How to exploit Bell non-locality to advance communication technologies?

Lecture 2: Bell non-locality

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1 Introduction

Bell test aims at distinguishing physical theories according to the principle of local causality – the idea that events can only be influenced by actions in their past light-cone, see Fig. 1. In particular, John Bell proposed an experiment in 1964 – known as a Bell test – to unequivocally show that correlations originating from local causality are bounded more strongly than in quantum theory.

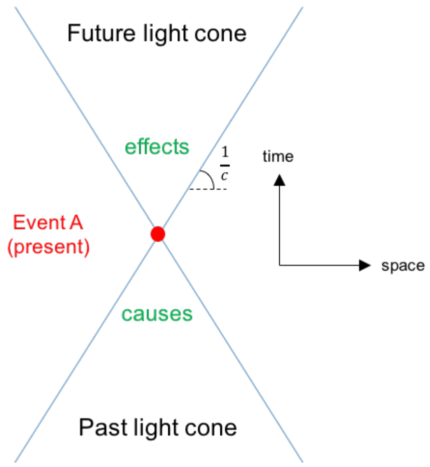

Figure 1: In the vicinity of a given event, space-time is divided into three separate zones delimited by the light cone: one in which the possible effects of the event can be found, one for its possible causes and the "elsewhere" in which there can be no signaling with the event. Local causality states that events cannot be influenced by actions outside the past light cone (any kind of influences, including the ones which do not lead to communication.)

As it aims at distinguishing different theories, a Bell test relies on an abstract – theory independent – description of a test with parties, each holding boxes delivering an output when provided an input. The test is concerned with correlations between the inputs and outputs. It can take several forms depending on the number of parties, inputs or outputs. We here consider the simplest Bell test – the CHSH test – and provide a derivation of the CHSH inequality which is fulfilled by any theory satisfying local causality.

2 The scenario

Let us consider the simplest Bell test with two parties – Alice and Bob. Each of them is given a box which accepts a binary input 0 or 1 and delivers a binary outcome -1 or $+1$ for each input choice. The test is divided in rounds. At each round, the parties are asked to choose an input – called x for Alice and y for Bob. They each receive an output – a and b respectively – for each input choice. The correlations between the input choices and outcomes that can be obtained by local causality are bounded by

$$S = \sum_{x,y=0}^{1} (-1)^{xy} \underbrace{(p(a=b|xy) - p(a \neq b|xy))}_{\langle xy \rangle} \leq 2 \tag{1}$$

where $p(a = b|xy)$ is the probability of Alice and Bob receiving the same outcomes conditioned on their choices of input x and y. Interestingly, there exists quantum implementations of this test violating the inequality given in Ineq. (1) and achieving values of S up to $2\sqrt{2}$.

3 Derivation of the CHSH inequality in a probability space

According to the rules on probabilities, $p(ab|xy)$ can be refined in terms of some additional (possibly unknown or "hidden") variables as

$$p(ab|xy) = \sum_{y} p(ab\lambda|xy). \tag{2}$$

The derivation of the CHSH inequality is then obtained from two assumptions. The locality assumption states that the outcome in one side is determined from the specification of the local input and the knowledge of λ only, that is

$$p(a|xyb\lambda) = p(a|x\lambda), \tag{3}$$

$$p(b|xy\lambda) = p(b|y\lambda). \tag{4}$$

The measurement independence assumption expresses that the additional variable λ is statistically independent from the settings x and y

$$p(\lambda|xy) = p(\lambda). \tag{5}$$

Equivalently, this condition states that the inputs are free random variables, that is, x and y are chosen independently from λ.

From these two assumptions, a notion of separability appears

$$\begin{aligned} p(ab|xy) &= \sum_\lambda p(\lambda|xy)p(ab|xy\lambda) \\ &= \sum_\lambda p(\lambda)p(ab|xy\lambda) \\ &= \sum_\lambda p(\lambda)p(a|xy\lambda b).p(b|xy\lambda) \\ &= \sum_\lambda p(\lambda)p(a|x\lambda).p(b|y\lambda) \end{aligned} \tag{6}$$

where the first and third equality use Bayes' rule. Introducing

$$A(x,\lambda) = p(a=+1|x\lambda) - p(a=-1|x\lambda) \tag{7}$$
$$B(y,\lambda) = p(b=+1|y\lambda) - p(b=-1|y\lambda) \tag{8}$$

and inserting the notion of separability given in Eq. (6) into the definition of S given in Eq. (1), we find $S = \sum_\lambda p(\lambda)S_\lambda$ with

$$S_\lambda = A(0,\lambda)\left(B(0,\lambda)+B(1,\lambda)\right) + A(1,\lambda)\left(B(0,\lambda)-B(1,\lambda)\right). \tag{9}$$

Considering $|S_\lambda|$, we have

$$\begin{aligned} |S_\lambda| &\leq |A(0,\lambda)\left(B(0,\lambda)+B(1,\lambda)\right)| + |A(1,\lambda)\left(B(0,\lambda)-B(1,\lambda)\right)| \\ &= |A(0,\lambda)||\left(B(0,\lambda)+B(1,\lambda)\right)| + |A(1,\lambda)||\left(B(0,\lambda)-B(1,\lambda)\right)| \\ &\leq |\left(B(0,\lambda)+B(1,\lambda)\right)| + |\left(B(0,\lambda)-B(1,\lambda)\right)| \\ &= 2|B(0,\lambda)| \\ &\leq 2 \end{aligned}$$

where the first inequality uses the triangle inequality and the third inequality uses $A(x,\lambda) \in [-1,+1]$. For the fourth equality, we have considered without loss of generality that $B(0,\lambda) \geq B(1,\lambda) \geq 0$. We thus deduce that $|S| \leq \sum_\lambda p(\lambda)|S_\lambda| \leq 2\sum_\lambda p(\lambda) = 2$, that is

$$-2 \leq S \leq 2. \tag{10}$$

In a probability space, a violation of this inequality – the CHSH inequality – leads unequivocally to a contradiction with the locality or measurement independence assumption, i.e. it rules out any model satisfying the conditions Eqs. (3)-(5).

3 CHSH inequality in space-time

Now consider a physical implementation of the CHSH test, where at each round, a source produces a pair of physical systems – a pair of particle for example – one for Alice and one for Bob. Alice and Bob then measure these particles with a measurement device, corresponding to the box in the abstract description. They choose a measurement setting – the input – and receive a measurement result, the output. As the particle might have interacted in the source (the events associated to the creation of the two particles have overlapping past-light cones), a formalization of local-causality consists in suggesting that we should be able to identify a set of past factors, a predetermined program, described by some variables λ, having a joint influence on Alice and Bob outcomes and fully accounting for the dependence between a and b. Setting the test with a space-like separation, mutual influences between Alice's outcome and Bob's input and outcome (as well as Bob's outcome and Alice's input) lie outside past light-cones. In this configuration, the principle of local causality forces the locality assumptions in Eqs. (3) to hold. A violation of the inequality given in Ineq. (1) then implies that no theory satisfying the principle of local causality is able to reproduce the observed correlations or that the condition related to the measurement independence is not satisfied. In summary, the proposed derivation of Bell's inequalities in the physical space can be summarized in the following relation:

$$\begin{cases} \text{space-like separation + principle of local causality} \\ \text{measurement independence} \end{cases} \Rightarrow \text{Bell inequalities} \tag{11}$$

6 Quantum non-locality

Now consider the quantum scenario where Alice and Bob use the following measurements

$$
\begin{aligned}
& x=0 \equiv \sigma_x,\ x=1 \equiv \sigma_z, \\
& y=0 \equiv \frac{1}{\sqrt{2}}\left(\sigma_x+\sigma_z\right),\ y=1 \equiv \frac{1}{\sqrt{2}}\left(\sigma_x-\sigma_z\right)
\end{aligned}
\tag{12}
$$

on the state $|\phi^+\rangle = \frac{1}{2}\left(|00\rangle + |11\rangle\right)$. We have

$$
\begin{aligned}
& \langle x=0, y=0\rangle+\langle x=0, y=1\rangle+\langle x=1, y=0\rangle-\langle x=1, y=1\rangle \\
& =\sqrt{2}\left(\langle\sigma_x \sigma_x\rangle+\langle\sigma_z \sigma_z\rangle\right)=\sqrt{2}(1+1)=2 \sqrt{2}>2 .
\end{aligned}
\tag{13}
$$

The Bell-CHSH inequality can thus be violated using valid measurements and states in quantum theory. This means that we can set an experiment producing correlations that no theory satisfying the principle of local causality can explained if the condition related to the measurement independence is satisfied.

The best one can do to support the assumption about the measurement independence, see Eq. (5), is to choose the inputs with a process that is likely to be uncorrelated with λ. Some chose to use a quantum random number generator. Others generated the inputs of a Bell test from fluctuations of radiation coming from very distant stellar objects, i.e. produced by matter that has not been in contact with earthly matter since inflation, if ever. Others have argued that the best way to enforce the assumption about the measurement independence is to generate the inputs from a hash function of the latest million tweets with the hashtag #quantum. In order to maintain the idea of local causality in case a Bell inequality violation is observed, we must believe that the behavior of some physical systems is correlated to millions of twitter messages.

Note that ensuring space-like separation is not a trivial task. Let me emphasize that the idea is to enforce that Alice's box produces the output before any information about Bob's inputs may have arrived at Alice's location. As information cannot propagate at the speed exceeding the speed of light, this is in principle possible by setting a distance between Alice's and Bob's location larger than the time interval between the input choice and the outcome multiplied by the speed of light. When this space-like separation is not realized, it is said that the locality loophole is open. This is all good but to define the time interval, we need to know when the input was actually available and when the outcome was actually known. There is some uncertainty there, ultimately related to the time at which a measurement ends. This means that we can only conclude about the impossibility of local causality to reproduce the results of Bell tests under the assumptions that a proper space-time description of events at hand is done and the input are chosen freely.

7 What is Bell inequality violation good for?

First of all, a Bell test can be seen as a test of Nature. If a violation of a Bell inequality is observed and the measurement independence assumption holds, we conclude that Nature can produce correlations that are stronger than the ones from locally causal theories (those satisfying the principle of local causality). Importantly, this holds independently of any physical theory, i.e. the derivation of a Bell inequality is based on the statistics of conditional results only. This means that any post-quantum theory aiming to describe Nature globally has to be non locally causal (or non-local in short).

Within the quantum formalism, we can conclude from a Bell inequality violation that the measured state is entangled. It is easy to show that separable states lead to probability distributions that admit a local model, i.e. Bell non-locality implies entanglement. Consider the case where ρ is separable, that is, there exists $p_\lambda \geq 0$ and subsystem states $\{\rho_A^\lambda\}$ and $\{\rho_B^\lambda\}$ such that

$$\rho_{\text{sep}} = \sum_\lambda p_\lambda \, \rho_A^\lambda \otimes \rho_B^\lambda. \tag{14}$$

The statistics $p(ab|xy) = \text{Tr}\, M_a^{(x)} \otimes M_b^{(y)} \rho_{\text{sep}}$ can be obtained by a local strategy in which the random variable λ distributed according to p_λ is given to Alice and Bob's measurement devices which they use to prepare $\rho_A^\lambda/\rho_B^\lambda$ and to measure them according to $M_a^{(x)}/M_b^{(y)}$. In this case

$$p(ab|x,\lambda) = \sum_\lambda p_\lambda \underbrace{\text{Tr}\, M_a^{(x)} \rho_A^\lambda}_{p(a|x,\lambda)} \underbrace{\text{Tr}\, M_b^{(y)} \rho_B^\lambda}_{p(b|y,\lambda)} \implies S \leq 2. \tag{15}$$

A violation of the CHSH inequality thus implies that ρ is entangled. Notice that CHSH inequality violation provides a device-independent way to certify the presence of entanglement, i.e. we conclude about entanglement without specifying the Hilbert space dimension, nor the direction of the measurement settings. Note also that some entangled state do not violate a Bell inequality, i.e. are local. The set of states violating a Bell inequality (Bell-correlated states) is thus smaller than the set of entangled states.

Note also that some correlations produced by Bell-correlated states can be reproduced by local models. In particular, consider the state $|\phi^+\rangle$ and measurements $x = 0 \equiv y = 1 \equiv \sigma_x$, $x = 1 \equiv y_0 \equiv \sigma_z$ leads to correlations that can be reproduced by the following local recipe. Let's assume that Alice and Bob share a vector $\vec{\lambda}$ which takes four equiprobable values: $\vec{\lambda}_{1,2} = \frac{\vec{x} \pm \vec{z}}{\sqrt{2}}$ and $\vec{\lambda}_{3,4} = -\vec{\lambda}_{1,2}$. Labeling $\vec{X}$ (respectively $\vec{Y}$) the unit vector in the direction of the measurement on Alice's side (Bob's side), we define the local strategy as $\text{sign}\left(\vec{\lambda}.\vec{X}\right)$ for Alice and -$\text{sign}\left(\vec{\lambda}.\vec{Y}\right)$ for Bob. This strategy would be indistinguishable from the quantum scenario in which measurements are performed on $|\phi^+\rangle$, i.e. the outcomes probabilities are the same. Proving that a state is Bell-correlated hence takes specific measurements.

We will also see in the next lectures that Bell non-locality is a resource for quantum device certification, randomness generation or quantum key distribution.