How to exploit Bell non-locality to advance communication technologies?

Lecture 3: Self-testing

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October 24/25, 2022

1 Introduction

The correlations that can be attained by theories satisfying the principle of local causality are weaker than those allowed by quantum theory. In particular, the CHSH score S defined in the previous lecture note can not exceed 2 for theories satisfying local causality. In this section, we shall turn the question around and ask: What does the CHSH score tell us about quantum models of devices used in the implementation of the CHSH test? As we will see, non-locality provides a compelling resource for device characterization.

The first next sections provide an introduction to self-testing. In particular, they show that a unique quantum model, i.e a unique set of state and observables $\{\rho, A_x, B_y\}$ leads to a CHSH score of $2\sqrt{2}$. Section 7 provides a discussion on robust self-testing, i.e. self-testing statements in case the CHSH score is smaller than $2\sqrt{2}$.

2 Goal of self-testing from the CHSH score

Let us first clarify on the meaning of quantum models in the CHSH test. The implementation of the CHSH test that we considered before is divided in rounds. Each round starts by the distribution of a pair of particles, one for Alice and one for Bob. Alice and Bob then choose a measurement setting x and y, apply the corresponding measurement and record the measurement results $a, b \in \{\pm 1\}$. The pair of particles is now described by a quantum state $\rho \in L(\mathcal{H}^A \otimes \mathcal{H}^B)$ and we assume that both Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ have a finite, but unknown dimension. The measurements of Alice and Bob now admit a description in term of POVM $x \leftrightarrow A_x$ and $y \leftrightarrow B_y$. The CHSH score can thus be expressed as

$$S = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \tag{1}$$

where $\langle A_x B_y \rangle$ is the expectation value of results for the measurements $A_x B_y$. As the measurements we consider have two outcomes ± 1, the quantum models of A_x is a POVM with two elements[1] that we label $\{M_-^{(x)} := M_{-1}^{(x)}, M_+^{(x)} := M_{+1}^{(x)}\}$ such that

$$A_x = (+1)M_+^{(x)} + (-1)M_-^{(x)} = M_+^{(x)} - M_-^{(x)}. \tag{2}$$

Note that the expression of the operator A_x fully specifies the quantum model of measurements (the POVM elements) as completeness tells us that

$$M_+^{(x)} = \frac{1}{2}(\mathbb{1} + A_x), \quad M_-^{(x)} = \frac{1}{2}(\mathbb{1} - A_x)$$

and similarly for B_y. Having a full quantum model of the test thus consists in specifying the state ρ and the observables A_0, A_1, B_0, and B_1.

Showing that $S > 2$ implies ρ is entangled is direct (see the note of lecture 2). Similarly if $S > 2$, then the observables A_0, A_1 do not commute[2]. However, one can say more about the state and measurements from the CHSH score. In particular, one can show that there is a single quantum strategy (state and measurements) leading to $S = 2\sqrt{2}$, that is, if $S = 2\sqrt{2}$ is observed in a given implementation, we can certify that the source and measurement devices used in this CHSH test produce a particular state and admit a particular description in term of projectors (up to local isometries,

[1]The POVM elements $\{M_\pm^{(x)}\}$ satisfy

1. Hermiticity $\left(M_+^{(x)}\right)^\dagger = M_+^{(x)}$
2. Positivity $M_+^{(x)} \geq 0$ that is, $\langle \psi | M_+^{(x)} | \psi \rangle \geq 0$ for any vector $|\psi\rangle$

3; Completness $M_+^{(x)} + M_-^{(x)} = \mathbb{1}$

[2]Consider the case where they are compatible, i.e.

$$[A_0, A_1] = 0. \tag{3}$$

In this case, there exists a basis $\{|k\rangle\}$ where both operators are diagonal and in particular

$$A_x = \sum_k |k\rangle\langle k|_A \; A_x \; |k\rangle\langle k|_A \, . \tag{4}$$

Consider a situation where Alice projects her subsystem in the basis $\{|k\rangle\}$ at each round of the Bell test and discard the measurement result. The resulting state is

$$\bar{\rho} = \sum_k (|k\rangle\langle k|_A \otimes \mathbb{1}_B)\, \rho \, (|k\rangle\langle k|_A \otimes \mathbb{1}_B) = \sum_{k=0} p_k \, |k\rangle\langle k|_A \otimes \rho_B^{(k)} \tag{5}$$

that is, the channel $\mathcal{E}$ such that $\mathcal{E}(\rho) = \bar{\rho}$ is entanglement breaking. Yet, the correlators see below). We first show that the maximum CHSH score obtained with a quantum strategy is $2\sqrt{2}$. We then prove that the observables not only anti-commute but also that they square to the identity. This will allow us to give a precise description of these measurements in term of Pauli measurements and of the measured state in term of a two-qubit maximally entangled state.

3 Tsirelson's bound

In the quantum framework, the CHSH score is the expected value of the Hermitian operator

$$\begin{aligned} \hat{S} &= A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 \\ &= A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1) \end{aligned} \tag{7}$$

on the state ρ, i.e. $S = \mathrm{Tr}\, \hat{S}\rho$. By introducing the following operators

$$\begin{aligned} L_0 &= \left(A_0 \otimes \mathbb{1}_B - \mathbb{1}_A \otimes \frac{B_0 + B_1}{\sqrt{2}} \right) \\ L_1 &= \left(A_1 \otimes \mathbb{1}_B - \mathbb{1}_A \otimes \frac{B_0 - B_1}{\sqrt{2}} \right). \end{aligned} \tag{8}$$

$\hat{S}$ can be expressed as a sum of squares

$$\hat{S} = \frac{1}{\sqrt{2}} \Big((A_0^2 + A_1^2) \otimes \mathbb{1}_B + \mathbb{1}_A \otimes (B_0^2 + B_1^2) - L_0^2 - L_1^2 \Big). \tag{9}$$

As the eigenvalues[3] of A_x and B_y all lay in the interval $[-1, 1]$, the eigenvalues of $\frac{1}{\sqrt{2}}\Big((A_0^2+A_1^2)\otimes\mathbb{1}_B+\mathbb{1}_A\otimes(B_0^2+B_1^2)\Big)$ are upper bounded by $4/\sqrt{2} = 2\sqrt{2}$. In

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in the CHSH score (see Eq. (1)) are given by

$$\begin{aligned} \mathrm{tr}\, \bar{\rho}\, A_x \otimes B_y &= \mathrm{tr} \left(\sum_{k=0} (|k\rangle\langle k|_A \otimes \mathbb{1}_B)\, \rho\, (|k\rangle\langle k|_A \otimes \mathbb{1}_B) A_x \otimes B_y \right) \\ &= \mathrm{tr}\rho \Big(\sum_{k=0} (|k\rangle\langle k|_A \otimes \mathbb{1}_B)\, A_x \otimes B_y\, (|k\rangle\langle k|_A \otimes \mathbb{1}_B \Big) \\ &= \mathrm{tr}\rho \Big(\sum_{k=0} |k\rangle\langle k|_A\ A_x\ |k\rangle\langle k|_A \otimes B_y \Big) \\ &= \mathrm{tr}\, \rho\, A_x \otimes B_y \end{aligned} \tag{6}$$

i.e. any correlations obtained with compatible measurement can be obtained with a separable state. The violation of the CHSH inequality thus implies that the measurements are locally incompatible.

[3]From Eq. (2) and the condition that the POVM elements sum up to identity, it follows

addition, as L_0^2 and L_1^2 are squares of Hermitian operators, their eigenvalues are positive. Hence, the eigenvalues of $\hat{S}$ are all upper bounded by $2\sqrt{2}$, meaning that for any quantum state and measurements, the CHSH score is upper bounded by $2\sqrt{2}$. This is known as the Tsirelson's bound.

4 Observable properties

Let us denote $\{|\Psi_i\rangle\} \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ the basis in which ρ is diagonal

$$\rho = \sum_i p_i \, |\Psi_i\rangle\!\langle\Psi_i| \,. \tag{11}$$

As the observed statistics only depends on the support of ρ, i.e. the states $|\Psi_i\rangle$ associated to $p_i \neq 0$, we also define

$$\mathcal{H}_* = \mathrm{span}\{|\Psi_i\rangle \,|\, \text{such that}\, p_i \neq 0\} \tag{12}$$

which is the relevant Hilbert space for describing the measurements (states outside $\mathcal{H}_*$ are never prepared). Similarly, we define a basis for the reduced states

$$\rho_{A(B)} = \mathrm{Tr}\, \rho_{B(A)} = \sum_i p_i^{A(B)} \, |\Psi_i^{A(B)}\rangle\!\langle\Psi_i^{A(B)}| \tag{13}$$

and the corresponding relevant Hilbert spaces

$$\mathcal{H}_*^{A(B)} = \mathrm{span}\{|\Psi_i^{A(B))}\rangle| \,\text{such that}\, p_i^{A(B)} \neq 0\}. \tag{14}$$

From the sum of squares decomposition given in Eq. (9) we have

$$\mathrm{tr}\,\rho\,\hat{S} = 2\sqrt{2} \implies \begin{cases} (i) & \mathrm{tr}\,\rho\, L_0^2 = \mathrm{tr}\,\rho\, L_1^2 = 0 \\ (ii) & \mathrm{tr}\,\rho\, A_x^2 \otimes \mathbb{1}_B = \mathrm{tr}\,\rho\, \mathbb{1}_A \otimes B_y^2 = 1. \end{cases} \tag{15}$$

This implies

(i) $\forall\, |\Psi_i\rangle$ such that $p_i \neq 0$, $\langle\Psi_i|\, L_{0(1)}^2 \,|\Psi_i\rangle = 0 \Rightarrow ||L_{0(1)}\,|\Psi_i\rangle|| = 0 \Rightarrow L_{0(1)}\,|\Psi_i\rangle = 0$. From the definitions of $L_{0(1)}$, we obtain that

$$\forall\, |\Psi\rangle \in \mathcal{H}_* \begin{cases} A_0 \otimes \mathbb{1}_B\, |\Psi\rangle & = \frac{1}{\sqrt{2}} \mathbb{1}_A \otimes (B_0 + B_1)\, |\Psi\rangle \,, \\ A_1 \otimes \mathbb{1}_B\, |\Psi\rangle & = \frac{1}{\sqrt{2}} \mathbb{1}_A \otimes (B_0 - B_1)\, |\Psi\rangle \,. \end{cases} \tag{16}$$

that

$$A_x = 2M_+^{(x)} - \mathbb{1}. \tag{10}$$

Since $M_+^{(x)}$ is a POVM element, it satisfies $0 \leq M_+^{(x)} \leq 1$, i.e. the eigenvalues of the Hermitian operator $M_+^{(x)}$ all lay in the interval $[0, 1]$. It follows that the eigenvalues of the Hermitian operator A_x all lay in the interval $[-1, 1]$.

(ii) $\mathrm{tr}\,\rho_A\, A_x^2 = \mathrm{tr}\,\rho_B\, B_y^2 = 1$. This means that for any state $|\Psi_i^A\rangle$ such that $p_i^A \neq 0$, we have

$$\langle\Psi_i^A|\, A_x^2\, |\Psi_i^A\rangle = 1. \tag{17}$$

For any vector $|\Psi_i^A\rangle$, $A_x^2\,|\Psi_i^A\rangle$ can be decomposed as

$$A_x^2\,|\Psi_i^A\rangle = \alpha\,|\Psi_i^A\rangle + \beta\,|\Psi_i^{A\perp}\rangle \tag{18}$$

where $|\Psi_i^{A\perp}\rangle$ accounts for all the components of $A_x^2\,|\Psi_i^A\rangle$ not overlapping with $|\Psi_i^A\rangle$. The condition (17) guarantees that $\langle\Psi_i^A|\, A_x^2\,|\Psi_i^A\rangle = \alpha = 1$. In addition, since the eigenvalues of A_x^2 are positive and upper-bounded by one, we also have

$$||A_x^2\,|\Psi_i^A\rangle\,|| \leq 1 \implies |\alpha|^2 + |\beta|^2 \leq 1 \implies \beta = 0. \tag{19}$$

Hence, we get

$$\forall\,|\Psi_A\rangle \in \mathcal{H}_*^A,\ A_x^2\,|\Psi_A\rangle = |\Psi_A\rangle\,, \tag{20}$$

A similar identity holds for B_y^2.

In summary, one can conclude that the operators describing the measurement observables leading to $S = 2\sqrt{2}$ satisfy

$$\begin{aligned}
&(a)\quad A_0 \otimes \mathbb{1}_B\,|\Psi\rangle = \frac{1}{\sqrt{2}}\mathbb{1}_A \otimes (B_0 + B_1)\,|\Psi\rangle \\
&(b)\quad A_1 \otimes \mathbb{1}_B\,|\Psi\rangle = \frac{1}{\sqrt{2}}\mathbb{1}_A \otimes (B_0 - B_1)\,|\Psi\rangle \\
&(c)\quad A_0^2\,|\Psi_A\rangle = A_1^2\,|\Psi_A\rangle = |\Psi_A\rangle \\
&(d)\quad B_0^2\,|\Psi_B\rangle = B_1^2\,|\Psi_B\rangle = |\Psi_B\rangle
\end{aligned}$$

$\forall\,|\Psi\rangle \in \mathcal{H}_*, |\Psi_A\rangle \in \mathcal{H}_*^A$ and $|\Psi_B\rangle \in \mathcal{H}_*^B$.

Let us show that the observables A_x anti-commutes in the support of ρ_A. First, we use Eqs. (a) and (b) to get

$$\begin{aligned}
A_0 A_1 \otimes \mathbb{1}_B\,|\Psi\rangle =& (A_0 \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes \frac{B_0 - B_1}{\sqrt{2}})\,|\Psi\rangle \\
=& (\mathbb{1}_A \otimes \frac{B_0 - B_1}{\sqrt{2}})(A_0 \otimes \mathbb{1}_B)\,|\Psi\rangle \\
=& (\mathbb{1}_A \otimes \frac{B_0 - B_1}{\sqrt{2}})(\mathbb{1}_A \otimes \frac{B_0 + B_1}{\sqrt{2}})\,|\Psi\rangle \\
=& \frac{1}{2}\mathbb{1}_A \otimes (B_0 - B_1)(B_0 + B_1)\,|\Psi\rangle\,.
\end{aligned} \tag{21}$$

Similarly, we get

$$A_1 A_0 \otimes \mathbb{1}_B |\Psi\rangle = \frac{1}{2}\mathbb{1}_A \otimes (B_0 + B_1)(B_0 - B_1) |\Psi\rangle . \tag{22}$$

Since[4] $\mathbb{1} \otimes B_0^2 |\Psi\rangle = \mathbb{1} \otimes B_1^2 |\Psi\rangle = |\Psi\rangle$, we have

$$\begin{aligned} A_0 A_1 \otimes \mathbb{1}_B |\Psi\rangle =& \frac{1}{2}\mathbb{1}_A \otimes (B_0^2 - B_1^2 + [B_0, B_1]) |\Psi\rangle \\ =& \frac{1}{2}\mathbb{1}_A \otimes [B_0, B_1] |\Psi\rangle \\ A_1 A_0 \otimes \mathbb{1}_B |\Psi\rangle =& -\frac{1}{2}\mathbb{1}_A \otimes [B_0, B_1] |\Psi\rangle . \end{aligned} \tag{23}$$

Combining the two previous equalities gives $A_0 A_1 \otimes \mathbb{1}_B |\Psi\rangle = -A_1 A_0 \otimes \mathbb{1}_B |\Psi\rangle$, that is $\{A_0, A_1\} \otimes \mathbb{1}_B |\Psi\rangle = 0$. It directly follows that

$$\{A_0, A_1\} |\Psi_A\rangle = 0 \tag{24}$$

for all states in the support of ρ_A[5].

5 Explicit expression of the CHSH operator

Alice's observables are thus described by a pair of operators satisfying

$$\begin{aligned} A_0^2 |\Psi_A\rangle = A_1^2 |\Psi_A\rangle = |\Psi_A\rangle \\ A_0 A_1 |\Psi_A\rangle = -A_1 A_0 |\Psi_A\rangle , \end{aligned} \tag{25}$$

$\forall \Psi_A \in \mathcal{H}_A^*$. Which operators can those be? Eq. (c) tells us that the eigenvalues of A_0 corresponding to eigenstates in $\mathcal{H}_A^*$ are ± 1. Consider any state $|0_k\rangle \in \mathcal{H}_{A*}$ satisfying[6] $A_0 |0_k\rangle = |0_k\rangle$. As there can be many such states, we introduced a label k to distinguish them. Next, we introduce

$$|1_k\rangle = A_1 |0_k\rangle \tag{26}$$

which satisfies $A_1 |1_k\rangle = A_1 A_1 |0_k\rangle = |0_k\rangle$. $|1_k\rangle$ is a well normalized state as $\langle 1_k | 1_k\rangle = \langle 0_k | A_1^2 |0_k\rangle = \langle 0_k | 0_k\rangle = 1$. Consider the action of A_0 on $|1_k\rangle$, we have

$$A_0 |1_k\rangle = A_0 A_1 |0_k\rangle = -A_1 A_0 |0_k\rangle = -A_1 |0_k\rangle = -|1_k\rangle . \tag{27}$$

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[4]Consider the Schmidt decomposition of $|\Psi\rangle = \sum_i \lambda_i |i^A\rangle |i^B\rangle$ with $\lambda_i \neq 0$. From the reduced state of B, $\rho_B = \sum_i \lambda_i^2 |i^B\rangle \langle i^B|$ we know that $\{|i^B\rangle\} \in \mathcal{H}_*^B$. From Eq. (d), we thus have $\mathbb{1} \otimes B_0^2 |\Psi\rangle = \sum_i \lambda_i |i^A\rangle B_0^2 |i^B\rangle = |\Psi\rangle$.

[5]This can be seen by using again the Schmidt decomposition of $|\Psi\rangle$, that is $|\Psi\rangle = \sum_i \lambda_i |i\rangle_A \otimes |i\rangle_B$. $\sum_i \lambda_i (\{A_0, A_1\} |i\rangle_A) \otimes |i\rangle_B = \sum_i \lambda_i |i\rangle_A \otimes |i\rangle_B$ implies that $\{A_0, A_1\} |i\rangle_A = |i\rangle_A$ for all $|i\rangle_A$.

[6]If there is no such state one can equivalently start with a state with eigenvalue -1.

Since

$$\langle 0_k|1_k\rangle = \langle 0_k|\, A_1 \,|0_k\rangle = \langle 0_k|\, A_1 A_0 \,|0_k\rangle = -\,\langle 0_k|\, A_0 A_1 \,|0_k\rangle = -\,\langle 0_k|\, A_1 \,|0_k\rangle \tag{28}$$

we deduce that the two states $|0_k\rangle$ and $|1_k\rangle$ are orthogonal. We have thus found two orthogonal states $|0_k\rangle$ and $|1_k\rangle$ for which

$$\begin{aligned} A_0\,|0_k\rangle &= |0_k\rangle \quad & A_0\,|1_k\rangle &= -\,|1_k\rangle \\ A_1\,|0_k\rangle &= |1_k\rangle \quad & A_1\,|1_k\rangle &= |0_k\rangle\,. \end{aligned} \tag{29}$$

In other words on the two-dimensional subspace spanned by $|0_k\rangle$ and $|1_k\rangle$, the operators A_0 and A_1 act as Pauli operators σ_z and σ_x. This procedure can be repeated for all k. Ultimately, it allows one to construct a basis in which the operators A_0 and A_1 are block diagonal with blocks of size two (for each pair $|0_k\rangle$ and $|1_k\rangle$), and act as Pauli operators σ_z and σ_x within each block

$$A_0 = \bigoplus_k \sigma_z^{(k)} \quad A_1 = \bigoplus_k \sigma_x^{(k)}. \tag{30}$$

The same procedure can be repeated for Bob's operators in which case we find that $\forall \Psi_B \in \mathcal{H}_*^B$

$$\begin{aligned} B_+^2\,|\Psi_B\rangle &= B_-^2\,|\Psi_B\rangle = |\Psi_B\rangle \\ B_+B_-\,|\Psi_B\rangle &= -B_-B_+\,|\Psi_B\rangle \end{aligned} \tag{31}$$

where $B_\pm = \frac{1}{\sqrt{2}}(B_0 \pm B_1)$. This suggests the choice of a rotated basis for each of the blocks

$$B_0 = \bigoplus_\ell \frac{\sigma_z^{(\ell)} + \sigma_x^{(\ell)}}{\sqrt{2}} \quad B_1 = \bigoplus_k \frac{\sigma_z^{(\ell)} - \sigma_x^{(\ell)}}{\sqrt{2}}. \tag{32}$$

The result is easier to interpret if we introduce a quantum number k on Alice's side labeling the subspace of each pair $|0_k\rangle = |0,k\rangle$ and $|1_k\rangle = |1,k\rangle$ (and similarly for Bob $|0_\ell\rangle = |0,\ell\rangle$ and $|1_\ell\rangle = |1,\ell\rangle$). The Hilbert spaces of Alice can then be decomposed as $\mathcal{H}^A = \mathbb{C}_A^2 \otimes \mathcal{H}_{L_A}$ – each state $|a,k\rangle$ is specified by the value $a = 0,1$ and the label k (and the same for Bob). With this notation, we obtain

$$\begin{aligned} \sigma_z^{(k)} &= |0,k\rangle\langle 0,k| - |1,k\rangle\langle 1,k| = \sigma_z \otimes |k\rangle\langle k| \\ \sigma_x^{(k)} &= |0,k\rangle\langle 1,k| + |1,k\rangle\langle 0,k| = \sigma_x \otimes |k\rangle\langle k| \end{aligned} \tag{33}$$

and hence

$$\begin{aligned}
A_0 &= \bigoplus_k \sigma_z^{(k)} = \sum_k \sigma_z \otimes |k\rangle\langle k| = \sigma_z \otimes \sum_k |k\rangle\langle k| = \sigma_z \otimes \mathbb{1}_{L_A} \\
A_1 &= \sigma_x \otimes \mathbb{1}_{L_A} \\
B_0 &= \frac{\sigma_z + \sigma_x}{\sqrt{2}} \otimes \mathbb{1}_{L_B} \\
B_1 &= \frac{\sigma_z - \sigma_x}{\sqrt{2}} \otimes \mathbb{1}_{L_B}.
\end{aligned} \tag{34}$$

$S = 2\sqrt{2}$ thus implies that Alice and Bob's Hilbert spaces can be decomposed in tensor products $\mathcal{H}^A = \mathbb{C}^2_A \otimes \mathcal{H}_{L_A}$ and $\mathcal{H}_B = \mathbb{C}^2_B \otimes \mathcal{H}_{L_B}$, where the qubit $\mathbb{C}^2_{A(B)}$ carries the state $|0\rangle$ and $|1\rangle$, and the Hilbert space $\mathcal{H}_{L_A(L_B)}$ (of unknown dimension) carries the label $|k\rangle$. In this decomposition, the measurement operators A_x and B_y act nontrivially on the qubit subsystem only and are precisely specified as described in Eq. (34).

We can now express the CHSH operator as

$$\hat{S} = \sqrt{2} \bigoplus_{k,\ell} (\sigma_z^{(k)} \otimes \sigma_z^{(\ell)} + \sigma_x^{(k)} \otimes \sigma_x^{(\ell)}) = \sqrt{2}(\sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x) \otimes \mathbb{1}_{L_A, L_B}, \tag{35}$$

which acts nontrivially on the qubit subsystems of Alice and Bob $\mathbb{C}^2_A \otimes \mathbb{C}^2_B$ only. The diagonalisation of $(\sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x)$ gives two non-zero eigenvalues with eigenvectors $\Phi^\pm$ (both associated to the eigenvalue 2), that is[7]

$$(\sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x) = 2\left(|\Phi^+\rangle\langle\Phi^+| - |\Psi^-\rangle\langle\Psi^-|\right) \tag{36}$$

Hence, we proved that

$$\hat{S} = 2\sqrt{2}\left(|\Phi^+\rangle\langle\Phi^+| - |\Psi^-\rangle\langle\Psi^-|\right) \otimes \mathbb{1}_{L_A, L_B}. \tag{37}$$

6 Self-tested state from CHSH

[7] This can be trivially seen by writing $(\sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x)$ in the basis $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$.

We can now specify the state. We have

$$
\begin{aligned}
2\sqrt{2} &= \mathrm{tr}\rho\hat{S} \\
&= 2\sqrt{2}\mathrm{tr}_{\mathbb{C}_A^2,\mathbb{C}_B^2}\mathrm{tr}_{\mathcal{H}_{L_A},\mathcal{H}_{L_B}}\rho\left(|\Phi^+\rangle\langle\Phi^+| - |\Psi^-\rangle\langle\Psi^-|\right)\otimes \mathbb{1}_{L_A,L_B} = \\
&= 2\sqrt{2}\,\mathrm{tr}_{\mathbb{C}_A^2,\mathbb{C}_B^2}\left(\mathrm{tr}_{\mathcal{H}_{L_A},\mathcal{H}_{L_B}}\rho\right)\left(|\Phi^+\rangle\langle\Phi^+| - |\Psi^-\rangle\langle\Psi^-|\right) \\
&= 2\sqrt{2}\,\mathrm{tr}\,\rho_{\mathbb{C}_A^2,\mathbb{C}_B^2}\left(|\Phi^+\rangle\langle\Phi^+| - |\Psi^-\rangle\langle\Psi^-|\right) \\
&\implies \rho_{\mathbb{C}_A^2,\mathbb{C}_B^2} = |\Phi^+\rangle\langle\Phi^+|
\end{aligned}
\tag{38}
$$

where $\rho_{\mathbb{C}_A^2,\mathbb{C}_B^2} = \mathrm{tr}_{\mathcal{H}_{L_A},\mathcal{H}_{L_B}}\rho \in L(\mathbb{C}_A^2 \otimes \mathbb{C}_B^2)$ is the reduced two-qubit state obtained by tracing out the part lying in $\mathcal{H}_{L_A}$ and $\mathcal{H}_{L_B}$. Because the two qubit state $\rho_{\mathbb{C}_A^2,\mathbb{C}_B^2}$ is pure, the global state is a product state, that is

$$
\rho = |\Phi^+\rangle\langle\Phi^+| \otimes \rho' \tag{39}
$$

with $\rho' \in \mathcal{H}_{L_A} \otimes \mathcal{H}_{L_B}$. We conclude that from the CHSH score $S = 2\sqrt{2}$, we can identify a subspace in which the state is a maximally entangled two qubit state $|\Phi^+\rangle$ and measurements are orthogonal Pauli operators.

Let us be precise for commenting the obtained statement. We have shown that there exists a basis in which the state leading to $S = 2\sqrt{2}$ is of the form $|\Phi^+\rangle\langle\Phi^+| \otimes \rho'$. There exist actually other bases where the state is different. To see this, consider the following quantum model

$$
\left((U_A \otimes U_B)\rho(U_A^\dagger \otimes U_B^\dagger), U_A A_x U_A^\dagger, U_B B_y U_B^\dagger\right), \tag{40}
$$

where U_A and U_B are local unitary operations. It is easy to see that it leads to the same measurement statistics than the quantum model that has been considered so far (ρ, A_x, B_y). Indeed, since the trace is cyclic, we have

$$
\mathrm{tr}(U_A \otimes U_B)\rho(U_A^\dagger \otimes U_B^\dagger)\,(U_A A_x U_A^\dagger) \otimes (U_B B_y U_B^\dagger) = \mathrm{tr}\rho A_x \otimes B_y. \tag{41}
$$

In particular, the two models lead to the same CHSH value. This means that self-testing is an identification of the state and the measurements up to local unitaries or more generally, up to local isometries (not only performing a unitary, but first adding an ancillary state and then applying a global unitary).

7 Robust self-testing from CHSH

For practical purpose, it is important to consider the case where S is not exactly $2\sqrt{2}$ and conclude that in this case the measured state is not exactly but resembles a maximally entangled two-qubit state. The derivation goes beyond this course, but we can show that the fidelity of the state with respect to a two-qubit maximally entangled state is lower bounded by the following fonction of the CHSH score

$$\mathcal{F} \geq f(S) = \frac{1}{2}\left(1 + \frac{S - S^*}{2\sqrt{2} - S^*}\right) \tag{42}$$

with $S^* = \frac{16+14\sqrt{2}}{17} \approx 2.11$. Non-trivial fidelities can thus be obtained as long as the observed CHSH score S is larger than ≈ 2.11. More precisely, if the CHSH score exceeds ≈ 2.11, there exists local extraction maps which output a state, when applied to the actual state, with a fidelity larger than $1/2$ with respect to $|\phi^+\rangle$. The fidelity then increases linearly up to 1 for $S = 2\sqrt{2}$.