Dynamics of a Quantum System

QM postulate: The time evolution of a state $|\psi\rangle$ of a closed quantum system is described by the **Schrödinger equation**

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

where H is the hermitian operator known as the **Hamiltonian** describing the closed system.

a **closed quantum system** does not interact with any other system

general solution:

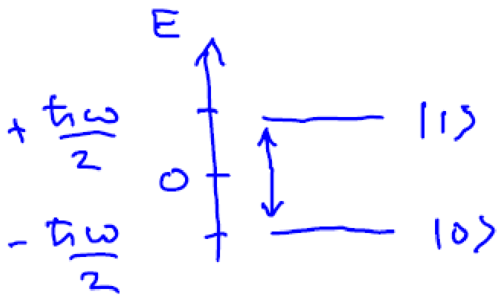
$$|\psi(t)\rangle = \exp\left[\frac{-iHt}{\hbar}\right]|\psi(0)\rangle$$

the **Hamiltonian**:

- H is hermitian and has a spectral decomposition
- with eigenvalues E
- and eigenvectors $|E\rangle$
- smallest value of E_0 is the ground state energy with the eigenstate $|E\rangle_0$

$$H = \sum_E E\,|E\rangle\langle E|$$

example:



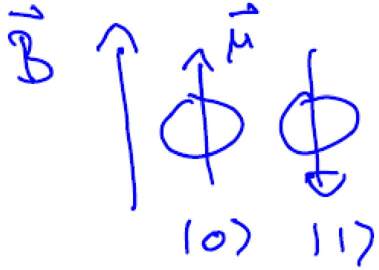
$$H = -\frac{\hbar\omega}{2} Z$$

$$H = -\frac{\hbar\omega}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)$$

$$|\psi(0)\rangle = |0\rangle \longrightarrow |\psi(t)\rangle = e^{\frac{i\omega}{2}t}|0\rangle$$

$$|\psi(0)\rangle = |1\rangle \longrightarrow |\psi(t)\rangle = e^{-\frac{i\omega}{2}t}|1\rangle$$

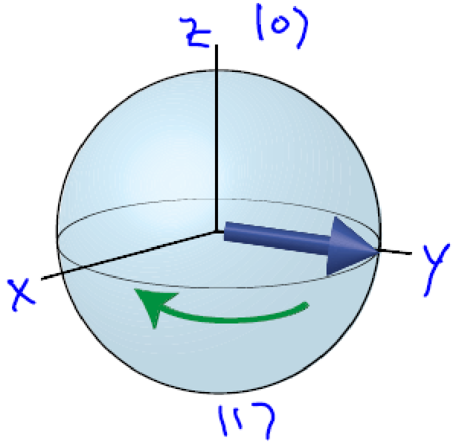
e.g. electron spin in a field:



$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \cancel{e^{\frac{i\omega}{2}t}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)$$

on the Bloch sphere:



$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

$$\Longrightarrow \quad \theta = \frac{\pi}{2}, \quad \varphi = -\omega t$$

this is a rotation around the equator with **Larmor precession frequency ω**

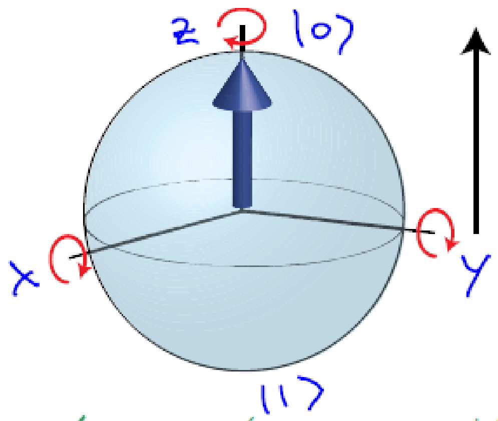
Rotation Operators

when exponentiated the Pauli matrices give rise to rotation matrices around the three orthogonal axis in 3-dimensional space.

$$R_x(\theta) = e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_y(\theta) = e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_z(\theta) = e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

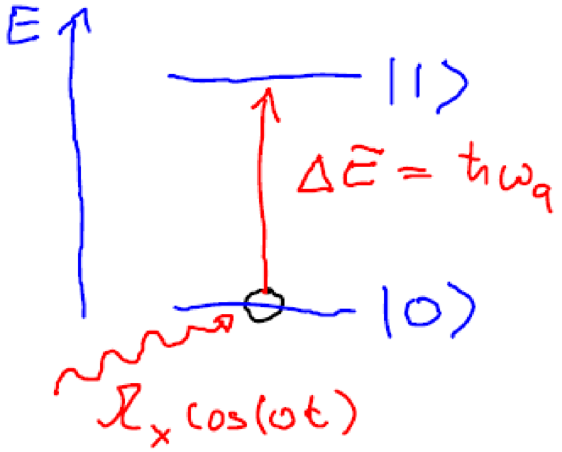


If the Pauli matrices X, Y or Z are present in the Hamiltonian of a system they will give rise to rotations of the qubit state vector around the respective axis.

exercise: convince yourself that the operators $R_{x,y,z}$ do perform rotations on the qubit state written in the Bloch sphere representation.

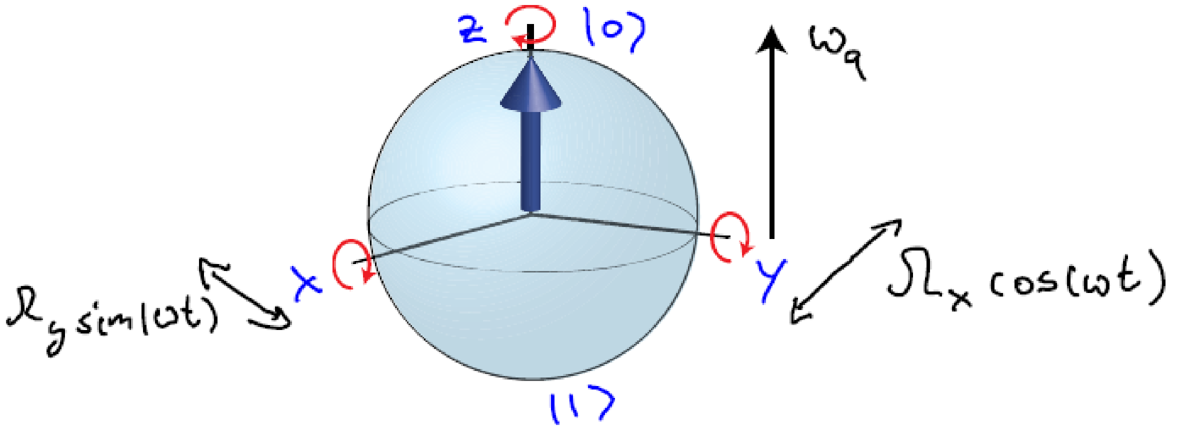
Control of Single Qubit States

by resonant irradiation:



qubit Hamiltonian with ac-drive:

$$H = \hbar\left[\frac{\omega_q}{2}\hat{Z} + \Omega_x \cos(\omega t)\,\hat{X} + \Omega_y \sin(\omega t)\,\hat{Y}\right]$$



ac-fields applied along the x and y components of the qubit state

Rotating Wave Approximation (RWA)

unitary transform: $H' = U H U^{+} - i U \dot{U}^{-1}$

with $U = e^{-i\frac{\omega}{2}t\hat{z}}$

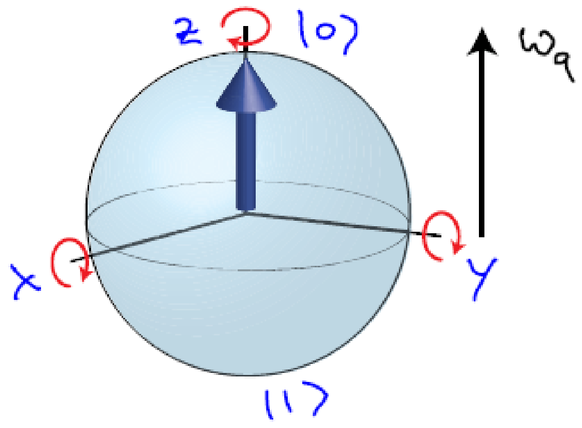
result: $H' = \hbar\left[\frac{\omega_a - \omega}{2}\hat{z} + \frac{\Omega_x}{2}\hat{x}\left(1 + e^{2i\omega t}\right) + \frac{\Omega_y}{2}\hat{y}\left(1 - e^{-2i\omega t}\right)\right]$

drop fast rotating terms (RWA): $H' \approx \frac{\hbar}{2}\left[\Delta\hat{z} + \Omega_x\hat{x} + \Omega_y\hat{y}\right]$

with detuning: $\Delta = \omega_a - \omega$

i.e. irradiating the qubit with an ac-field with controlled amplitude and phase allows to realize arbitrary single qubit rotations.

preparation of qubit states:



initial state $|0\rangle$:

prepare excited state by rotating around **x** or **y** axis:

X_π pulse: $\Omega_x t = \pi$; $|0\rangle - \boxed{X_\pi} - |1\rangle$

Y_π pulse: $\Omega_y t = \pi$; $|0\rangle - \boxed{Y_\pi} - -i|1\rangle$

preparation of a superposition state:

$X_{\pi/2}$ pulse: $\Omega_x t = \frac{\pi}{2}$ $\qquad |0\rangle - \boxed{X_{\pi/2}} - \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$Y_{\pi/2}$ pulse: $\Omega_y t = \frac{\pi}{2}$ $\qquad |0\rangle - \boxed{Y_{\pi/2}} - \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$

in fact such a pulse of chosen length and phase can prepare any single qubit state, i.e. any point on the Bloch sphere can be reached