Introduction to Quantum Science and Technology

Final exam	Assignment date: January 17, 2023, 9h15
Fall term 2022	Due date: January 17, 2023, 12h15

QUANT 400 – Exam problems – Course edition 2022

These are 6 problems given in the exam of the 2022 edition. In the 2023 edition there will be 5 problems based on the material taught this semester.

Problem 1. *Module 1 - Zoe Holmes*

(a) The Hadamard (H) and Phase (S) gates takes the form

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Show that,

i. $X = HZH$ and $Y = SHZHS^\dagger$, where X, Y and Z are Pauli operators. (2 marks)

ii. Hence describe how to measure in the X basis and Y basis on a quantum computer that can only measure in the computational (Z) basis. (2 marks)

(b) i. The circuit shown below is called the Hadamard test. Show that it can be used to measure $\Re[\langle\psi|U|\psi\rangle]$. (3 marks)

ii. How can this circuit be modified to measure $\Im[\langle\psi|U|\psi\rangle]$? (Please draw your proposed circuit). (3 marks)

(c) Suppose you know how to prepare $|\phi_1\rangle$ and $|\phi_2\rangle$, i.e. you know circuits R_{ϕ_1} and R_{ϕ_2} such that $R_{\phi_1}|0\rangle = |\phi_1\rangle$ and $R_{\phi_2}|0\rangle = |\phi_2\rangle$, but are interested in computing $\langle\chi|M|\chi\rangle$ for $|\chi\rangle = \alpha|\phi_1\rangle + \beta|\phi_2\rangle$ (for any real α and β) where M is an arbitrary observable.

i. Show that $\langle\chi|M|\chi\rangle = \alpha^2\langle\phi_1|M|\phi_1\rangle + \beta^2\langle\phi_2|M|\phi_2\rangle + 2\alpha\beta\,\Re[\langle\phi_2|M|\phi_1\rangle]$. (1 mark)

ii. How could you compute the terms $\langle\phi_1|M|\phi_1\rangle$ and $\langle\phi_2|M|\phi_2\rangle$ on a quantum computer? You may assume you know a circuit R_M to rotate M into the computational basis, e.g. $M = R_M Z R_M^\dagger$. (1 mark)

iii. How could you compute $\Re[\langle\phi_2|M|\phi_1\rangle]$ on a quantum computer? You may assume that you know how to decompose M into a sum of unitary operators, e.g. $M = \sum_j c_j U_j$ where the $\{c_j\}$ are scalars and the $\{U_j\}$ are a set of unitaries. (3 marks).

iv. Hence sketch the circuit diagrams required to compute $\langle\chi|M|\chi\rangle$ when $|\phi_1\rangle = |{+}{+}\rangle$, $|\phi_2\rangle = |00\rangle$, $M = Z \otimes Y$, $\alpha = 2$ and $\beta = 7$. (No need to decompose the required gates into basic single and two qubit gates). (3 marks)

(d) Suggest a possible application of the Hadamard test and comment on the suitability of the Hadamard test for near-term quantum computers. (2 marks)

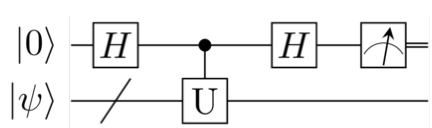

Figure 1: Hadamard Test

Solution to Problem 1:

(a) i.

$$HZH = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = X\,.$$

(1 mark)

$$SHZHS^\dagger = SXS^\dagger = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = Y\,.$$

(1 marks)

ii. $\langle\psi|X|\psi\rangle = \langle\psi|HZH|\psi\rangle$. Hence, it is possible to measure in the X basis by first applying H and then measuring in the computational basis. (1 mark)

Similarly, $\langle\psi|Y|\psi\rangle = \langle\psi|SHZHS^\dagger|\psi\rangle$. It is possible to measure in the Y basis by first applying the *inverse* of the phase gate $S^\dagger$, then H and then measuring in the computational basis. (1 mark)

(b) i. $|0\,\psi\rangle \to |+\psi\rangle \to \frac{1}{\sqrt{2}}(|0\,\psi\rangle + U|1\,\psi\rangle) := |\psi_{\rm out}\rangle$. (1 mark)

The circuit measures

$$\begin{aligned}\langle\psi_{\rm out}|X|\psi_{\rm out}\rangle &= \frac{1}{2}\big(\langle 0|1\rangle + \langle 1|0\rangle + \langle 0|0\rangle\langle\psi|U|\psi\rangle + \langle 1|1\rangle\langle\psi|U^\dagger|\psi\rangle\big)\\ &= \frac{1}{2}\big(\langle\psi|U|\psi\rangle + \langle\psi|U^\dagger|\psi\rangle\big) = \Re[\langle\psi|U|\psi\rangle]\end{aligned}$$

(2 marks)

ii. The trick is to work the previous calculation backwards.

$$\begin{aligned}\Im[\langle\psi|U|\psi\rangle] &= \frac{1}{2}\big(\langle\psi|U|\psi\rangle - \langle\psi|U^\dagger|\psi\rangle\big)\\ &= \frac{1}{2}\big(\langle 0|1\rangle + \langle 1|0\rangle + \langle 0|0\rangle\langle\psi|U|\psi\rangle - \langle 1|1\rangle\langle\psi|U^\dagger|\psi\rangle\big) = \langle\psi_{\rm out}|M|\psi_{\rm out}\rangle\end{aligned}$$

with

$$M = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = -iY\,.$$

(2 marks)

Hence it is possible to measure $\Im[\langle\psi|U|\psi\rangle]$ by applying H to the ancilla and then $C-U$ as in the original Hadamard test but then measuring Y. That is, running the standard Hadamard test but replacing the final Hadamard gate with $S^\dagger H$. (1 marks).

(c) i. By basic linear algebra we have

$$\begin{aligned}\langle\chi|M|\chi\rangle &= \alpha^2\langle\phi_1|M|\phi_1\rangle + \beta^2\langle\phi_2|M|\phi_2\rangle + 2\Re[\alpha\beta\langle\phi_2|M|\phi_1\rangle] \\ &= \alpha^2\langle\phi_1|M|\phi_1\rangle + \beta^2\langle\phi_2|M|\phi_2\rangle + 2\alpha\beta\,\Re[\langle\phi_2|M|\phi_1\rangle]\end{aligned}$$

(1 mark)

ii. The term $\langle\phi_1|M|\phi_1\rangle$ can be measured simply by preparing $|\phi_1\rangle$ and then measuring M (e.g. by applying $R_M^\dagger$ then measuring in the computational basis). Measuring the term $\langle\phi_2|M|\phi_2\rangle$ is entirely analogous. (1 mark)

iii. To see how to measure the term $\Re[\langle\phi_2|M|\phi_1\rangle]$ note that

$$\Re[\langle\phi_2|M|\phi_1\rangle] = \sum_j c_j \Re[\langle 0|R_{\phi_2}^\dagger U_j R_{\phi_1}|0\rangle]\,.$$

Each of the terms $\Re[\langle 0|R_{\phi_2}^\dagger U_j R_{\phi_1}|0\rangle]$ can be measured using a Hadamard test. (3 marks.)

ii. The circuits that need to be run in this case are:

$(I \otimes HS^\dagger)(H \otimes H)|00\rangle$,

$(I \otimes HS^\dagger)|00\rangle$,

$(H \otimes I \otimes I)\,\mathrm{C}{-}(HZ \otimes HY)\,(H \otimes I \otimes I)|000\rangle$

(followed by computational basis measurements). Then we can compute $\langle\chi|M|\chi\rangle$ using post processing.

(3 marks)

(d) Potential applications: Computing the inner product between a pair of states. Simulating a linear combination of quantum states, this could be useful to model more complex quantum states that are hard to prepare on a quantum computer or to simulate non-physical states (e.g. non-normalized states). (Or anything sensible). (1 mark)

It only requires one ancilla unitary but implementing controlled unitaries such as $C-U$ can require a large number of two qubit gates (depending on the depth/structure of U). (Or other sensible comments demonstrating an understanding of the constraints of near-term hardware). (1 mark)

Problem 2. *Module 1 - Nicolas Macris*

The problem has two independent parts.

Part 1: imperfect BB84. (10 marks) We let $|\alpha\rangle = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle$, $|\alpha_\perp\rangle = -\sin\alpha\,|0\rangle + \cos\alpha\,|1\rangle$, and $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ the Hadamard matrix. Alice and Bob use the BB84 protocol to generate a one-time pad, *but they do not know that their respective encoding and decoding basis are misaligned by a (small) angle* α. We remind the first steps of the protocol:

- Alice generates two independent random bit sequences x_i and e_i, $i = 1, \dots, n$. Each sequence is i.i.d with uniform probabilities 1/2 for each bit. If $e_i = 0$ she sends (to Bob) a qubit in state $|x_i\rangle$. If $e_i = 1$ she sends a qubit in state $H\,|x_i\rangle$.
- Bob generates a random sequence $d_1 \dots d_N$ of i.i.d bits with uniform probabilities 1/2. If $d_i = 0$ Bob measures the received state with the basis $|\alpha\rangle$, $|\alpha_\perp\rangle$, and if $d = 1$ he measures with the basis $H\,|\alpha\rangle$, $H\,|\alpha_\perp\rangle$. When measurements output $|\alpha\rangle$ or $H\,|\alpha\rangle$ he registers $y_i = 0$, and when measurements output $|\alpha_\perp\rangle$ or $H\,|\alpha_\perp\rangle$ he registers $y_i = 1$.

(a) Compute $\mathbb{P}(x_i = y_i | e_i = d_i = 0, x_i = 0)$; $\mathbb{P}(x_i = y_i | e_i = d_i = 0, x_i = 1)$; and $\mathbb{P}(x_i = y_i | e_i = d_i = 1, x_i = 0)$; $\mathbb{P}(x_i = y_i | e_i = d_i = 1, x_i = 1)$. Deduce $\mathbb{P}(x_i = y_i | e_i = d_i)$

(b) Explain the rest of the protocol and in particular explain how Alice and Bob can evaluate the misalignment of their basis thanks to the security test (assuming for some reason they know the protocol is noiseless and there is no eavesdropper).

Part 2: imperfect dense coding. (10 marks) Alice and Bob use dense coding to communicate 2 classical bits. *But they dont know they share state* $|S\rangle = \frac{1}{\sqrt{3}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B + |0\rangle_A \otimes |1\rangle_B)$ *instead of the usual Bell state.* We recall the steps used by Alice and Bob:

- In order to send message $ij \in \{00, 01, 10, 11\}$ Alice applies the unitary $Z_A^i X_A^j$ to her qubit and then sends it to Bob. We recall $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- Once in possession of the pair Bob performs a measurement in the Bell basis $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle$, $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle$. Bob's decoding map is $|\Psi^+\rangle \to 00$, $|\Phi^+\rangle \to 01$, $|\Psi^-\rangle \to 10$, $|\Phi^-\rangle \to 11$.

(a) Prove that $|S\rangle$ is an *entangled* state.

(b) For each message ij, what is the state of the pair of qubits in Bob's lab *just after* receiving Alice's qubit?

(c) Suppose now Alice intends to send message 10. What are the possible measurement outcomes of Bob and their respective probabilities?

(d) Still assuming Alice's message is 10, what is the probability of a transmission error?

Solution to Problem 2:

Part 1

(a)
- For $e_i = 0$ Alice sends qubits $|x_i\rangle = |0\rangle, |1\rangle$. And for $d_i = 0$ Bob's measurement basis is $|\alpha\rangle, |\alpha_\perp\rangle$. Thus applying Born's rule:

$$P(x_i = y_i | e_i = d_i = 0, x_i = 0) = |\langle\alpha| |0\rangle|^2 = (\cos\alpha)^2$$
$$P(x_i = y_i | e_i = d_i = 0, x_i = 1) = |\langle\alpha_\perp| |1\rangle|^2 = (\cos\alpha)^2$$

- Similarly, for $e_i = 1$ Alice sends qubits $H|x_i\rangle = |+\rangle, |-\rangle$. And for $d_i = 1$ Bob's measurement basis is $H|\alpha\rangle, H|\alpha_\perp\rangle$. Thus applying Born's rule:

$$P(x_i = y_i | e_i = d_i = 1, x_i = 0) = |\langle\alpha| H^\dagger H |0\rangle|^2 = (\cos\alpha)^2$$
$$P(x_i = y_i | e_i = d_i = 1, x_i = 1) = |\langle\alpha_\perp| H^\dagger H |1\rangle|^2 = (\cos\alpha)^2$$

Therefore since $P(x_i = 0) = P(x_i = 1) = 1/2$

$$P(x_i = y_i | e_i = d_i = 0) = \frac{1}{2}(\cos\alpha)^2 + \frac{1}{2}(\cos\alpha)^2 = (\cos\alpha)^2$$

and

$$P(x_i = y_i | e_i = d_i = 1) = \frac{1}{2}(\cos\alpha)^2 + \frac{1}{2}(\cos\alpha)^2 = (\cos\alpha)^2$$

Finally,

$$\begin{aligned} P(x_i = y_i | e_i = d_i) &= P(x_i = y_i | e_i = d_i, e_i = 0)P(e_i = 0) + P(x_i = y_i | e_i = d_i, e_i = 1)P(e_i = 1) \\ &= \frac{1}{2}(\cos\alpha)^2 + \frac{1}{2}(\cos\alpha)^2 = (\cos\alpha)^2 \end{aligned}$$

We check that when $\alpha = 0$ this probability is one (ideal BB84). Other sanity checks are for $\alpha = \pi/2$ this prob is zero and for $\alpha = \pi/4$ this prob is 1/2 (should be intuitive).

(b) Once the quantum communication and measurement phases are finished, Alice and Bob reveal on a public channel their basis choices e_i and d_i. Each time $e_i \neq d_i$ they discard bits x_i and y_i. The set P of other bits forms their one-time pad. They check agreement $x_i = y_i$ with security test: they reveal on a public channel a small fraction of bits in P and (assuming the protocol is noiseless and there is no eavesdropper) the fraction of tested bits which agree is $(\cos\alpha)^2 \approx 1 - \alpha^2/2$ (for α small). From this fraction they get an estimate of α.

Part 2

(a) Proof by contradiction: suppose $|S\rangle$ is not entangled:

$$|S\rangle = \big(a\,|0\rangle + b\,|1\rangle\big) \otimes \big(c\,|0\rangle + d\,|1\rangle\big) = ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle\,.$$

Then $ac = ad = bd = \frac{1}{\sqrt{3}}$ and $bc = 0$. This implies $b \neq 0$ since $bd \neq 0$; and $c = 0$ since $b \neq 0$ and $bc = 0$. But then $ac = 0$ and since we also have $ac = \frac{1}{3}$ we find a contradiction. In conclusion $|S\rangle$ must be entangled.

(b) Here are the states in Bob's lab once he receives Alice's qubit:

- Alice's message 00: Bob gets $|S\rangle = \frac{1}{\sqrt{3}}\big(\,|00\rangle + |11\rangle + |01\rangle\,\big)$
- Alice's message 01: Bob gets $X\,|S\rangle = \frac{1}{\sqrt{3}}\big(\,|10\rangle + |01\rangle + |11\rangle\,\big)$
- Alice's message 10: Bob gets $Z\,|S\rangle = \frac{1}{\sqrt{3}}\big(\,|00\rangle - |11\rangle + |01\rangle\,\big)$
- Alice's message 11: Bob gets $ZX\,|S\rangle = \frac{1}{\sqrt{3}}\big(-|10\rangle + |01\rangle - |11\rangle\,\big)$

(c) Alice sends 10 so Bob possesses the pair

$$Z\,|S\rangle = \frac{1}{\sqrt{3}}\big(\,|00\rangle - |11\rangle + |01\rangle\,\big).$$

The measurement outcomes and probabilities for Bob are

- $|\Psi^+\rangle$ with probability $|\,\langle\Psi^+|\,Z\,|S\rangle\,|^2 = \frac{1}{2}\cdot\frac{1}{3}(1-1)^2 = 0$
- $|\Psi^-\rangle$ with probability $|\,\langle\Psi^-|\,Z\,|S\rangle\,|^2 = \frac{1}{2}\cdot\frac{1}{3}(1+1)^2 = \frac{2}{3}$
- $|\Phi^+\rangle$ with probability $|\,\langle\Phi^+|\,Z\,|S\rangle\,|^2 = \frac{1}{2}\cdot\frac{1}{3}(1)^2 = \frac{1}{6}$
- $|\Phi^-\rangle$ with probability $|\,\langle\Phi^-|\,Z\,|S\rangle\,|^2 = \frac{1}{2}\cdot\frac{1}{3}(1)^2 = \frac{1}{6}$

Note that probabilities indeed sum to one.

(d) According to the decoding map of Bob:

$$P(\text{transmission error}) = P(\text{Bob doesnt get }\Psi^-) = 0 + \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Problem 3. *Module 2 - Giuseppe Carleo*

Consider the following Hamiltonian defined on two qubits:

$$\hat{\mathcal{H}} = \hat{\sigma}_1^x \hat{\sigma}_2^x + \hat{\sigma}_1^y \hat{\sigma}_2^y,$$

where we use the standard definition for the Pauli matrices:

$$\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. Write $\hat{\mathcal{H}}$ as a 4x4 matrix in the standard basis $|0,0\rangle$, $|0,1\rangle$, $|1,0\rangle$, $|1,1\rangle$. Find the smallest eigenvalue and the corresponding eigenstate.

2. Consider the problem of simulating quantum dynamics starting from the initial state $|0,0\rangle$ using a quantum computer. Show that the FSIM(θ, ϕ) gate (as implemented on Google hardware, for example) can be used to obtain $|\psi(t)\rangle = \exp(-it\hat{\mathcal{H}})|0,0\rangle$. Determine what values of θ and ϕ are necessary. Recall that the FSIM gate is defined as

$$\mathrm{FSIM}(\theta, \phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -i\sin(\theta) & 0 \\ 0 & -i\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{bmatrix}$$

3. Now consider the problem of approximating the ground state of $\hat{\mathcal{H}}$ using a variational ansatz. We will consider the ansatz

$$|\Psi(\gamma_1, \gamma_2)\rangle = \mathrm{CNOT} \times \mathrm{RY}_1(\gamma_1) \times \mathrm{RY}_2(\gamma_2)|0,0\rangle$$

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\mathrm{RY}(\gamma) = \exp(-i\hat{\sigma}^y\gamma/2) = \begin{bmatrix} \cos(\gamma/2) & -\sin(\gamma/2) \\ \sin(\gamma/2) & \cos(\gamma/2) \end{bmatrix}$$

Find the expression of the energy as a function of the parameters γ_1 and γ_2. For what values of the parameters do you recover the exact ground-state energy computed at point 1?

Solution to problem 3:

1. We have that

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   The matrix has a non-zero block among the states $|0,1\rangle$ and $|1,0\rangle$. In this subspace, it coincides with $2\hat{\sigma}^x$, thus we have two eigenvalues, -2 and 2. The eigenvalue -2 thus corresponds to the eigenstate $\frac{|0,1\rangle - |1,0\rangle}{\sqrt{2}}$.

2. We have that

$$\exp(-it\hat{\mathcal{H}}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2t) & -i\sin(2t) & 0 \\ 0 & -i\sin(2t) & \cos(2t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   thus it can be simulated with $\mathrm{FSIM}(\theta = 2t, \phi = 0)$.

3. The expected value of the energy reads

$$E(\gamma_1, \gamma_2) = 2\sin(\gamma_1/2)^2 \sin(\gamma_2),$$

   thus it attains the ground-state energy for $E(\gamma_1 = \pi, \gamma_2 = 3/2\pi) = -2$.

Problem 4. *Module 2 - Pasquale Scarlino*

<u>**Superconducting qubits**</u> (*10 marks*)

1. **General questions** (try to give short answers and go straight to the point)

   (a) Why can't we use a harmonic oscillator as a qubit? (*1 mark*)

   (b) Explain briefly what is a Cooper Pair Box (CPB) qubit. Draw its circuit schematic and the profile of the two lowest energy levels. (*1 mark*)

   (c) What is the main issue of the CPB qubit? How can we partially improve its coherence time? (*1 mark*)

2. **Exercise**
   Single-qubit gates are generally performed by driving a qubit (for example with a laser) close to resonance. A drive with Rabi frequency Ω is detuned by δ from the qubit transition. The resulting Hamiltonian (in the rotating frame of the drive) is given by:

$$\tilde{H} = \begin{pmatrix} \frac{1}{2}\hbar\delta & \frac{1}{2}\hbar\Omega \\ \frac{1}{2}\hbar\Omega & -\frac{1}{2}\hbar\delta \end{pmatrix}$$

   The corresponding propagator is given by:

$$\tilde{U} = \exp[-i \begin{pmatrix} \frac{1}{2}\delta & \frac{1}{2}\Omega \\ \frac{1}{2}\Omega & -\frac{1}{2}\delta \end{pmatrix} t]$$

   which can be simplified to

$$\tilde{U} = \begin{pmatrix} \cos(\Omega_\delta t/2) - i(\delta/\Omega_\delta)\sin(\Omega_\delta t/2) & -i(\Omega/\Omega_\delta)\sin(\Omega_\delta t/2) \\ -i(\Omega/\Omega_\delta)\sin(\Omega_\delta t/2) & \cos(\Omega_\delta t/2) + i(\delta/\Omega_\delta)\sin(\Omega_\delta t/2) \end{pmatrix}$$

   where $\Omega_\delta = \sqrt{\Omega^2 + \delta^2}$ is known as the generalised Rabi frequency.

   (a) An initial state $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ evolves under $\tilde{H}$. Calculate the probability $P_1(t, \delta)$ to find the system in the state $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ after a time t. (*2 marks*)

   Hint: $P_1(t, \delta) = \left| \langle 1| \tilde{U} |0\rangle \right|^2$

   (b) Plot $P_1(t, \delta)$ for $\delta = 0$. At what time t is P_1 maximum ? (*1 mark*)

   (c) For any δ, what is the maximum value P_1 can achieve? What is the name of this probability distribution? Draw it and compute its full-width at half-maximum (FWHM). (*2 marks*)

(d) We wish to use this drive to apply a single-qubit X gate, which flips $|0\rangle \to |1\rangle$. A laser is available which has a sufficient intensity to give a Rabi frequency $\Omega = 2\pi \times 1$ MHz for the transition. The qubit can be prepared in the lower state with very close to 100% reliability. How precisely must the laser frequency be tuned to the atomic resonance if the final population of state $|1\rangle$ is above 99.99%.
Hint: Use the result of the previous questions. (*2 marks*)

Spin Qubits in Semiconductor Quantum Dots (*10 marks*)

1. **General Questions** (try to give short answers and go straight to the point)

   (a) What is a 2-dimension electron gas (2-DEG)? Illustrate briefly at least one solid state platform that can host a 2-DEG. (*1 mark*)

   (b) Suppose the 2-DEG is self-accumulated in such a platform. This means that no (positive) voltages are needed to accumulate the 2-DEG, which is already present. What is needed to confine electrons in all the 3 dimensions?
   Sketch a possible design (top view) of a device where electrons can be confined in a single Quantum Dot (QD). (*1 mark*)

   (c) Illustrate at least one possible computational basis which can be used to implement a spin qubit in semiconductor QDs. To operate with QDs as spin qubits is necessary to locally tune the energy of the confined electrons. How can this aspect be implemented experimentally? Refer also to the sketch of the previous point. (*1 mark*)

2. **Exercise**
   During the exercise session we ended up with two useful expressions for the total change in *electrostatic energy*, respectively when injecting one electron from the source into the dot and from the dot into the drain (for $V_{DS} > 0$):

$$\Delta E_{tot,S\to dot} = \frac{e}{C_{dot}}((N + \frac{1}{2})e - C_G V_G - C_2 V_{DS}),$$
$$\Delta E_{tot,dot\to D} = \frac{e}{C_{dot}}(-(N - \frac{1}{2})e + C_G V_G - (C_1 + C_G) V_{DS}),$$

   where C_G, C_1, C_2, C_{dot} are respectively the gate, dot-to-source, dot-to-drain and total dot capacitance, V_G and V_{DS} the gate and drain-to-source (bias) voltage, e the electron charge (absolute value) and N the number of electrons inside the dot.

   (a) From the two expressions above, find for which values of V_{DS} is energetically favorable to inject one electron from the source into the dot and from the dot into the drain, as a function of V_G, N and all the different capacitances. (*2 marks*)

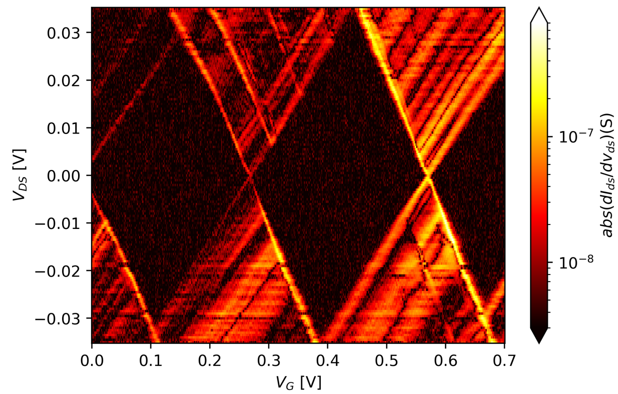

Figure 2: Charge stability diagram of a single QD for an arbitrary number of electrons N.

(b) Which energy contributions from the two equations above represent the variation of *charging energy* ΔE_C? Find an expression for the (N-independent) *charging energy* E_C. (*2 marks*)

(c) Focus now on one diamond from the ones shown in Figure 2. Estimate the charging energy and the lever arm α of the gate V_G. *Remember*: $\alpha = e\frac{C_G}{C_{dot}}$. (*1 mark*)

Now, a static external magnetic field is applied along the z-direction to split the electrons energy levels into two sub-levels spin-up and -down. The Hamiltonian of an electron inside a magnetic field can be written as follows:

$$H = g\mu_B \vec{B} \cdot \vec{S},$$

where $\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$ is the magnetic field and $\vec{S} = \frac{1}{2}\begin{pmatrix} \hat{\sigma_x} \\ \hat{\sigma_y} \\ \hat{\sigma_z} \end{pmatrix}$ the spin matrices.

(d) In order to manipulate the qubit state, a time-varying magnetic field with frequency ω and phase ϕ is applied. The AC magnetic field is directed along the x-direction, whereas the static one always along z. Write down the full Hamiltonian of the system in matrix form (in the lab frame). (*1 mark*)

(e) In the rotating frame of the drive, the Hamiltonian reads exactly as the one in the Exercise of the superconducting part. At resonance, find the expression of the time length of the pulse to apply to flip the population from ground to excited state. (*1 mark*)

Solution to problem 4: **Superconducting qubits**

1. **General questions**

   (a) Why can't we use a harmonic oscillator as a qubit? (*1 mark*)

   (b) Explain briefly what is a Cooper Pair Box (CPB) qubit. Draw its circuit schematic and the profile of the two lowest energy levels. (*1 mark*)

   (c) What is the main issue of the CPB qubit? How can we partially improve its coherence time? (*1 mark*)

2. **Exercise**

   (a) The probability is defined as, $P_1(t,\delta) = \left|\langle 1|\tilde{U}|0\rangle\right|^2$, we can simply brute-force apply it:

$$\begin{aligned}
P_1(t,\delta) &= \left|\langle 1|\tilde{U}|0\rangle\right|^2 \\
&= \left|\langle 1| \begin{pmatrix} \cos(\Omega_\delta t/2) - i(\delta/\Omega_\delta)\sin(\Omega_\delta t/2) & -i(\Omega/\Omega_\delta)\sin(\Omega_\delta t/2) \\ -i(\Omega/\Omega_\delta)\sin(\Omega_\delta t/2) & \cos(\Omega_\delta t/2) + i(\delta/\Omega_\delta)\sin(\Omega_\delta t/2) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2 \\
&= \left|\langle 1| \begin{pmatrix} \cos(\Omega_\delta t/2) - i(\delta/\Omega_\delta)\sin(\Omega_\delta t/2) \\ -i(\Omega/\Omega_\delta)\sin(\Omega_\delta t/2) \end{pmatrix}\right|^2 \\
&= \left|\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\Omega_\delta t/2) - i(\delta/\Omega_\delta)\sin(\Omega_\delta t/2) \\ -i(\Omega/\Omega_\delta)\sin(\Omega_\delta t/2) \end{pmatrix}\right|^2 \\
&= \left|-i(\Omega/\Omega_\delta)\sin(\Omega_\delta t/2)\right|^2 \\
&= (\Omega/\Omega_\delta)^2 \sin^2(\Omega_\delta t/2)
\end{aligned}$$

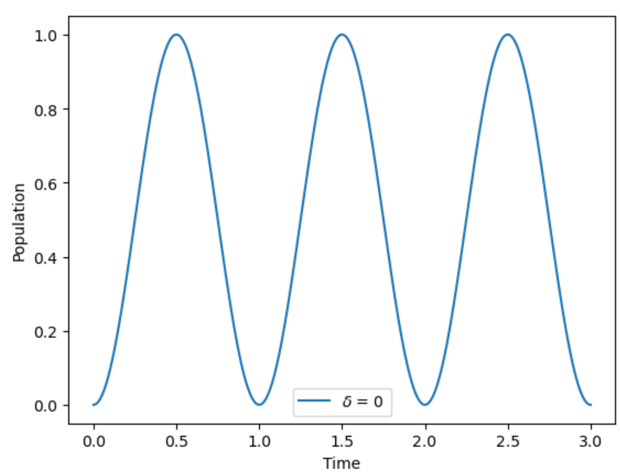

Figure 3: Rabi oscillations at $\delta = 0$

(b) At resonance, $P_1(t, 0) = \sin^2(\Omega t/2)$. $P_1(t)$ is maximum when, $t = (n+1)\frac{\pi}{\Omega}$.

(c) The maximum value P_1 can achieve is 1. The name of this probability distribution is a Lorentzian.
!!!ADD DRAWING !!!
The Half Width at Half Maximum (HWHM), is defined as: $P_1(t, \delta) = 1/2$.
In this case, we have:

$$\begin{aligned}\frac{\Omega^2}{\Omega^2+\delta^2} &= \frac{1}{2}\\ 2\Omega^2 &= \Omega^2+\delta^2\\ \Omega^2 &= \delta^2\end{aligned}$$

Hence, the FWHM is 2Ω.

(d)

$$\begin{aligned}\frac{\Omega^2}{\Omega^2+\delta^2} &= 0.9999\\ \Omega^2(1-0.9999) &= 0.9999\delta^2\\ \Omega^2\frac{1-0.9999}{0.9999} &= \delta^2\\ \delta &= \sqrt{\frac{1-0.9999}{0.9999}}\Omega\\ \delta &= 10 \text{ kHz}\end{aligned}$$

Spin Qubits in Semiconductor Quantum Dots

1. **General questions**

(a) What is a 2-dimension electron gas (2-DEG)? Illustrate briefly at least one solid state platform that can host a 2-DEG. (*1 mark*)

(b) Suppose the 2-DEG is self-accumulated in such a platform. This means that no (positive) voltages are needed to accumulate the 2-DEG, which is already present. What is needed to confine electrons in all the 3 dimensions?
Sketch a possible design (top view) of a device where electrons can be confined in a single Quantum Dot (QD). (*1 mark*)

(c) Illustrate at least one possible computational basis which can be used to implement a spin qubit in semiconductor QDs. To operate with QDs as spin qubits is necessary to locally tune the energy of the confined electrons. How can this aspect be implemented experimentally? Refer also to the sketch of the previous point. (*1 mark*)

2. **Exercise**

(a)

$$\Delta E_{tot,S\to dot} < 0 \to V_{DS} > -\frac{C_G}{C_2}V_G + (N+\frac{1}{2})\frac{e}{C_2}. \quad (1)$$

$$\Delta E_{tot,dot\to D} < 0 \to V_{DS} > \frac{C_G}{C_1+C_G}V_G - (N-\frac{1}{2})\frac{e}{C_1+C_G}. \quad (2)$$

(b) The charging energy is represented by the two terms: $\Delta E_{C,S\to dot} = \frac{e^2}{C_{dot}}(N+\frac{1}{2})$ and $\Delta E_{C,dot\to D} = -\frac{e^2}{C_{dot}}(N-\frac{1}{2})$, which indicate the energy to be overcome in order to respectively inject one extra electron from the source into the dot, if N electrons are already present into the dot, and to inject one electron from the dot into the drain (always if N electrons are present into the dot). This is due to Coulomb repulsion and these energies are provided by the two generators V_G and V_{DS}, which constitute the remaining terms of $\Delta E_{tot,S\to dot}$ and $\Delta E_{tot,dot\to D}$.

$$E_C = \Delta E_{C,S\to dot}(N+1) - \Delta E_{C,S\to dot}(N) = \frac{e^2}{C_{dot}}. \quad (3)$$

(Or alternatively: $E_C = \Delta E_{C,dot\to D}(N) - \Delta E_{C,dot\to D}(N-1) = \frac{e^2}{C_{dot}}$).

(c) The charging energy can be read directly from the height of one of the *Coulomb diamonds*. In fact, the energy provided by the voltage generator V_{DS} is converted directly in energy for the electrons in the dot: $E_C = eV_{DS}$. If students do not remember this from the exercise class, they can easily verify it by expressing Eq.(1) and (2) as a function of V_G and then equating them. On the other hand, the energy provided by the voltage generator V_G is converted in electrons energy through the *lever arm* $\alpha = e\frac{C_G}{C_{dot}} = \frac{e^2/\Delta V_G}{e^2/E_C} = \frac{E_C}{\Delta V_G}$.
Focusing on the central diamond gives:
$E_C \approx e \cdot 30\ mV \approx 30\ meV$ and $\alpha \approx \frac{30\ meV}{300\ mV} \approx 0.1\ meV/mV$.

(d)

$$H = g\mu_B \vec{B}\cdot\vec{S} = \frac{1}{2}g\mu_B(B_z\sigma_z + B_x cos(\omega t+\phi)\sigma_x) =$$
$$= \frac{1}{2}g\mu_B \begin{pmatrix} B_z & B_x cos(\omega t+\phi) \\ B_x cos(\omega t+\phi) & -B_z \end{pmatrix}.$$

(e) Since in the rotating frame of the drive the Hamiltonian reads exactly like the one of the superconducting part, we can recycle the result of the previous part and write directly that, at zero detuning δ, the probability of being in the excited state is maximum when: $\Omega t = \pi$. From which:

$$t = \frac{\pi}{\Omega} = \frac{\pi}{\frac{1}{2}\frac{g\mu_B B_x}{\hbar}} = \frac{h}{g\mu_B B_x}. \quad (4)$$

Problem 5. *Module 3 - Adrian Ionescu*

Questions 1 and 2 below are "Multiple Choice Questions" where you have to select the correct statements. Every good answer selection is +1pt and every wrong answer selection is -1pt. In each question, if the selected bad answers are more than the good ones, you will get 0 (zero) points. Your final score is then calibrated on a total of 10+10 = 20 marks.

Question 1: Single Electron Transistors (SET) are a discrete charge tunneling devices using a conductive nanodot as central island and three other electrodes as source, drain and gate to control the tunneling barriers. Choose the correct properties and characteristics of this device among the following statements.

1. The Coulomb gap in the output characteristics, $I_d - V_d$, of a SET transistor correspond to the region of drain voltages where the device is in off state; this region of drain voltages is dependent on the applied gate voltage;
2. The same SET device can have both positive and negative transconductance ($g_m = dI_d/dV_g$), depending only on the value of applied gate voltage, at same drain voltage value, therefore one can built a complementary logic using the same device for the equivalent n- and p-type as for instance, building the functionality of an inverter;
3. The background charge effect on SET can affect the periodicity of the SET transfer characteristics.
4. The theoretical R-SET (R=resistive) device, exploiting a stronger electrical potential coupling of the gate to the island, is expected to have increased background charge effect.
5. Under the effect of Coulomb blockade, the transfer characteristics, $I_d - V_g$, of a SET transistor are periodic, with a period equal to e^2/C_Σ, where C_Σ is the total SET capacitance to the ground and e is the elementary charge.
6. If one is using a metallic central island of 1 nm radius surrounded by three metal electrodes (gate, source and drain) with similar radius size, the resulting SET shows Coulomb blockade is expected to be effective at room temperature ($T = 300K$) to build a SET inverter.
7. The intrinsic frequency operation (dictating the speed of the intrinsic device) of a SET can be higher that GHz because of the very fast tunneling processes.
8. An inverter based on two SETs is consuming both static and dynamic power, which is dependent on temperature.
9. One can engineer with controlled strain induced by thermal oxidation silicon gated nanowires to build SET devices without physical oxide barriers between the central island and the drain and source contacts.

10. SETMOS is hybrid equivalent device, made out of a SET and MOSFET achieving high peaks of the current (micro-Amps) with periodic $I_d - V_g$ transfer characteristics due to Coulomb blockade.

Question 2: Select the correct statements about the semiconducting Tunnel FETs using quantum mechanical band-to-band tunneling conduction mechanisms, from the list below.

1. Tunnel FETs inherit some technology booster (technological parameters that can improve their switch performance) from MOSFETs; among these, one can cite: use of high-k dielectrics, abrupt junctions and thinner semiconducting device bodies.
2. At low voltage and low current levels Tunnel FETs can offer higher analog amplification than MOSFET due to the smaller subthreshold slope.
3. A Tunnel FET with a silicon channel and a germanium source is a heterojunction tunneling device.
4. Trap-Assisted Tunneling (TAT) is a phenomenon that depends on temperature.
5. Trap-Assisted Tunneling (TAT) is a phenomenon that depends on the density of electrically-active traps at both at the tunneling junctions and the oxide-to-channel interfaces.
6. Trap-Assisted Tunneling (TAT) is reduce significantly when the temperature is increasing.
7. Tunnel FET can be used to design more energy efficient hybrid CMOS-TFET multi-core processor architecture. For instance, the computing tasks assigned to Tunnel FETs are the ones requiring high-performance (HP) (faster operation) specifications.
8. The leakage current, I_{off}, of Tunnel FETs decreases at cryogenic (sub-77K) temperatures.

9 In an ideal tunnel FET where the Band-to-Band Tunneling is largely predominant over the trap-Assisted tunneling, the device subthreshold slope is expected to be of 60mV/decade at room temperature (300K).

10. One can build a Single Electron Transistor using Band-to-Band tunneling junctions.

Solution to problem 5:

Problem 6. *Module 3 - Edoardo Charbon*

Di Vincenzo Criteria and the quantum stack (*10 marks*)

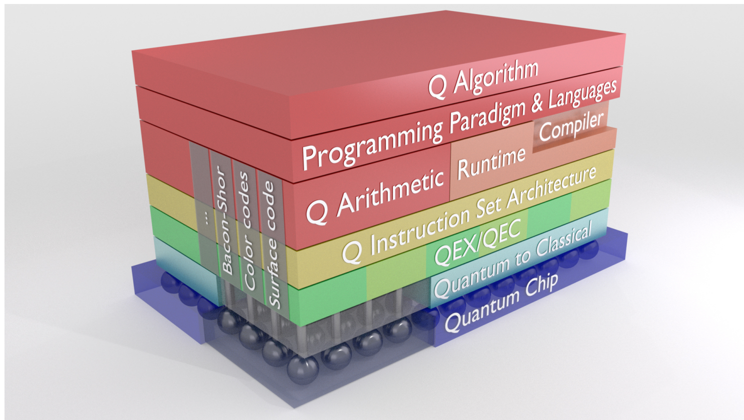

Figure 4: Quantum stack.

1. **General questions:**
    (a) Describe the first five criteria and provide an example of a quantum algorithm (1 mark)
    (b) What do you need to ensure during the algorithm's completion? (1 mark)
    (c) How do you implement a quantum algorithm in hardware using qubits of your choice? (1 mark)
    (d) What is the purpose of the QEX unit in the quantum stack? (1 mark)

2. **Exercise:** In a quantum processor, individual qubits may be accessed using a variety of techniques involving radiofrequency signals applied to the qubits through a passive circuit operating partially at high temperatures and partially at cryogenic temperatures.
    (a) Explain the components of the passive circuit and their function. Identify their preferred temperature of operation. (2 marks)
    (b) Explain the purpose and implementation of thermalization in the above scheme. (1 mark)
    (c) How do you perform simultaneous control of multiple qubits maintaining specificity? (1 mark)
    (d) Can you propose a block diagram that enables to achieve this functionality at cryogenic temperature? (1 mark)
    (e) For a total bandwidth of 1GHz how many qubits can you control individually assuming a minimum frequency separation of 100MHz between qubits? How can you relate the spectral purity to the fidelity of the control of the qubits? (1 mark)

Cryo-CMOS electronics (*10 marks*)

1. **General questions:**

   (a) What is the cause of the change in mobility in silicon as a function of temperature? (0.5 marks)

   (b) What are the other effects on transistors and why? (0.5 marks)

   (c) In digital circuits, what are the consequences of these changes? (1 mark)

   (d) In analog circuits, what are the consequences of these changes? (1 mark)

2. **Exercise:**

   (a) Write the equation of the gain in a single-ended amplifier, assuming harmonic distortion. Assume the input is sinusoidal. Write second harmonic distortion (HD_2) as a function of the harmonic components of your gain equation. (1.5 marks)

   (b) How does the THD relate to HD_2? (1 mark)

   (c) Why is intermodulation important? How do you compute IM_3 based on the components of the gain equation? (0.5 marks)

   (d) Consider the plot of Figure 5. Is the N-well resistive increase a problem for the correct behavior of a transistor? If so, what happens to an inverter based on such transistor? (1 mark)

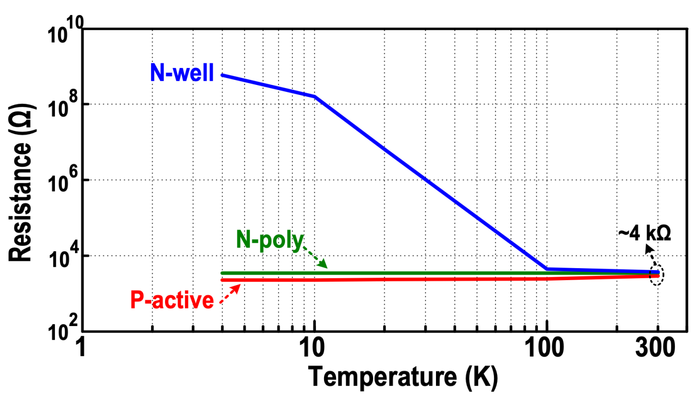

Figure 5: Plot 1.

   (e) Consider Figure 6. Identify the pros and cons for each of the two amplifier front-ends when operated at cryogenic temperature? (1 mark)

   (f) Using the same harmonic components of the gain equation, explain which component(s) will be suppressed and what consequences you expect for HD_2 and HD_3. (0.5 marks)

(g) Which of the amplifier front-ends is better suited for low IM_3? Why? (0.5 marks)

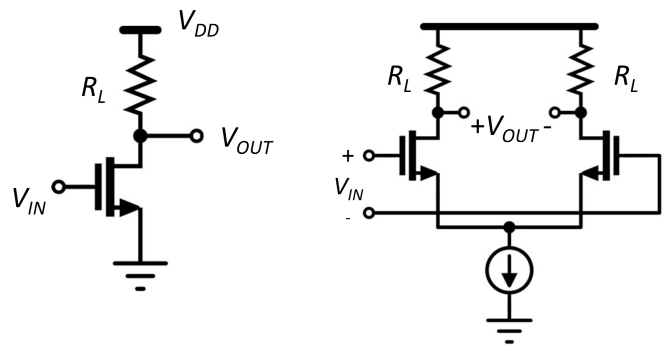

Figure 6: Amplifier front-end: single-ended (left) and differential (right).

(h) What is the dominant source of the input-referred noise in the two amplifier front-ends? Which amplifier front-end will have more noise, assuming identical sizing of the transistors? (1 mark)

Solution to problem 6: