Solutions

Exercise 1:

Given the single cubit quantum gates:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the qubits:

$$|\Psi_1\rangle = |1\rangle$$
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

a) Show that X, Y, Z and H are unitary and hermitian.

$XX^\dagger = YY^\dagger = ZZ^\dagger = HH^\dagger = I$ - *unitary*

$X = X^\dagger;\ Y = Y^\dagger;\ Z = Z^\dagger;\ H = H^\dagger$ - *hermitian*

b) Show that $XYZ = iI$.

c) Show that $HXH = Z$.

d) Show that $HZH = X$ without explicitly multiplying the matrices.

$HXH = Z$ *and* H *is unitary and hermitian, so*

$HXH = Z <=> HHXHH = HZH <=> IXI = HZH <=> X = HZH$

e) Show that $HXHZHXH = Z$ without explicitly multiplying the matrices.

$HXH = Z$ *and* Z *is unitary and hermitian, so*

$HXHZHXH = ZZZ = IZ = Z$

f) What is the result of applying Y to $|\Psi_2\rangle$?

$$Y\,|\Psi_2\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(-i\,|0\rangle + i\,|1\rangle)$$

Exercise 2:

If $\hat{n} = (n_X, n_Y, n_Z)$ is a real unit vector in three dimensional space, we can define a rotation by θ around the $\hat{n}$ axis as:

$$R_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)(n_X X + n_Y Y + n_Z Z)$$

a) Show that $R_{\hat{x}}(\pi) = -iX$, where $\hat{x} = (1,0,0)$.

b) What is the result of applying $R_{\hat{z}}(\pi)$, where $\hat{z} = (0,0,1)$, to $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$?

$$R_{\hat{z}}(\pi)\,|\Psi\rangle = \left[\cos\left(\tfrac{\pi}{2}\right) I - i\sin\left(\tfrac{\pi}{2}\right) Z\right]|\Psi\rangle = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} =$$
$$\frac{1}{\sqrt{2}}(-i\,|0\rangle - |1\rangle)$$

c) Find $\hat{n}$ so that $H = iR_{\hat{n}}(\pi)$.

$$\hat{n} = \frac{1}{\sqrt{2}}(1,0,1)$$

d) Show that $H = iR_{\hat{x}}(\pi)R_{\hat{y}}(\frac{\pi}{2})$, where $\hat{x} = (1,0,0)$ and $\hat{y} = (0,1,0)$.

$$iR_{\hat{x}}(\pi)R_{\hat{y}}(\tfrac{\pi}{2}) = i\,(-iX)\left(\tfrac{1}{\sqrt{2}}I - \tfrac{i}{\sqrt{2}}Y\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = H$$

e) Show that $R_{\hat{z}}(\theta) Y R_{\hat{z}}(\theta)^\dagger = \cos\theta Y - \sin\theta X$.

$$
\begin{aligned}
R_{\hat{z}}(\theta) Y R_{\hat{z}}(\theta)^\dagger &= \left(\cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) Z\right) Y \left(\cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) Z\right)^\dagger \\
&= \left(\cos\left(\frac{\theta}{2}\right) Y - i\sin\left(\frac{\theta}{2}\right) ZY\right) \left(\cos\left(\frac{\theta}{2}\right) I + i\sin\left(\frac{\theta}{2}\right) Z\right) \\
&= \cos^2\left(\frac{\theta}{2}\right) Y + \sin^2\left(\frac{\theta}{2}\right) ZYZ + \\
&+ i\left(\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) YZ - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) ZY\right) \\
&= \left[\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right] Y \\
&+ i\left[\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) iX - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)(-iX)\right] \\
&= \cos(\theta) Y - 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) X \\
&= \cos(\theta) Y - \sin(\theta) X
\end{aligned}
$$

knowing that:

$$
\begin{aligned}
&cos(\theta) = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \\
&sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\
&YZ = iX \\
&ZYZ = -Y \\
&YZ = -ZY
\end{aligned}
$$