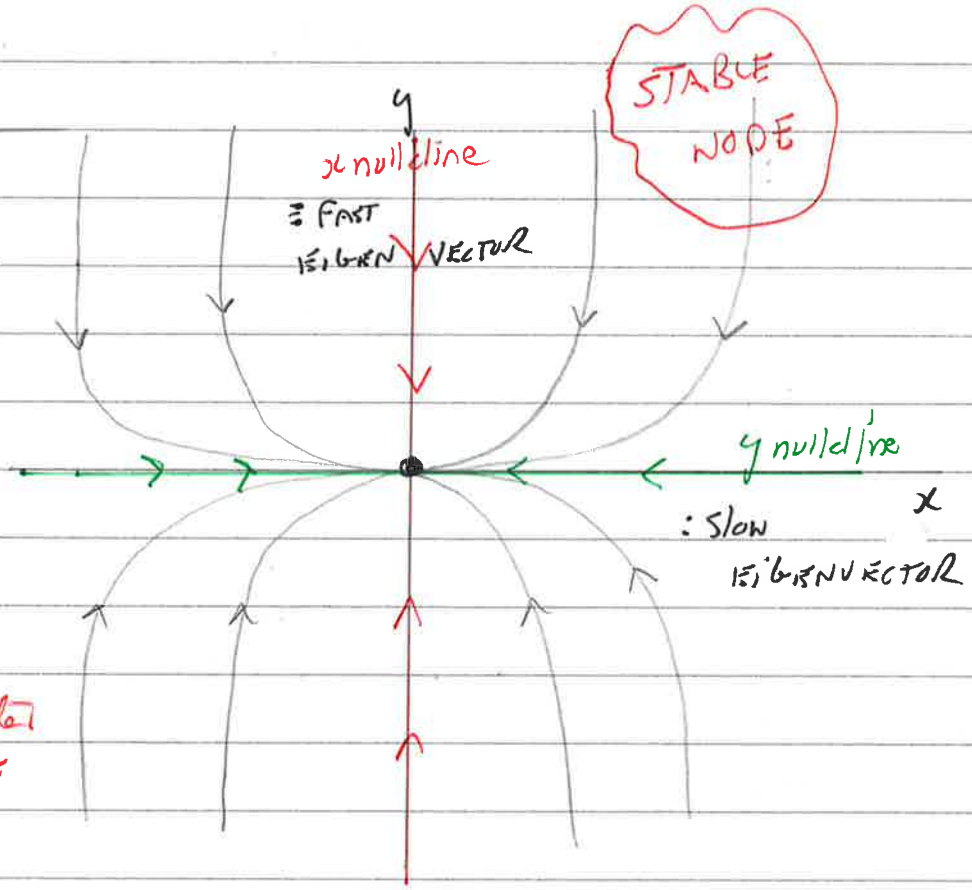
$\Rightarrow x(t) = x_0$

$y(t) = y_0 e^{-t}$

Every x_0 is a fixed point. Here they are STABLE as trajectories approach $y = 0$ as $t \to +\infty$.

Case 3: $-1 < a < 0$

$$\underline{\underline{M}} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix}$$



$\Rightarrow \dot{x} = -\frac{1}{2} x$

$\dot{y} = -y$

$\Rightarrow x(t) = x_0 e^{-\frac{1}{2}t}$

$y(t) = y_0 e^{-t}$ // Decays faster than $e^{-\frac{1}{2}t}$

x nullcline is $x = 0$, and $\dot{y} = -y$ along it

y nullcline is $y = 0$, and $\dot{x} = -\frac{1}{2}x$ along it

NB slow and fast eigenvectors do NOT always coincide with nullclines NOR the axes.

Case 4: $a = -1$

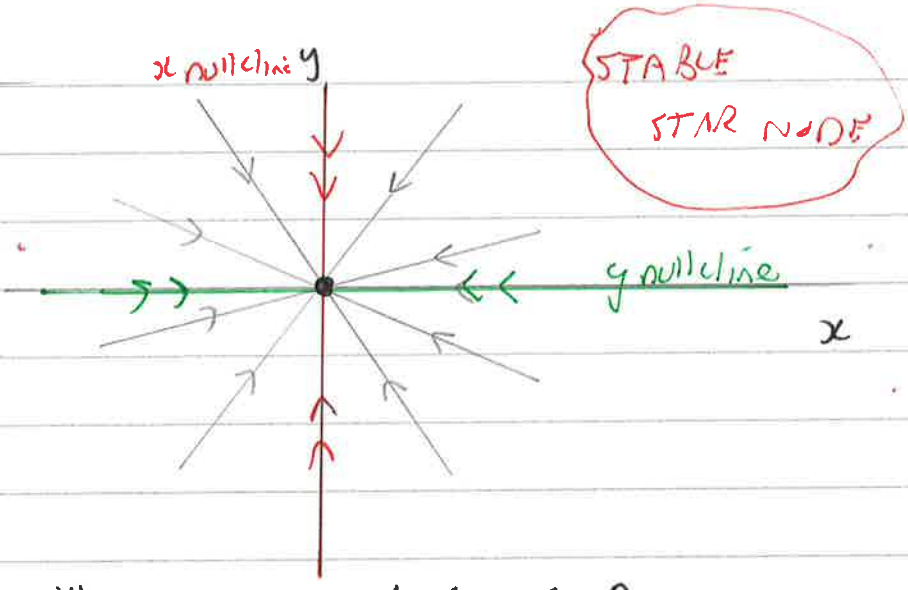
$$\underline{\underline{M}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow \dot{x} = -x$

$\dot{y} = -y$

$\Rightarrow x(t) = x_0 e^{-t}$

$y(t) = y_0 e^{-t}$ $\Bigg\} \Rightarrow \dfrac{y(t)}{x(t)} = \dfrac{y_0}{x_0} =$ constant $= \tan\theta_0$



Every straight line through origin is a trajectory; it is a STABLE STAR node because both exponents are negative.

Case 5: $a < -1$

$$\underline{\underline{M}} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

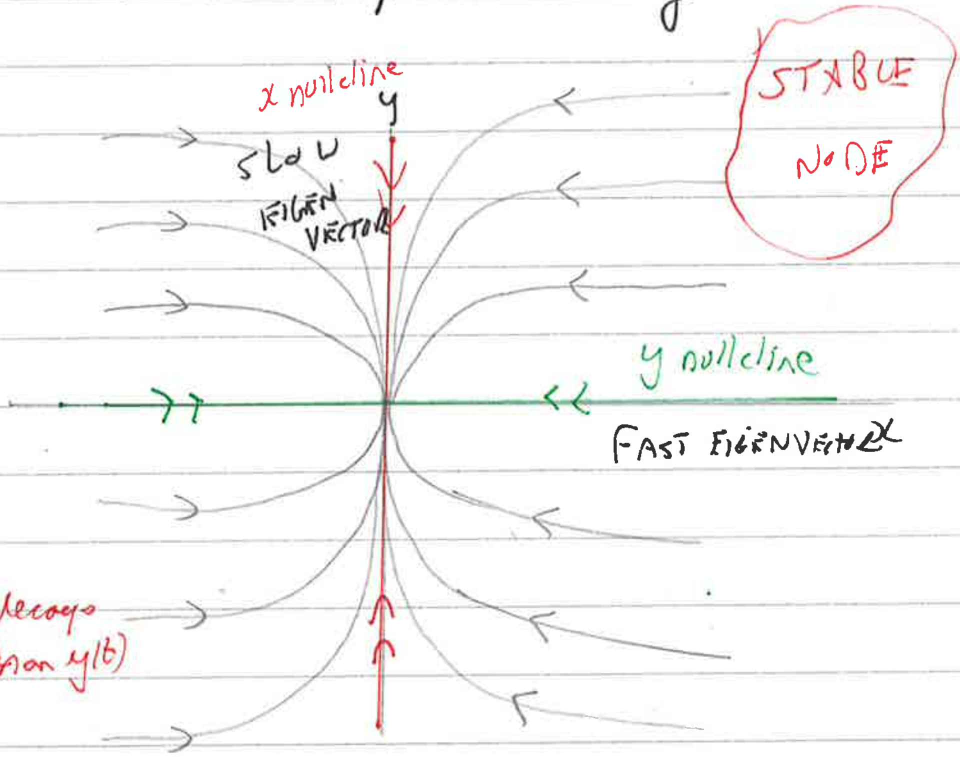
$\Rightarrow \dot{x} = -2x$

$\dot{y} = -y$

$\Rightarrow x(t) = x_0 e^{-2t}$

$y(t) = y_0 e^{-t}$

// Now $x(t)$ decays faster than $y(t)$



General Solution to Linear 2D Dynamical System

$\dot{\underline{x}} = \underline{\underline{M}}\,\underline{x}$ is a linear equation, so we try the solution:

$$\boxed{\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2}$$

where λ_1, λ_2 are eigenvalue of $\underline{\underline{M}}$

$\underline{v}_1, \underline{v}_2$ are the corresponding eigenvectors.

Why is this a solution?

$$\underline{\underline{M}}\,\underline{x} = \underline{\underline{M}}\left(c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2\right)$$

$$= c_1 e^{\lambda_1 t}\underbrace{\underline{\underline{M}}\,\underline{v}_1}_{\lambda_1 \underline{v}_1} + c_2 e^{\lambda_2 t}\underbrace{\underline{\underline{M}}\,\underline{v}_2}_{\lambda_2 \underline{v}_2}$$

$$= \lambda_1 c_1 e^{\lambda_1 t}\underline{v}_1 + \lambda_2 c_2 e^{\lambda_2 t}\underline{v}_2$$

$$\therefore \underline{\underline{M}}\,\underline{x} = \dot{\underline{x}}$$

c_1, c_2 are determined by initial condition (x_0, y_0):

$t = 0 \Rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \underline{v}_1 + c_2 \underline{v}_2$ and we can solve these simultaneous equations to get c_1, c_2 in terms of x_0, y_0.

NOTE. 1) If $\lambda_1, \lambda_2 < 0$, it is a stable node

$\lambda_1, \lambda_2 > 0$, " " unstable "

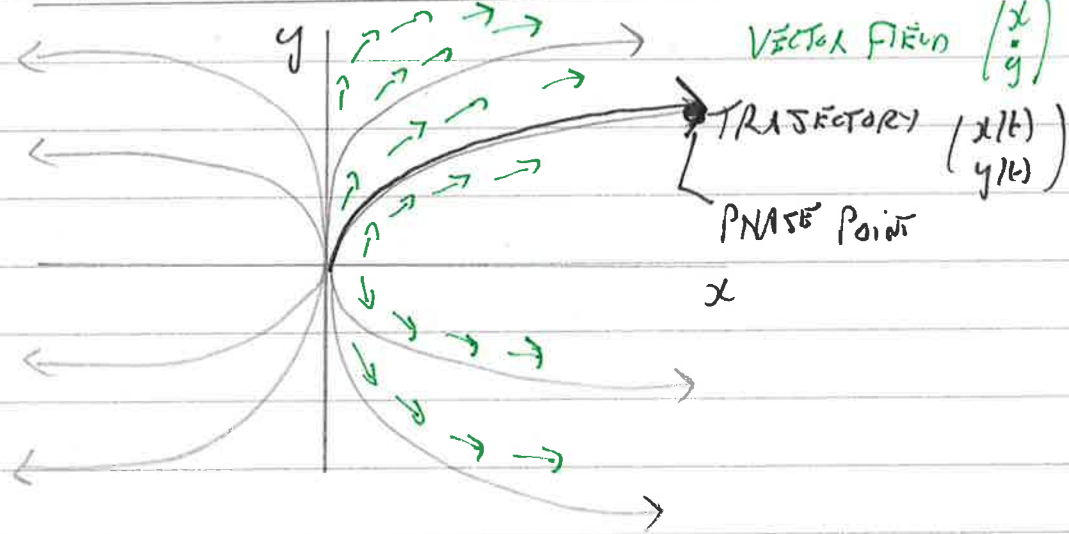
λ_1, λ_2 have opposite sign, it is a saddlepoint.

while if λ_1, λ_2 are complex, they must be complex conjugate.

2) The eigenvalues control the rate of growth, while eigenvectors set the direction of approach or receding from the fixed point.

NOMENCLATURE

PHASE PORTRAIT = 2D PHASE SPACE



NULLCLINE = a curve along which $\dot{x} = 0$ OR $\dot{y} = 0$

FIXED POINT = intersection of the nullclines, i.e. $\dot{x} = \dot{y} = 0$

NODE ≡ { STABLE NODE = fixed point towards which trajectories flow as $t \to +\infty$.

UNSTABLE NODE = fixed point from which trajectories flow away as $t \to +\infty$ i.e. trajectories move towards it as $t \to -\infty$. }

SLOW EIGENVECTOR = The eigenvector with the smallest MAGNITUDE eigenvalue. Trajectories approach ($\lambda < 0$) or leave ($\lambda > 0$) the fixed point along this eigenvector as $t \to +\infty$.

FAST EIGENVECTOR = The eigenvector with the largest magnitude eigenvalue. Trajectories become parallel to this eigenvector as $t \to -\infty$.

SADDLE POINT = A fixed point with eigenvalues of opposite sign; Trajectories approach or recede but don't touch it. (Except for the manifolds, see next)

STABLE MANIFOLD = Set of initial points (x_0, y_0) that have $\underline{x}(t) \to \underline{x}^*$ as $t \to +\infty$.

UNSTABLE MANIFOLD = Set of initial points that have $\underline{x}(t) \to \underline{x}^*$ as $t \to -\infty$.

Eigenvalues and $\tau - \Delta$ Plot

Given $\dot{x} = ax + by$

$\dot{y} = cx + dy$

Let $\underset{=}{M} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

The characteristic equation: $\det |\underset{=}{M} - \lambda \underset{=}{I}| = 0$

$\Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$ or $(a-\lambda)(d-\lambda) - bc = 0$

$\Rightarrow \lambda^2 - \underbrace{(a+d)}_{\tau}\lambda + \underbrace{ad - bc}_{\Delta} = 0$

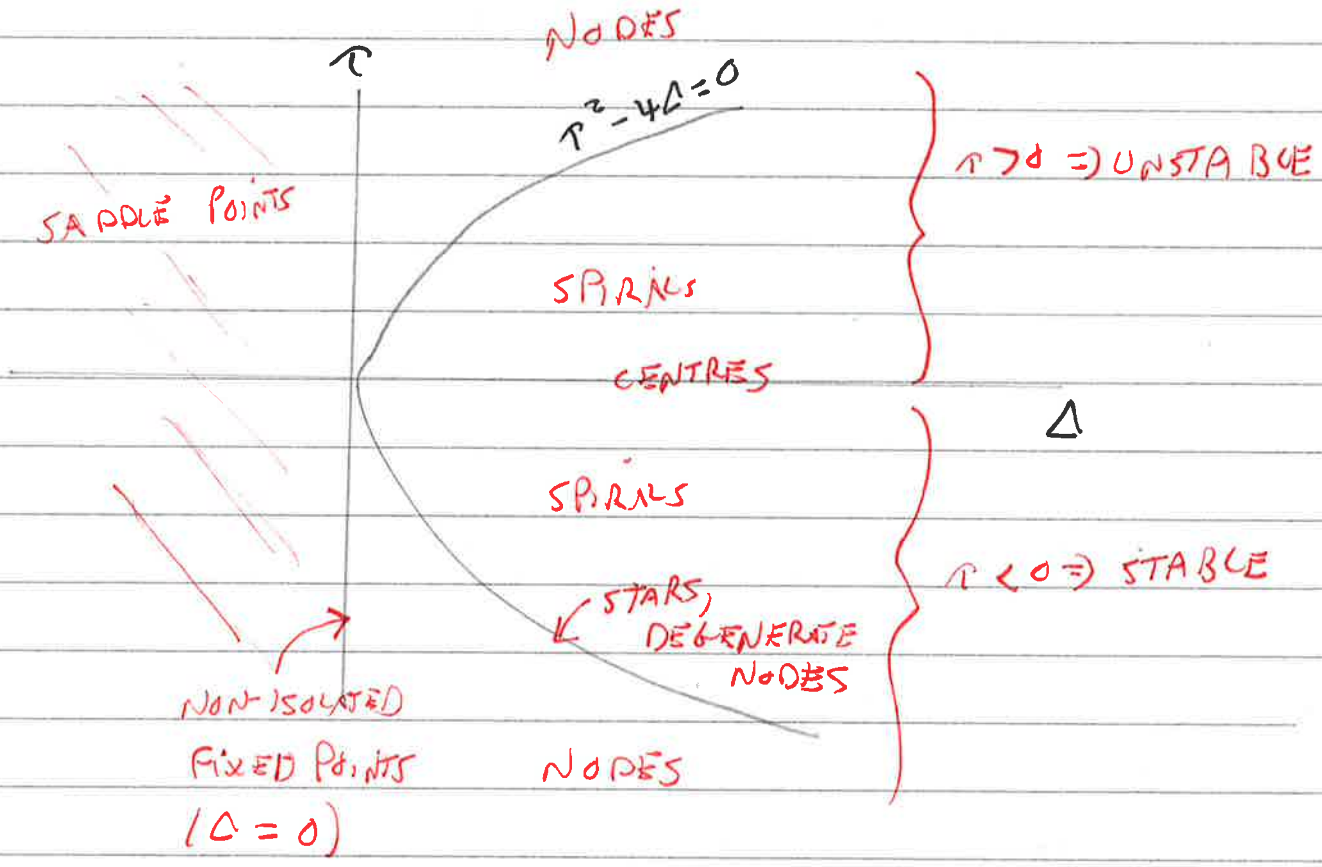
$\therefore \lambda^2 - \tau\lambda + \Delta = 0$ with solutions: $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$

So, the two eigenvalues are:

$\underline{\lambda_1 = \dfrac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}}$, $\underline{\lambda_2 = \dfrac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}}$

Because $\underset{=}{M}$ is a real matrix, $\tau = a + d \in \mathbb{R}$ so eigenvalues must be real or complex conjugates if $\tau^2 < 4\Delta$

$$\boxed{\begin{aligned} \tau &= \lambda_1 + \lambda_2 \\ \Delta &= \lambda_1 * \lambda_2 \end{aligned}}$$



NODES
τ
$\tau^2 - 4\Delta = 0$
$\tau > 0 \Rightarrow$ UNSTABLE
SADDLE POINTS
SPIRALS
CENTRES
Δ
SPIRALS
$\tau < 0 \Rightarrow$ STABLE
STARS, DEGENERATE NODES
NON-ISOLATED FIXED POINTS ($\Delta = 0$)
NODES

Recipe for 2D Linear System

1. Given the equation: $\underline{\dot{x}} = \underline{\underline{M}}\,\underline{x}$ with $\underline{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

2. find $\tau = \text{Trace } \underline{\underline{M}}$
   $\Delta = \det \underline{\underline{M}}$

3. Draw the τ-Δ plot, add the quadratic equation $\tau^2 - 4\Delta = 0$ to the rhs

4. Plot τ, Δ on the graph, and read off the type and stability of the fixed point.
   (stop if all you need is the FP)

5. Find the eigenvalues, eigenvectors of $\underline{\underline{M}}$.

6. on the phase portrait, draw the nullclines $\dot{x} = 0$, $\dot{y} = 0$, note where they cross
   (Fixed point is always at $(0,0)$ for linear system.)

7. Draw the eigenvectors on the phase portrait, and sketch the vector field.
   (stop here for qualitative solution)

8. write the general solution as: $\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{v}_1 + c_2 e^{\lambda_2 t} \underline{v}_2$

   and find c_1, c_2 from the initial condition $(x_0, y_0) = c_1 \underline{v}_1 + c_2 \underline{v}_2$