Some Information on superconductivity

Before describing the Josephson effect, let's consider some basic properties of superconductors.

(1) Infinite conductivity. The major property of supercond. is the flow of dissipationless current. This assumes that its resistance equates to zero when the temperature is below T_c critical temperature. Typical values for critical current density is $\sim 10^6\ A/cm^2$. Niobium and Aluminum have $T_c \sim 1.2K$ and $9.2K$ respectively.

(2) Macroscopic wave function of the condensate.

The conventional superconductivity is explained with the Bardeen-Cooper-Schrieffer theory by coupling electrons, which form the so-called "Cooper pairs" with zero spin. The size of a Cooper-pair is characterized by it "coherence length" ξ_0. The aggregate of such bosons forms the Bose-Einstein condensate: all those paired electrons are situated on the same quantum level and they can be described by a single wave function

$\psi(\vec{r}) = |\psi(\vec{r})|\, e^{i\varphi(\vec{r})}$. This global wavefunction is normalized by the density of the Cooper pairs $|\psi(\vec{r})| = \sqrt{n_s}$, where n_s represents the density.

For the current density of Cooper pairs with charge $2e$ and mass $2m$, we have
$$\vec{J}_s = (2e)\left(\psi^* \frac{\vec{p} - 2e\vec{A}/c}{2(2m)}\psi + c.c.\right)$$
$$= -i\frac{e\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \frac{2e^2}{mc}|\psi|^2\vec{A} =$$
$$= 2e\,|\psi|^2\,\frac{\hbar\nabla\varphi - 2e\vec{A}/c}{2m} \equiv 2e\,n_s\,\vec{V}_s$$

This means that the phase gradient $\nabla\varphi$ defines the current in the system via the superfluid velocity $\vec{V}_s$. Here it was taken into account that the presence of a magnetic field corresponding to $\vec{p} \longrightarrow \vec{p} - (2e)\vec{A}/c$ and $\vec{p} = -i\hbar\nabla$.

So the supercurrent density is related to the phase gradient as follows: $\vec{J}_s = 2e\,n_s\,\dfrac{\hbar\nabla\varphi - 2e\vec{A}/c}{2m}$

(3) Perfect diamagnetism.

A magnetic field does not penetrate into bulk of superconductors, unlike normal metals, but it is rather expelled from them. This is related to the appearance of the surface current, of which the field shields the external magnetic field, known as "Meissner effect".

In order to understand this, let's consider a homogeneous superconductor in a weak magnetic field and in equilibrium.

From the Eq. of the superc. density, we get

$$\mathrm{rot}\,\vec{J} = -\frac{2e^2 n_s}{mc} B \qquad (\text{London equation})$$

By adding the Maxwell equations $\mathrm{rot}\,\vec{B} = \frac{4\pi}{c}\vec{J}$; $\mathrm{div}\,\vec{B} = 0$ and using that $\mathrm{rot}\,\mathrm{rot}\,\vec{B} = \nabla(\mathrm{div}\vec{B}) - \Delta\vec{B}$, we get

$$\mathrm{rot}\,\vec{J} = \frac{c}{4\pi}\,\mathrm{rot}\,\mathrm{rot}\,\vec{B} = -\frac{c}{4\pi}\Delta\vec{B} \hat{=} -\frac{2e^2 n_s}{mc}\vec{B} \Rightarrow \Delta\vec{B} = \frac{1}{\lambda^2}\vec{B}$$

with $\lambda^2 = \frac{mc^2}{8\pi e^2 n_s}$ is the so called "field penetration depth".

The magnetic field penetrates only to the depth $\sim\lambda$ ($\sim 0.1\,\mu m$).

(4) Magnetic flux quantization.

Consider a doubly connected supercond. (a ring).
Let's consider a circuit C inside the ring and integrate along it the eq. for the superc. density, we will get:

$$0 = \int_C d\vec{l}\,(\hbar \nabla\varphi - 2e\vec{A}/c)$$. For Stokes theorem

$$\int_C d\vec{l}\,\vec{A} = \int_{S_C} d\vec{S}\cdot\nabla\times\vec{A} = \vec{B}\int_S d\vec{S} = BS = \Phi \left\{ \begin{array}{l} \text{magnetic flux} \\ \text{inside the ring} \end{array} \right.$$

We obtain $\Phi = \frac{\hbar c}{2e}\int_C d\vec{l}\,\nabla\varphi$. On the other hand, the requirement of the wavefunction to be single-valued gives that over the full path-tracing $\int_C d\vec{l}\,\nabla\varphi = \delta\varphi = -2\pi n$, $n \in \mathbb{N}$

We obtain $\Phi = n\Phi_0$; $\Phi_0 = \frac{\hbar c}{2|e|}$

This means that the magnetic flux trapped by the superc. ring, $\Phi = BS$, can take only values multiple of the "flux quantum"
$\Phi_0 \simeq 2\cdot 10^{-15}$ Wb. Note that this is also a way to change the phase of the superconducting wavefunction.

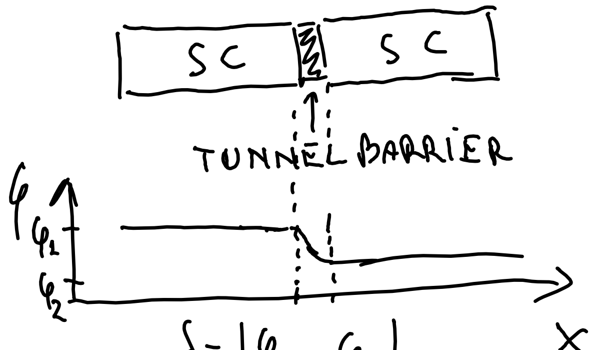
(5) Quasiparticle excitations.

In the non-stationary regime, the response of a superconductor is defined both by the superconductive condensate and the quasiparticle excitations over the ground state. The excitations are separated by the gap $2\Delta_s$ from the ground state, and therefore, they can be neglected at sufficiently low temp., $k_B T \ll \Delta_s$. The value $2\Delta_s$ corresponds to the energy, which is necessary for splitting one Cooper pair.

Another condition on the absence of quasiparticle excitations is the absence of high-freq. fields with $\hbar\omega \sim \Delta_s$, which means that it is necessary also to have

$$\omega \ll \Delta_s / \hbar.$$

The Josephson Junction as a NON-LINEAR INDUCTOR



$\delta = |\varphi_1 - \varphi_2|$

INDUCTION LAW: $V = -L\dot{I}$ (6)

JOSEPHSON RELATIONS $\rightarrow$ (1) : $I = I_0 \sin\delta$

(supercurrent across the junction)

(2) : $V = \dfrac{\Phi_0}{2\pi}\dot{\delta}$

(time-dependent phase if voltage is applied)

We can find (from (1))

$$\dot{I} = I_0 \cos\delta\,\dot{\delta} \Rightarrow \dot{\delta} = \dot{I}/I_0\cos\delta$$

and placing it into the (2) Josephson relation:

$$V = \frac{\Phi_0}{2\pi}\frac{1}{I_0}\frac{1}{\cos\delta}\dot{I} \equiv L_J\,\dot{I} \quad \text{with} \quad L_J = \frac{\Phi_0}{2\pi I_0}\frac{1}{\cos\delta}$$

This relation allows us to interpret the quantity L_J (proportionality constant between V across the junction and the $\frac{d}{dt}(I)$ through the junction) as an inductor L_J.

$$L_J = L_{J_0} \frac{1}{\cos\delta} \quad \text{with } L_{J_0} = \frac{\Phi_0}{2\pi I_0}$$ specific Josephson inductance

$\hookrightarrow$ NON-LINEARITY

N.B. I_0 is the "critical current" of the Josephson junction

Φ_0 is the reduced flux quantum $= \hbar/2e$

The two Josephson relations relate the current and voltage across a Josephson junction to the superconducting phase-difference δ between the two superconducting islands.

Defining an effective flux variable $\tilde{\Phi} = \Phi_0 \delta$ and the Josephson inductance $L_J = \Phi_0 / I_0$ we obtain

$$I = \frac{\Phi_0}{L_J} \sin(\tilde{\Phi}/\Phi_0) = \frac{\tilde{\Phi}}{L_J} - \frac{1}{6} \frac{\tilde{\Phi}^3}{L_J \Phi_0^2} + \dots$$ for $\delta \ll 1$; i.e. $I \ll I_0$

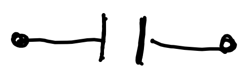
$$V = \dot{\tilde{\Phi}}$$

The phase difference δ across the junction can be regarded as a normalized magnetic flux.

- In this form the Josephson equations are analogous to the relations between V, I and the magnetic flux in a linear inductor.
- However, the current depends nonlinearly on $\tilde{\Phi}$.

For our purposes, the Josephson junction thus acts like a nonlinear inductor.

- Furthermore, current flows without any dissipation, a fundemental feature for their use in quantum coherent devices.

- Combining the Josephson junctions, which we can schematically represent by the following symbol , with inductors  and capacitors  allows us to build nonlinear electrical circuits, which can be operated in a quantum mechanical regime.

Josephson Inductance and Josephson Energy

- Josephson energy

$$E_J = \int V\, I\, dt =$$

$$= \int \frac{\Phi_0}{2\pi} \dot{\delta}\, I_0 \sin\delta\, dt =$$

$$= \frac{\Phi_0}{2\pi} I_0 \cos\delta$$

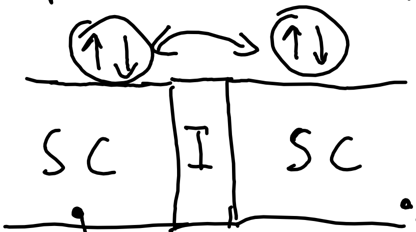
$$= E_{J0} \cos\delta \quad \text{with } E_{J0} = \frac{\Phi_0}{2\pi} I_0$$

- Typical parameters: $I_0 \simeq 100\ nA$

$$\Rightarrow L_{J0} = \frac{\Phi_0}{2\pi I_0} \simeq 3\ nH \quad \left\{ \begin{array}{l} \text{equivalent} \\ \text{of a wire} \\ \text{of} \sim 3\,mm \end{array} \right.$$

$$\Rightarrow E_{J0} = \frac{\Phi_0}{2\pi} I_0 \simeq 50\ GHz$$

JOSEPHSON EFFECT (1962)



GAP
OXIDE
LAYER

$$\psi_1 = \sqrt{\frac{n_{s1}}{2}}\, e^{i\Phi_1}$$

$$\psi_2 = \sqrt{\frac{n_{s2}}{2}}\, e^{i\Phi_2}$$

HYBRIDIZATION

$$\hat{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \; ; \; H = \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix}$$

$$i\hbar \partial_t \hat{\psi} = H \hat{\psi}$$

$$\Longrightarrow \frac{dn_{s1}}{dt} = -\frac{dn_{s2}}{dt} \propto K \sin(\Phi_2 - \Phi_1)$$

$$I = I_c \sin \Delta\Phi$$

CURRENT-PHASE RELATION
FOR DC JOSEPHSON JUNCTION

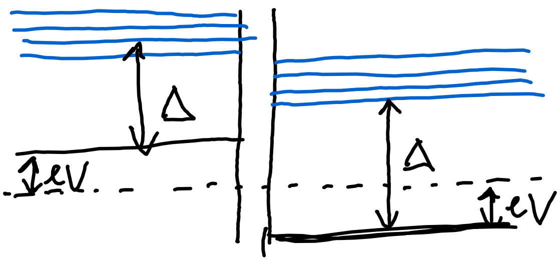
IF VOLTAGE IS APPLIED:

$$H = \begin{bmatrix} eV & B \\ B & -eV \end{bmatrix}$$

$$(\Delta\dot{\Phi}) = \frac{2e}{\hbar} V$$

(AC JOSEPHSON EFFECT)

Assume now that the electric potential difference V is applied to the junction.



The two states of the junction banks form a two-level system with the energy levels defined by the shift of the chemical potential and equal to $\pm eV$.

Let us define the vector-states for the junction so that they form the basis of a two-level system

$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let's expand the wave function in the basis $|\psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle = \begin{pmatrix} \sqrt{n_{s1}}\, e^{i\varphi_1} \\ \sqrt{n_{s2}}\, e^{i\varphi_2} \end{pmatrix}$

The system's Hamiltonian has the diagonal elements which are equal to the system's energy in the respective state: $H_{11} = \langle\psi_1|H|\psi_1\rangle = eV$; $H_{22} = -eV$

The off-diagonal elements describe transitions between the levels, which are related to the tunneling through the barrier, which we characterize by the value B: $H_{12} = H_{21} = B$

$$H = \begin{pmatrix} eV & B \\ B & -eV \end{pmatrix} = eV\sigma_z + B\sigma_x$$

(12)

And from the Schrödinger equation $i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$ we obtain

$$\begin{cases} \dot{n}_{s_1} = -\dot{n}_{s_2} = \dfrac{2Bn_{s0}}{\hbar}\sin\varphi & (1) \\ \dot{\varphi}_1 = -\dfrac{B}{\hbar}\cos\varphi - \dfrac{eV}{\hbar} & (2) \\ \dot{\varphi}_2 = -\dfrac{B}{\hbar}\cos\varphi + \dfrac{eV}{\hbar} & (3) \end{cases}$$

n_{s0} = density of Cooper pairs in the banks without the junction.

Since the current through the tunneling junction is $\propto$ to the rate of change of electron density, $I_s \propto dn_s/dt$ we get $I_J = I_c \sin\varphi$ [stationary Josephson effect]

with I_c the junction critical current.

Subtracting (2) - (3) we get $\dot{\varphi} = \dfrac{2eV}{\hbar}$ [non-stationary Josephson effect]

So $\varphi(t) = \varphi_0 + \dfrac{2e}{\hbar}Vt \Rightarrow I_J = I_c \sin\left(\varphi_0 + \dfrac{2e}{\hbar}Vt\right)$

By applying the voltage results in flowing of alternating current with the angular freq. $\omega = \frac{2eV}{\hbar}$

Note that the average power, consumed from the external source for the supercurrent drive, is zero, $\overline{I_J V} = 0$

This means that also this alternating supercurrent does not dissipate energy!

If we invert $V = \frac{\hbar}{2e}\dot{\varphi}$ it becomes clear that the non-stationary Josephson effect consists in the appearence of the dc-voltage on the junction, if the phase difference linearly depends on time.

SIMPLEST SUPERCONDUCTING DEVICE

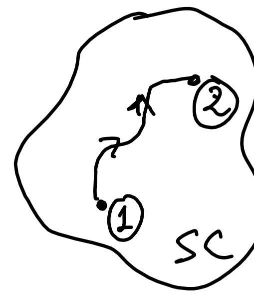
SQUID (Superconducting Quantum Interference Device)

- $J_s = -\frac{e^2 n_s}{m}\left(\vec{A} - \frac{\hbar}{2e}\nabla\Phi\right)$

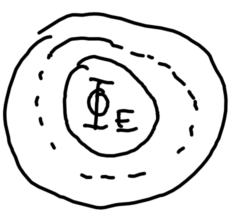
  ↳ It is zero inside a superconductor

  So $\vec{A} = \frac{\hbar}{2e}\nabla\Phi \Rightarrow \int_1^2 A\,dl = \frac{\Phi_0}{2\pi}\int_1^2 \nabla\Phi =$

  $= \frac{\Phi_0}{2\pi}\left[\Phi_2 - \Phi_1\right]$



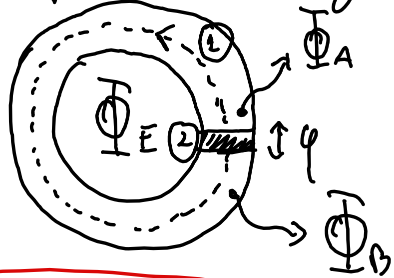
- Superconducting loop $\Rightarrow$ QUANTIZED FLUX



$$\oint A\,dl = \frac{\Phi_0}{2\pi}\oint\nabla\Phi = \frac{\Phi_0}{2\pi}2\pi n \qquad n\in\mathbb{Z}$$

$$\boxed{\Phi_E = \Phi_0\, n}$$

- Superconducting Loop with one Junction (RF-SQUID)

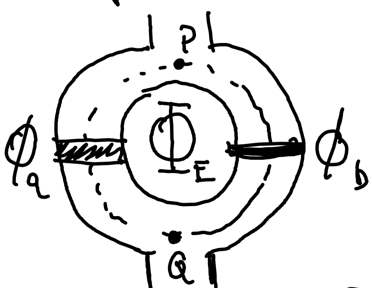


$$\oint \nabla \Phi = 2\pi m$$

$$\oint \nabla \Phi = \underset{①}{\int_A^B \nabla \Phi} + \underset{②}{\int_B^A \nabla \Phi} = \frac{2\pi}{\Phi_0} \underbrace{\int_A^B A\, dl}_{\Phi_E} + \varphi$$

$$\boxed{2\pi m = \frac{2\pi}{\Phi_0}\Phi_E + \varphi}$$

- Superconducting Loop with Two Junctions (DC-SQUID)



$$\Phi_Q - \Phi_P = \Phi_a + \frac{2\pi}{\Phi_0}\int_P^Q \vec{A}\cdot d\vec{l} = \Phi_b + \frac{2\pi}{\Phi_0}\int_P^Q \vec{A}\, d\vec{l}$$

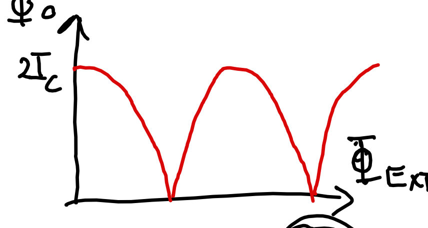
$$\Phi_a - \Phi_b = \frac{2\pi}{\Phi_0}\oint \vec{A}\cdot d\vec{l} = \frac{2\pi}{\Phi_0}\Phi_E$$

$$I_{TOT} = I_c\left[\sin\Phi_a + \sin\Phi_b\right]$$

$$\Phi = \Phi_a - \pi\frac{\Phi_E}{\Phi_0} \quad ; \quad \Phi = \Phi_b - \pi\frac{\Phi_E}{\Phi_0}$$

$$I = 2I_c \cos\left[\pi \Phi_E/\Phi_0\right] \sin\varphi$$

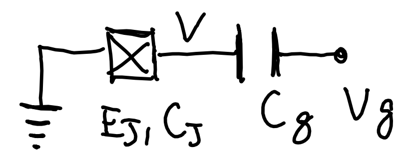
THE CRITICAL CURRENT IS MODULATED BY THE EXTERNAL FIELD



Superconducting island - charge qubit

Consider the charge superconducting qubit. It is based on a superconducting island, or the so-called Cooper-pair box (CPB).

The island is formed by a Josephson junction (with the energy E_J, capacitance C_J, and phase difference φ) and the gate capacitance C_g, by means of which the island is connected to the gate electrode with the voltage V_g. The smallness of the island is in the regime called of "Coulomb blockade", where the Cooper pairs can tunnel discretely



and the charge of the island is a well-defined value.

The electrostatic energy of the circuit, shown in Fig., can be written for the island voltage $V = (\hbar/2e)\dot{\varphi}$, as:

$$\frac{C_J V^2}{2} + \frac{C_g (V_g - V)^2}{2} \rightarrow 4E_c\left(\frac{C_\Sigma V}{2e} - \frac{C_g V_g}{2e}\right) \equiv 4E_c (n - n_g)^2 .$$

Here we have defined the total capacitance of the island $C_\Sigma = C_J + C_g$ and the characteristic charging energy $E_C = e^2/2C_\Sigma$; (2)

We can also defined the number of the Cooper pairs on the island, $n = C_\Sigma V/(2e) = C_\Sigma \hbar \dot{\varphi}/(4e^2) = \hbar\dot{\varphi}/(8E_C)$ and the dimensionless voltage on the gate electrode, $n_g = C_g V_g/2e$.

Subtracting the junction Josephson energy, we obtain the system Lagrangian

$$L(\varphi, \dot{\varphi}) = 4E_C\left(\frac{\hbar}{8E_C}\dot{\varphi} - n_g\right)^2 + E_J \cos\varphi.$$

Coming to the canonical momentum, $p = \frac{\partial L}{\partial \dot{\varphi}} = \hbar(n - n_g)$ we can get the Hamiltonian

$$H(\varphi, p) = 4E_C(n - n_g)^2 - E_J\cos\varphi = \frac{4E_C}{\hbar^2}p^2 - E_J\cos\varphi.$$

which we can now quantize.

We write down the Hamiltonian in the charge basis (3)
that is the basis of the charge-operator eigenstates
$\hat{n}|n\rangle = n|n\rangle$. From here we have the expression
for the charge operator, plus the completeness condition
for the respective projectors:

$$\hat{n} = \sum_n n|n\rangle\langle n|, \quad \sum_n |n\rangle\langle n| = \mathbb{I}$$

We can now recall that, for circuits, the conjugate variables
are φ and n, so their operators saddisfies $[\varphi, n] = i$).
A particle wavefunction in the coordinate representation
with momentum $\vec{p}$ has the form $\psi_{\vec{p}}(\vec{r}) \equiv \langle \vec{r}|\vec{p}\rangle = e^{i\vec{r}\cdot\vec{p}/\hbar}$
In our case, the role of the generalized coordinate is
played by φ, and instead of the momentum, we use n.
Then we have the wavefunction $\langle\varphi|n\rangle = e^{in\varphi}$
From this, we obtain $|\varphi\rangle = \sum_n |n\rangle\langle n|\varphi\rangle = \sum_n e^{-in\varphi}|n\rangle$.
with the inverse transformation $|n\rangle = \frac{1}{2\pi}\int d\varphi\, e^{in\varphi}|\varphi\rangle$.

Having in mind that we need to derive an expression for $\cos\varphi$, we have

(4)

$$|n\pm 1\rangle = e^{\pm i\varphi}|n\rangle \Rightarrow \left(e^{i\varphi}+e^{-i\varphi}\right)|n\rangle = |n+1\rangle + |n-1\rangle$$

Here, in particular, we observed that the effect of the operator $\exp(i l \varphi)$ is analogous to the finite-displacement operator $T_{\vec{a}} = \exp\left(\frac{i}{\hbar}\vec{a}\cdot\hat{\vec{p}}\right)$ such that $T_{\vec{a}}\psi(\vec{r}) = \psi(\vec{r}+\vec{a})$

So, for the terms in the Hamiltonian, in the charge representation

$$(n-n_g)^2 = \left(\sum_n n|n\rangle\langle n| - n_g\right)^2 =$$

$$= \left(\sum_n n|n\rangle\langle n|\right)^2 - 2n_g\sum_n n|n\rangle\langle n| - n_g^2\sum_n |n\rangle\langle n| =$$

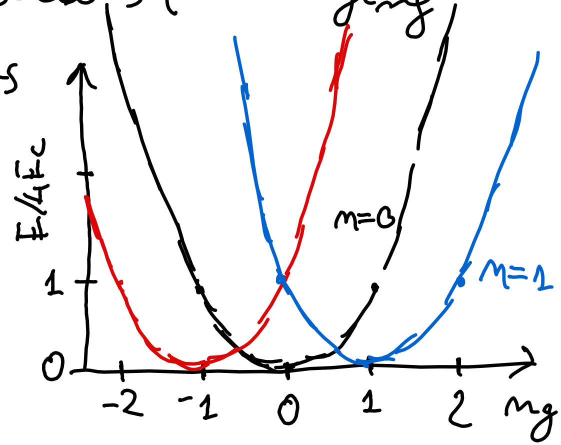
$$= \sum_n (n-n_g)^2|n\rangle\langle n|$$

$$\cos\hat{\varphi} = \frac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)\sum_n|n\rangle\langle n| = \frac{1}{2}\sum_n\left(|n+1\rangle\langle n| + |n-1\rangle\langle n|\right)$$

The Hamiltonian in the charge-repres.

$$H = \sum_n\left\{4E_c(n-n_g)^2|n\rangle\langle n| - \frac{E_J}{2}\left(|n+1\rangle\langle n| + |n\rangle\langle n+1|\right)\right\}$$

Here, the first, dominating term describes the charging energy. The respective energy levels $4E_c(n-n_g)^2$ are shown in Fig



Here it is convenient to consider the excess Cooper-pair number on the island, rather than their total number.

For this consideration, assume that the voltage is changed around some integer value $n_g \sim \tilde{n}_g$, then $n - n_g = (n - \tilde{n}_g) - (n_g - \tilde{n}_g) \equiv N - N_g$, and in order to avoid introducing new variables, we change $N \to n$ and $N_g \to n_g$.

In the two-level approximation, the Hamiltonian will be

$$H = 4E_c\left\{n_g^2 |0\rangle\langle 0| + (1-n_g)^2 |1\rangle\langle 1|\right\} - \frac{E_J}{2}\left\{|0\rangle\langle 1| + |1\rangle\langle 0|\right\}.$$

Making use of the completeness condition $|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{I}$ we obtain

$$2|1\rangle\langle 1| = |1\rangle\langle 1| + |1\rangle\langle 1| + |0\rangle\langle 0| - |0\rangle\langle 0| = \mathbb{I} + |1\rangle\langle 1| - |0\rangle\langle 0|.$$

Omitting the constant term, we get the expression

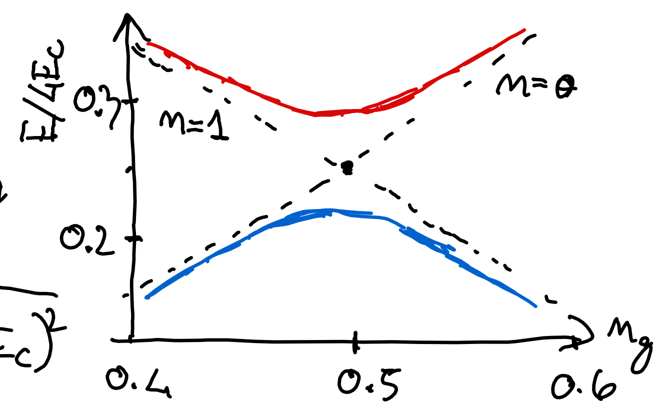
$$H = -2E_c(1-2n_g)\{|0\rangle\langle 0| - |1\rangle\langle 1|\} - \frac{E_J}{2}\{|0\rangle\langle 1| + |1\rangle\langle 0|\}$$

Introducing the Pauli matrices

$$H = -\frac{\Delta}{2}\sigma_x - \frac{\varepsilon}{2}\sigma_z \quad ; \quad \Delta = E_J ; \quad \varepsilon = 4E_c(1-2n_g)$$

So, the superconducting island can be described as a two-level system with controllable parameters. This is the charge qubit.

The energy levels of the qubit can be obtained by diagonalizing the Hamiltonian.



$$E_\pm = \pm\frac{1}{2}\sqrt{\Delta^2 + \varepsilon^2} = \pm 2E_c\sqrt{(1-2n_g)^2 + (E_J/4E_c)^2}$$

In experiments, it is better to use the charge qubits with two junctions, embedded in a loop with an external magnetic flux. This allows the Josephson energy to be made tunable.

Consider for simplicity the case with two identical junctions with the critical currents I_c and phase difference φ_1 and φ_2. We have $\varphi_1 - \varphi_2 = 2\pi \Phi / \Phi_0$.

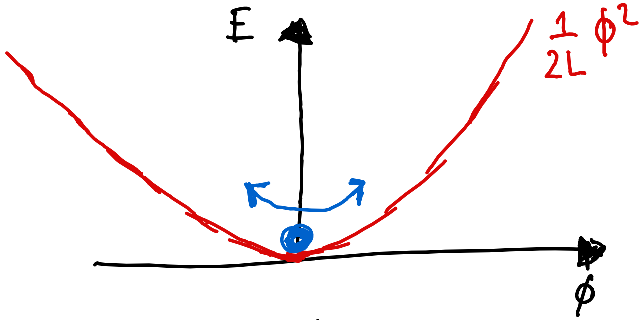
The total current can then be written in the form

$$I = I_1 + I_2 = I_c \left(\sin\varphi_1 + \sin\varphi_2 \right) = \tilde{I}_c \sin\varphi$$

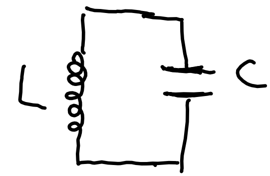
$$\tilde{I}_c = 2 I_c \cos\pi \frac{\Phi}{\Phi_0} \quad , \quad \varphi = \frac{\varphi_1 + \varphi_2}{2}$$

So, the loop with two junctions is described as a single junction with the critical current $\tilde{I}_c = \tilde{I}_c(\Phi)$, which corresponds to the effective Josephson energy $E_J(\Phi) = \hbar \tilde{I}_c / 2e$. This allows the two parameters in the Hamiltonian to be changed: $\varepsilon = \varepsilon(V_g)$ and $\Delta = \Delta(\Phi)$.

PHASE SPACE VS. CHARGE SPACE



QUANTUM LC OSCILLATOR



$$[\hat{\varphi}, \hat{m}] = i$$

$$H = 4E_C m^2 + \frac{1}{2}E_L \phi^2$$

- THE PHASE EIGENSTATES:

$$\hat{\varphi}\,|\varphi_0\rangle = \varphi_0\,|\varphi_0\rangle \implies$$ the particle is located at $\varphi = \varphi_0$

$$\langle\varphi_1|\varphi_0\rangle = \delta(\varphi_1 - \varphi_0)$$

- THE CHARGE EIGENSTATES:

$$\hat{m}\,|m_0\rangle = m_0\,|m_0\rangle \implies$$ the particle has momentum of m_0

Relation between the two spaces $\Rightarrow$ FOURIER TRANFORMATION

$$\hat{n}|n\rangle = n|n\rangle$$

$$\langle\varphi|\hat{n}|n\rangle = \langle\varphi|n|n\rangle = n\langle\varphi|n\rangle$$

$$\langle\varphi|-i\partial_\varphi|n\rangle = n\langle\varphi|n\rangle$$

$$-i\partial_\varphi\langle\varphi|n\rangle = n\langle\varphi|n\rangle$$

$$\langle\varphi|n\rangle = \frac{1}{\sqrt{2\pi}}e^{i\varphi n}$$

$$|n\rangle = \int d\varphi|\varphi\rangle\langle\varphi|n\rangle = \frac{1}{\sqrt{2\pi}}\int d\varphi\, e^{i\varphi n}|\varphi\rangle$$

$$|n\rangle = \frac{1}{\sqrt{2\pi}}\int d\varphi\, e^{i\varphi n}|\varphi\rangle$$

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}}\int dn\, e^{-i\varphi n}|n\rangle$$

IN COOPER PAIR BOX:

$$H = 4E_c(\hat{n} - n_g)^2 - E_J \cos\varphi$$

$H(\varphi) = H(\varphi + 2\pi) \longrightarrow$ PHASE IS 2π PERIODIC

$\Rightarrow \psi(\varphi) = \psi(\varphi + 2\pi) \longrightarrow \varphi$ is a COMPACT VARIABLE DEFINED ON A LOOP

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \int dn\, e^{-i\varphi n} |n\rangle$$

$$|\varphi\rangle = |\varphi + 2\pi\rangle = \frac{1}{\sqrt{2\pi}} \int dn\, \underbrace{e^{-i2\pi n}}_{1} e^{-i\varphi n} |n\rangle$$

$\Rightarrow n = \ldots, -2, -1, 0, +1, +2, \ldots$

CHARGE STATES ARE INTEGER:

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \sum_n e^{i\varphi n} |n\rangle$$

Qubit states are the superposition of charge states:

$$|\psi_0\rangle = \sum_n C_n^0 |n\rangle \quad ; \quad |\psi_1\rangle = \sum_n C_n^1 |n\rangle$$

COS φ IN CHARGE REPRESENTATION

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \sum_n e^{in\varphi} |n\rangle \; ; \quad \langle\varphi| = \frac{1}{\sqrt{2\pi}} \sum_n e^{-in\varphi} \langle n|$$

$$\cos\varphi |\varphi\rangle = \frac{e^{i\varphi}+e^{-i\varphi}}{2} \frac{1}{\sqrt{2\pi}} \sum_n e^{in\varphi} |n\rangle =$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \left\{ \sum_n e^{i(n+1)\varphi} |n\rangle + \sum_n e^{i(n-1)\varphi} |n\rangle \right] =$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \left\{ \sum_{n'} e^{in'\varphi} |n'-1\rangle + \sum_{n'} e^{in'\varphi} |n'+1\rangle \right] =$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \sum_n e^{in\varphi} \left[|n-1\rangle + |n+1\rangle \right]$$

$\int d\varphi \ldots |\varphi\rangle\langle\varphi|$

$$\underbrace{\int d\varphi \cos\varphi |\varphi\rangle\langle\varphi|}_{\cos\varphi} = \frac{1}{2}\frac{1}{\sqrt{2\pi}} \underbrace{\int d\varphi \sum_{n,m} e^{i(n-m)\varphi}}_{\delta_{n,m}} \left[|n-1\rangle\langle m| + |n+1\rangle\langle m| \right] =$$

$$= \frac{1}{2}\left[|n-1\rangle\langle n| + |n+1\rangle\langle n| \right]$$

$$H_{PHASE} = 4E_C\left(-i\partial_\varphi - n_g\right)^2 - E_J \cos\varphi$$

$$H_{CHARGE} = 4E_C\left(\hat{n} - n_g\right)^2 - \frac{E_J}{2}\left[|n-1\rangle\langle n| + |n+1\rangle\langle n| \right]$$

Hamiltonian in the charge representation 12

$$\hat{H} = \underbrace{E_c(\hat{N}-N_g)^2}_{\text{electrostatic}} - \underbrace{E_J \cos\hat{\varphi}}_{\text{magnetic energy}} \rightarrow \frac{E_J}{2}\left(e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}\right)$$

gate charge : $N_g = \frac{C_g V_g}{2e}$

charging energy

$$E_c = \frac{(2e)^2}{2C_\Sigma}$$

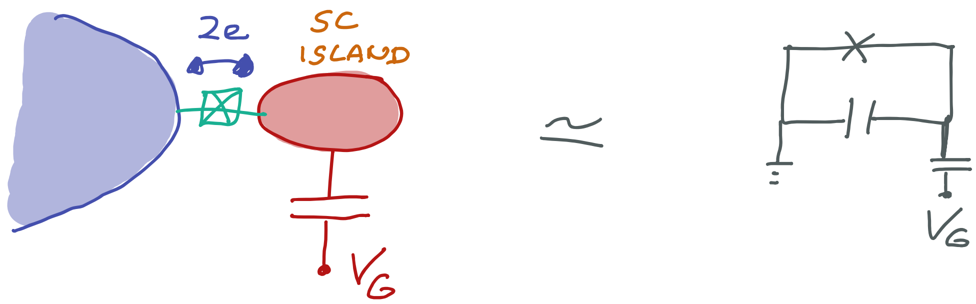
Josephson Coupling energy

$$E_J = \frac{\Phi\, I_c}{2\pi}$$

$$\hat{H} = E_c(\hat{N}-N_g)^2 |N\rangle\langle N| - E_J/2 \sum_N \left(|N+1\rangle\langle N| + |N\rangle\langle N+1|\right)$$

$$\hat{H} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & E_c(-1-N_g)^2 & -E_J/2 & 0 & \cdots \\ \cdots & -E_J/2 & E_c(0-N_g)^2 & -E_J/2 & \cdots \\ \cdots & 0 & -E_J/2 & E_c(1-N_g)^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

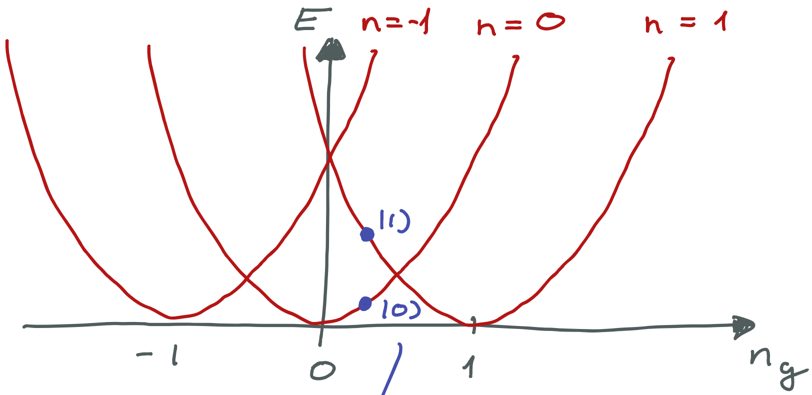
COOPER PAIR BOX ≈ MACROSCOPIC QUANTUM DOT



$$H = 4E_c(n - n_g)^2 - E_J \cos\varphi$$

① WITHOUT TUNNELING ($E_J = 0$)

$$H = 4E_c(n - n_g)^2$$



IS THIS A QUBIT ?

IT IS A TWO-LEVEL SYSTEM

BUT NOT A QUBIT

THERE IS NO TUNNELING

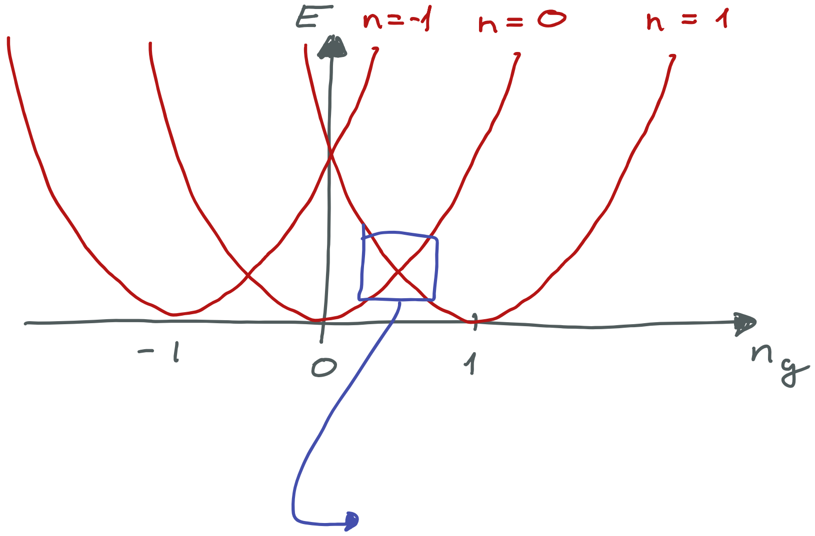
→ NO SUPERPOSITION

→ CLASSICAL BIT

② WITH TUNNELING $(E_J > 0)$

$$H_T = -\frac{E_J}{2}\sum_n \left(|n+1\rangle\langle n| + |n-1\rangle\langle n|\right)$$

→ COOPER PAIRS HOPPING



TRUNCATING THE HAMILTONIAN FOR 2 CHARGE STATES

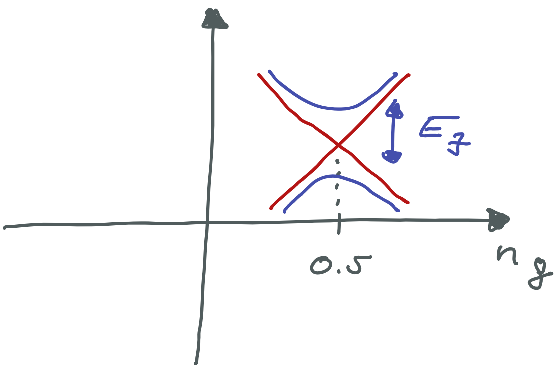
$$H = 4E_c(0-n_g)^2\,|0\rangle\langle 0| + 4E_c(1-n_g)^2\,|1\rangle\langle 1| +$$

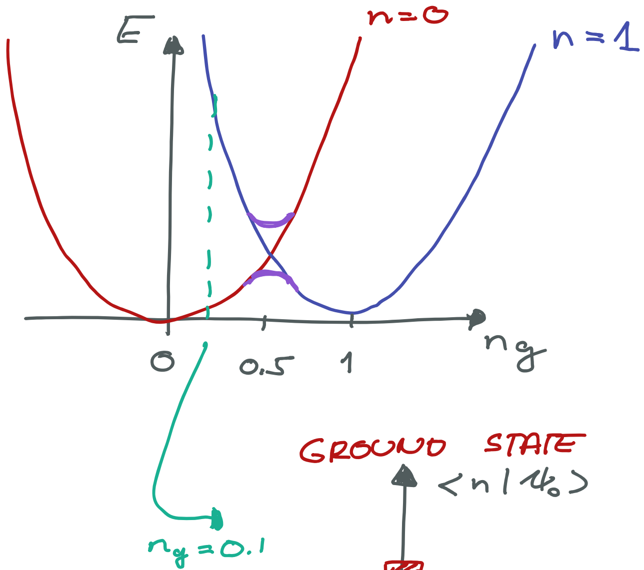
$$-\frac{E_J}{2}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) =$$

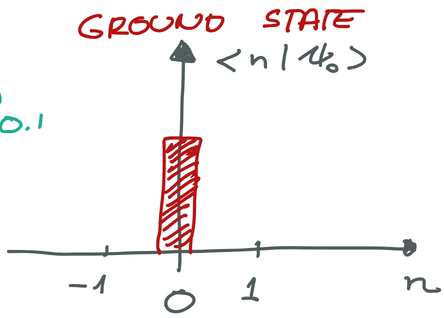
$$= \begin{bmatrix} 4E_c n_g^2 & -E_J/2 \\ -E_J/2 & 4E_c - 8E_c n_g + 4E_c n_g^2 \end{bmatrix} =$$

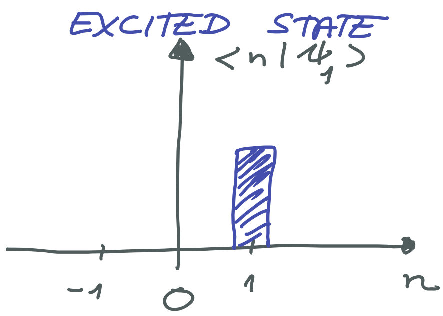
$$= \begin{bmatrix} -(2E_c - 4E_c n_g) & -E_J/2 \\ -E_J/2 & 2E_c - 4E_c n_g \end{bmatrix} =$$

$$= \left[4E_c n_g - 2E_c\right] \sigma_z - E_J/2\, \sigma_x$$

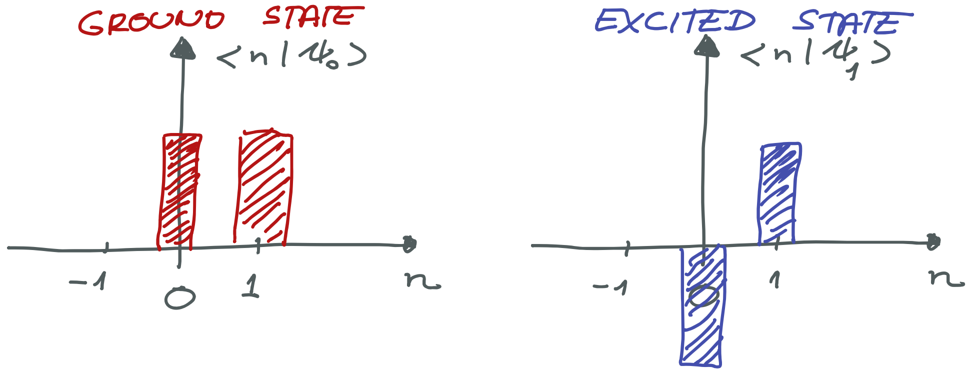


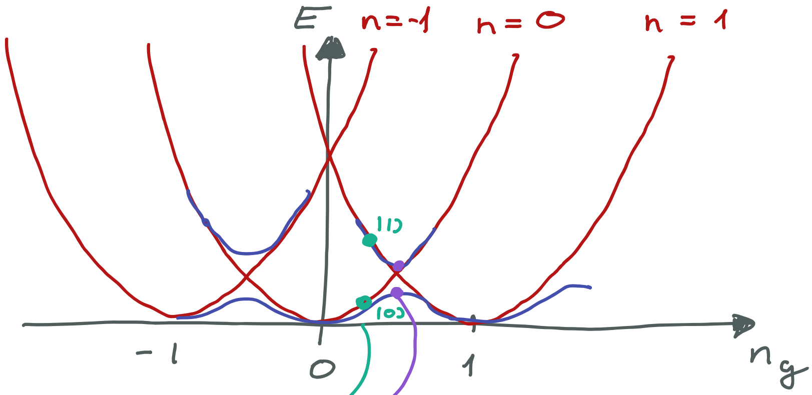






$n_g = 0$





COOPER PAIR BOX	(1998)	$T_2 = 0.1\,\text{ns}$	CHARGE-NOISE
QUANTRONIUM	(2002)	$T_2 = 50\,\text{ns}$ ↓	FIRST-ORDER INSENSITIVE TO CHARGE NOISE
	(2005)	$T_2 = 500\,\text{ns}$	