How to exploit Bell non-locality to advance communication technologies?

Lecture 1: State, Evolution and Measurement

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1 Introduction

As for any physical theory, quantum theory aims to predict the results of measurements. This takes a description of the system state, its evolution and the measurement. The goal of this lecture is to provide a general quantum description of these three ingredients. We will first see how they are described in an ideal formalism, intended to characterize the behavior of the entire universe. We will then see that in practice, when we are interested in describing a small part of the entire universe only, the formalism needs to be adapted. Fondamental principles, like the superposition principle and the corresponding notion of entanglement for multipartite systems will also be discussed.

2 The axioms of quantum theory

Associated to any isolated physical system is an Hilbert space, a complex vector space with an inner product known as the state space.

- **States** The system state is described by its state vector, i.e. a unit column vector in the system's state space. It is represented by a "ket" $|\psi\rangle$ (Dirac notation).

  The "bra" $\langle\psi|$ denotes a raw vector composed of conjugated elements of $|\psi\rangle$. The inner product $\langle\psi|\psi\rangle$ is thus a positive real number, whose square root is called the norm of $|\psi\rangle$. All physical states have a unit norm.

  More generally, the scalar product of two states $\langle\phi|\psi\rangle$ gives a complex number from any pair of states $\{|\phi\rangle, |\psi\rangle\}$.

Note that $|\psi\rangle$, $\langle\psi|$ and $|\psi\rangle\langle\psi|$ have the same information content and provide a complete description of the system state.

From the linearity of the Hilbert space it follows that, given two states of the Hilbert space $|\phi\rangle$ and $|\psi\rangle$, we can form a new element $|\zeta\rangle = \alpha|\phi\rangle + \beta|\psi\rangle$, where α and β are complex numbers, which belongs to the Hilbert space. The "superposition principle" says that if $|\zeta\rangle$ is normalized $\langle\zeta|\ \zeta\rangle = 1$, it also corresponds to a valid system state. If the states $|\phi\rangle$ and $|\psi\rangle$ are orthogonal, i.e. $\langle\phi|\psi\rangle = 0$, $\langle\xi|\xi\rangle = 1$ means that α and β satisfy $|\alpha|^2 + |\beta|^2 = 1$. The physical picture behind a superposition state, e.g. $\frac{1}{\sqrt{2}}(|\phi\rangle + |\psi\rangle)$ is *not* that the described system is either in state $|\phi\rangle$ or in state $|\psi\rangle$, it is in state $|\phi\rangle$ AND in state $|\psi\rangle$. Note that two elements of the Hilbert space $|\phi\rangle$ and $|\psi\rangle$ that are related by a global phase factor $|\psi\rangle = e^{i\,\delta}\,|\phi\rangle$, correspond to the same physical state. This follows from the observation that such two states give the same measurement results for all possible measurements.

- **Observable and measurement** An *observable* corresponding to a physical quantity (e.g. energy) is described by an operator M which is hermitian $M^\dagger = M$ (where † is the conjugate transpose). The expectation value of the observable for a system prepared in a state $|\phi\rangle$ is given by

$$\langle M\rangle = \langle\phi|\, M\, |\phi\rangle\,. \tag{1}$$

The fact that M is Hermitian ($M = M^\dagger$) ensures that the expected value $\langle M\rangle = \overline{\langle M\rangle}$ is real (physical quantities are real) because eigenvalues of hermitian operators are real. In addition, it ensures that the matrix can be diagonalized, i.e. there exists an orthonormal basis $\{|1\rangle, \ldots, |n\rangle\}$ in which M is expressed as

$$M = \sum_{m=1}^{n} \lambda_m |m\rangle\langle m|. \tag{2}$$

We can interpret this as follows. The eigenvalue λ_m corresponds to one possible outcome of the observable M. $|m\rangle$ is the corresponding eigenvector and $P_m = |m\rangle\langle m|$ is the corresponding projector. A projector satisfies $P_m P_k = \delta_{k,m} P_m$ and $P_m^\dagger = P_m$.

For a system prepared in a state $|\phi\rangle$, the probability to observe the outcome λ_m with such a measurement is given by the Born rule

$$p_m = |\langle\phi|m\rangle|^2 = \langle\phi|m\rangle\langle m|\phi\rangle = \langle\phi| P_m |\phi\rangle . \tag{3}$$

Note that the probability to get all the outcomes must sum to one, that is $\sum_k |\langle k|\phi\rangle|^2 = \sum_k |\alpha_k|^2 = 1$ where $|\phi\rangle = \sum_k \alpha_k |k\rangle$ which explains why we want physical states to be well normalized.

Right after the outcome λ_m is obtained, the system is in state

$$\frac{P_m |\psi\rangle}{\sqrt{\langle\psi|P_m|\psi\rangle}}. \tag{4}$$

The previous formula essentially tells us that the system state is projected into $|m\rangle$ after the measurement (and independently of the state of the system before the measurement). This follows from the fact that if the same measurement is repeated right after the first measurement, we expect the outcomes to be exactly the same, that is

$$\langle m| P_m |m\rangle = |\langle m|m\rangle|^2 = 1. \tag{5}$$

This means that (projective) measurement strongly perturbs the quantum system. This is fundamentally different from classical systems, which can be measured without disturbance.

Note that p_m is equivalently given by

$$p_m = \mathrm{Tr}\left(|\phi\rangle\langle\phi|\, |m\rangle\langle m|\right) \tag{6}$$

where the trace $\mathrm{Tr}\,(...)$ is defined as $\sum_k \langle k| \ldots |k\rangle$ for any choice of basis $\{|k\rangle\}$. The trace is cyclic $\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(BCA) = \mathrm{Tr}\,(CAB)$ and linear $\mathrm{Tr}\,(\alpha A + \beta B) = \alpha\,\mathrm{Tr}\,(A) + \beta\,\mathrm{Tr}\,(B)$. We also have $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A).\,\mathrm{Tr}(B)$.

- **Evolution** The time evolution of an isolated quantum system is linear and described by a unitary transformation

$$U : |\phi\rangle \mapsto U |\phi\rangle , \tag{7}$$

with a matrix U such that $U^\dagger U = UU^\dagger = \mathbb{1}$.

The unitarity of the transformation is imposed because the evolution has to preserve the normalization of states, i.e. map physical states to physical states. This requires

$$1 = \langle\phi|\phi\rangle = (|\phi\rangle)^\dagger |\phi\rangle \mapsto (U |\phi\rangle)^\dagger U |\phi\rangle = \langle\phi| U^\dagger U |\phi\rangle = 1, \quad (8)$$

that is $U^\dagger U = \mathbb{1}$.

3 General description of a quantum state

In many cases, a pure state $|\psi\rangle$ does not give an appropriate description of the system state. Instead the state must be modeled by a *density matrix*. We here consider an exemple in which information about a state preparation is partially known and show that a pure state description is insufficient in this case. After a quick presentation of the properties of density matrices, we will discuss the physics of bipartite systems including the concept of entanglement and we will see that the notion of density matrices is needed when considering the reduced state of some entangled states.

3.1 Mixture of pure states

Consider the following state preparation scheme. Bob throws a biased coin, that gives heads with probability p and tails with probability $1-p$. He then prepares the system A in the state $|\psi\rangle\langle\psi|$ or $|\psi'\rangle\langle\psi'|$ depending on the result of the thrown. Finally, he gives the quantum system to Alice without revealing the result of the coin flip to her. The best description of the system state for Alice is

$$|\psi\rangle\langle\psi| \quad \text{with probability} \quad p \qquad \text{or} \qquad |\psi'\rangle\langle\psi'| \quad \text{with probability} \quad 1-p.$$

Further consider that after Alice's system undergoes some unitary evolution U, she measures it in some basis. The probability that she observes the measurement outcome associated with some projector P_i is naturally given by

$$\begin{aligned} p_i &= p \,\mathrm{tr}\big(P_i\, U\, |\phi\rangle\langle\phi|\, U^\dagger\big) + (1-p)\,\mathrm{tr}\big(P_i\, U\, |\phi'\rangle\langle\phi'|\, U^\dagger\big) \\ &= \mathrm{tr}\big(P_i U (p\, |\phi\rangle\langle\phi| + (1-p)\, |\phi'\rangle\langle\phi'|) U^\dagger\big) \equiv \mathrm{tr}\big(P_i U\, \rho\, U^\dagger\big) \end{aligned}$$

where the second equality uses the linearity of the trace. The last equality (equivalence) suggests to describe Alice's state with an operator ρ

$$\rho = p\, |\phi\rangle\langle\phi| + (1-p)\, |\phi'\rangle\langle\phi'|\,. \quad (9)$$

which brings us to the notion of density matrix.

3.2 Density matrix

In the most general case, the state of a quantum system is described by a density matrix (or density operator) ρ satisfying

$$\rho = \rho^\dagger \tag{10}$$
$$\rho \geq 0 \tag{11}$$
$$\operatorname{tr} \rho = 1. \tag{12}$$

The three conditions insure that the probabilities of measurement outcomes are real, non-negative and sum up to one [1].

3.3 Purity of a state

The purity of a state ρ is defined as

$$\text{Purity}(\rho) = \operatorname{tr} \rho^2. \tag{15}$$

It is equal to one if and only if the density operator is a projector $\rho = |\psi\rangle\langle\psi|$[2], in other words if the state ρ is pure.

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[1] ρ is Hermitian also implies that there always exists a basis $\{|\psi_i\rangle\}_{i=1}^d$ in which it is diagonal

$$\rho = \sum_{i=1}^{d} p_i \, |\psi_i\rangle\langle\psi_i| \, . \tag{13}$$

However, there are infinitely many ways to decompose ρ as a mixture of pure states. Any basis $\{|\psi_i\rangle\}_{i=1}^d$ provides a partition of identity $\mathbb{1} = \sum_{i=1}^d |\psi_i\rangle\langle\psi_i|$ and hence

$$\rho = \sqrt{\rho}\,\mathbb{1}\sqrt{\rho} = \sqrt{\rho}\sum_{i=1}^{d} |\psi_i\rangle\langle\psi_i| \sqrt{\rho} = \sum_{i=1}^{d} p_i' \, |\psi_i'\rangle\langle\psi_i'| \, , \tag{14}$$

to give a decomposition of ρ in pure states $|\psi_i'\rangle = \frac{1}{\sqrt{p_i'}}\sqrt{\rho}\,|\psi_i\rangle$ and $p_i' = \operatorname{tr}\rho\,|\psi_i\rangle\langle\psi_i|$. Note that the states $\{|\psi_i'\rangle\}_{i=1}^d$ are not orthonormal in general.

[2] The proof that ρ is a projector, i.e. $\rho = |\phi\rangle\langle\phi|$ implies that $\operatorname{Tr}\rho^2 = 1$ is direct as if $\rho = |\phi\rangle\langle\phi|$, $\rho^2 = \rho$ and $\operatorname{Tr}\rho = 1$. For the proof that $\operatorname{Tr}\rho^2 = 1$ implies that $\rho = |\phi\rangle\langle\phi|$, we write ρ in a diagonal from $\rho = \sum_i \lambda_i \, |i\rangle\langle i|$ where the eigenvalues are real and non-negative and $\sum_i \lambda_i = 1$. We have

$$\operatorname{Tr}\rho^2 = 1 = \sum_i (\lambda_i)^2 = (\sum_i \lambda_i)^2 - 2\sum_{i<j}\lambda_i\lambda_j = 1 - 2\sum_{i<j}\lambda_i\lambda_j. \tag{16}$$

The only way to fulfill the equality $1 - 2\sum_{i<j}\lambda_i\lambda_j = 1$ given that $0 \leq \lambda_i \leq 1$ is by imposing all the λ_is to be 0 except one which takes the value 1. Hence ρ is a projector.

3.4 Physics of bipartite systems

Consider a system of two qubits. If the qubit A is prepared in a state $|\psi\rangle_A \in \mathbb{C}^2$ and the qubit B is prepared in the state $|\phi\rangle_B \in \mathbb{C}^2$, we can describe the state of the global system by giving the pair $|\psi\rangle_A$ and $|\phi\rangle_B$. Mathematically, we write the global state as

$$|\Psi\rangle = |\psi\rangle_A \otimes |\phi\rangle_B \in \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2, \tag{17}$$

where $\otimes$ is the tensor product ($\mathbb{C}^2 \otimes \mathbb{C}^2$ is the tensor product Hilbert space). This is a linear operation, that is

$$\begin{aligned}&(\lambda |\psi\rangle_A + \lambda' |\psi'\rangle_A) \otimes (\mu |\phi\rangle_B + \mu' |\phi'\rangle_B) \qquad (18)\\ &= \lambda\mu |\psi\rangle_A \otimes |\phi\rangle_B + \lambda\mu' |\psi\rangle_A \otimes |\phi'\rangle_B + \lambda'\mu |\psi'\rangle_A \otimes |\phi\rangle_B + \lambda'\mu' |\psi'\rangle_A \otimes |\phi'\rangle_B \,.\end{aligned}$$

An operator O_A acting on A alone is represented as

$$(O_A \otimes \mathbb{1}_B)(|\psi\rangle_A \otimes |\phi\rangle_B) = (O_A |\psi\rangle_A) \otimes |\phi\rangle_B \,. \tag{19}$$

Given the bases $\{|0\rangle, |1\rangle\}$ for Alice's and Bob's qubits, the canonical basis for $\mathcal{H}$ can be constructed as

$$\{|0\rangle_A \otimes |0\rangle_B , |0\rangle_A \otimes |1\rangle_B , |1\rangle_A \otimes |0\rangle_B , |1\rangle_A \otimes |1\rangle_B\} \tag{20}$$

with $(\langle i|_A \otimes \langle j|_B)(|i'\rangle_A \otimes |j'\rangle_B) = \langle i|i'\rangle\langle j|j'\rangle = \delta_{ii'}\delta_{jj'}$. Hence, the dimension of a tensor product Hilbert space is the product of the dimensions $dim(\mathcal{H}_A \otimes \mathcal{H}_B) = dim(\mathcal{H}_A)dim(\mathcal{H}_B)$. In our case $dim(\mathbb{C}^2 \otimes \mathbb{C}^2) = 4$. Two qubit states can be represented as 4 component vectors with

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ a_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix} . \tag{21}$$

Similarly, operators are represented as 4×4 matrices, where the tensor product of operators is obtained as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_{21} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{22} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{pmatrix} . \tag{22}$$

Just as for a single system, the measurement probabilities are computed with the Born rule in the multipartite case. Let A and B share a two qubit state $\rho_{AB} = |\Psi\rangle\langle\Psi|$, i.e. ρ_{AB} is here seen as an alternative description of the joint state of Alice and Bob. (The precise form of $|\Psi\rangle$ is here unknown, i.e. the individual state of Alice and Bob are not given). Alice measures her qubit in a basis $\{|\lambda_0\rangle_A, |\lambda_1\rangle_A\}$ and Bob measures his qubit in the basis $\{|\mu_0\rangle_B, |\mu_1\rangle_B\}$. The probability to observe outcomes "k" for Alice and "j" for Bob is given by

$$p_{k,j} = \mathrm{Tr}\,\rho_{AB}(|\lambda_k\rangle\!\langle\lambda_k|_A \otimes |\mu_j\rangle\!\langle\mu_j|_B). \tag{23}$$

Just as for a single system the measurement perturbs the state of Alice's and Bob's qubit, such that their post-measurement state conditional on k and j is

$$\rho_{AB}|_{k,j} = |\lambda_k\rangle\!\langle\lambda_k|_A \otimes |\mu_j\rangle\!\langle\mu_j|_B\,, \tag{24}$$

ensuring that if the same measurement is performed again the same outcomes are observed. Note that both the probability of the outcome and the post-measured state can be obtained from the operator

$$\rho_{A,B}(k,j) = (|\lambda_k\rangle\!\langle\lambda_k|_A \otimes |\mu_j\rangle\!\langle\mu_j|_B)\rho_{AB}(|\lambda_k\rangle\!\langle\lambda_k|_A \otimes |\mu_j\rangle\!\langle\mu_j|_B), \tag{25}$$

by

$$p_{k,j} = \mathrm{Tr}\,\rho_{A,B}(k,j) \tag{26}$$

$$\rho_{AB}|_{k,j} = \frac{\rho_{A,B}(k,j)}{\mathrm{Tr}\,\rho_{A,B}(k,j)}. \tag{27}$$

So one can understand $\rho_{A,B}(k,j)$ as the "unnormalized" post-measurement state of Alice and Bob

$$\rho_{A,B}(k,j) = p_{k,j}\,\rho_{AB}|_{k,j}. \tag{28}$$

The next question is given the bipartite state $\rho_{AB} = |\Psi\rangle\!\langle\Psi|$, what is the reduced state of Bob? This is a priori not easy as we don't know much about $|\Psi\rangle$, i.e. the state of Alice and Bob are not specified separately. But consider the situation where Alice measures her qubit only. In this case, the probability to observe the outcome "k" (associated to the projector $|\lambda_k\rangle\!\langle\lambda_k|_A$) is obtained by computing

$$p_k = \mathrm{Tr}\,\rho_{AB}(|\lambda_k\rangle\!\langle\lambda_k|_A \otimes \mathbb{1}_B). \tag{29}$$

The post-measured state is given by

$$\rho_{AB}|_k = \frac{(|\lambda_k\rangle\langle\lambda_k|_A \otimes \mathbb{1}_B)\rho_{AB}(|\lambda_k\rangle\langle\lambda_k|_A \otimes \mathbb{1}_B)}{\mathrm{Tr}(|\lambda_k\rangle\langle\lambda_k|_A \otimes \mathbb{1}_B)\rho_{AB}(|\lambda_k\rangle\langle\lambda_k|_A \otimes \mathbb{1}_B)} \tag{30}$$

$$= \frac{1}{p_k} |\lambda_k\rangle\langle\lambda_k|_A \otimes \langle\lambda_k|_A \rho_{AB} |\lambda_k\rangle_A . \tag{31}$$

Here it is natural to conclude that the state of Alice is $|\lambda_k\rangle\langle\lambda_k|_A$, as expected from the measurement result, meaning that Bob's state is

$$\rho_B|_k = \frac{1}{p_k} \langle\lambda_k|_A \rho_{AB} |\lambda_k\rangle_A . \tag{32}$$

When considering the state $\tilde{\rho}_{AB}$ that Alice and Bob gets in average

$$tilde\rho_{AB} = \sum_k p_k |\lambda_k\rangle\langle\lambda_k| \otimes \rho_B|_k = \sum_k |\lambda_k\rangle\langle\lambda_k| \otimes \langle\lambda_k|_A \rho_{AB} |\lambda_k\rangle_A \tag{33}$$

a natural guess for Bob's state is

$$\rho_B = \sum_k \langle\lambda_k|_A \rho_{AB} |\lambda_k\rangle_A \tag{34}$$

which corresponds to what is called the partial trace over A, i.e. $\rho_B = \mathrm{Tr}_A(\rho_{AB})$. We can check that (34) defines a proper density matrix. Indeed, for any state $|\phi\rangle$, we have

$$\langle\phi| \rho_B |\phi\rangle = \sum_k \langle\lambda_k|_A \otimes \langle\phi| \rho_{AB} |\lambda_k\rangle_A \otimes |\phi\rangle \geq 0 \tag{35}$$

since $\rho_{AB} \geq 0$. Moreover,

$$\mathrm{Tr}_B\, \rho_B = \sum_{k,j} \langle\lambda_k|_A \otimes \langle\mu_j| \rho_{AB} |\lambda_k\rangle_A \otimes |\mu_j\rangle = \mathrm{Tr}\,(\rho_{AB}) = 1. \tag{36}$$

Finally, the check that $\rho_B^\dagger = \rho_B$ is obtained directly from the constraint that ρ_{AB} is hermitian, i.e.

$$\rho_B^\dagger = \sum_k \langle\lambda_k|_A \rho_{AB}^\dagger |\lambda_k\rangle_A = \sum_k \langle\lambda_k|_A \rho_{AB} |\lambda_k\rangle_A = \rho_B. \tag{37}$$

In full generality, given a bipartite state ρ_{AB}, the partial state of Bob is obtained from the partial trace

$$\rho_B = \mathrm{Tr}_B\, \rho_{AB} \tag{38}$$

which is obtained by choosing any basis $\{|\phi_i\rangle_A\}_{i=1}^d$ of Alice's Hilbert space and computing

$$\rho_B = \sum_i \mathrm{tr}_A(|\phi_i\rangle\langle\phi_i|_A \otimes \mathbb{1}_B)\rho_{AB} = \sum_i \langle\phi_i|_A \, \rho_{AB} \, |\phi_i\rangle_A \,. \tag{39}$$

This is the sum of all Bob's states $\rho_{B|i} = \frac{1}{p_i}\mathrm{tr}_A(|\phi_i\rangle\langle\phi_i|_A \otimes \mathbb{1}_B)\rho_{AB}$ conditioned on Alice measuring her subsystem in the basis $\{|\phi_i\rangle_B\}_{i=1}^d$ and observing the outcome i. So from Bob's perspective, this situation is equivalent to the one where Alice prepares his state $\rho_{B|i}$ based on a value of a random variable distributed accordingly to p_i.

Before presenting measurement beyond projective measurement, let us quickly discuss the notion of entangled states. We have seen that product states of two qubits are represented as elements $|\psi\rangle_A \otimes |\phi\rangle_B$ on the global Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. The superposition principle implies that superpositions of product states also represent physical states. In fact, any state

$$|\psi\rangle = \sum_{i,j=0,1} \alpha_{ij} \, |i\rangle_A \otimes |j\rangle_B \tag{40}$$

with $\sum_{ij} |\alpha_{ij}|^2 = 1$ is a valid state. Remarkably, some of these states can not be written as a tensor product of qubit states

$$\langle\Psi| \, (|\psi\rangle_A \otimes |\phi\rangle_B) < 1 \quad \forall \, |\psi\rangle_A \,, |\phi\rangle_B \,. \tag{41}$$

This is the case of the state $|\phi^+\rangle$ which we discussed in the first exercise sheet. Any such pure state is called entangled.

Note that for pure bipartite product states, the reduced states are necessarily pure. This can be shown by the Schmidt decomposition which tells us that for any state $|w\rangle = \sum_{jk} a_{jk} \, |j\rangle \, |k\rangle$ in the tensor product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, there exist orthonormal states $|i\rangle_A$ for system A, and orthonormal states $|i_B\rangle$ of system B such that $|w\rangle = \sum_i \lambda_i \, |i\rangle_a \, |i\rangle_b$ where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$[3]. Bipartite product states have

[3] This can be easily seen by considering two orthonormal bases $|j\rangle$ and $|k\rangle$ for A and B. Then the tensor product form a basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ and we can write $|\omega\rangle = \sum_{j,k} a_{jk} \, |j\rangle \, |k\rangle$ for a matrix a with the complex coefficient a_{jk}. The singular decomposition of a gives $a = udv$ where u and v are unitaries and d is a diagonal matrix with real non-negative elements. This gives $|\omega\rangle = \sum_{i,j,k} u_{ji} d_{ii} v_{ik} \, |j\rangle \, |k\rangle$. Defining $|i\rangle_a = \sum_j u_{ji} \, |j\rangle$ and $|i\rangle_b = \sum_k v_{ik} \, |k\rangle$ and $\lambda_i \equiv d_{ii}$, we get $|w\rangle = \sum_i \lambda_i \, |i\rangle_a \, |i\rangle_b$. Importantly, the singular values $\lambda_i = d_{ii}$ are uniquely determined by the eigenvalues of $aa^\dagger$.

a Schmidt rank 1[4] (only one value λ_i is non zero) which implies that the reduced states are pure. For a pure bipartite state, one can thus recognize entanglement by simply showing that the reduced density operators are not pure.

The notion of entanglement can be extended to mixed states : A state is said to be separable if and only if it can be written in the form

$$\rho = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B, \text{ with } \lambda_i > 0 \text{ and } \sum_i \lambda_i = 1 \tag{42}$$

If a state is not separable then it is entangled. However, determining when a mixed state is entangled is a much more difficult problem in general. The Peres criterion, entanglement witnesses and Bell inequalities are examples of tools that can be used in practice to detect entanglement.

The intuition behind the definition of separability is that separable states must be classically correlated. More precisely, the production of a separable state only takes local operations and classical communication. Alice and Bob can, by classical communication, share a random number generator that produces the outcomes i with probabilities p_i. For each of the outcomes, they can agree to produce the state $\rho_i^A \otimes \rho_i^B$ locally. By this procedure, they produce the statistical mixture $\sum_i \lambda_i \rho_i^A \otimes \rho_i^B$, with $\lambda_i > 0$ and $\sum_i \lambda_i = 1$. This procedure is not specific for quantum theory, which justifies the notion of classical correlations. Otherwise, if a state is entangled, the correlations cannot originate from the classical procedure described above. In this sense entangled states are a typical feature of quantum theory.

4 General description of a measurement

It is the case that when a measurement involves a coupling to an auxiliary system and projections of this auxiliary system, it cannot be described by a projective measurement. In full generality, a measurement is described by a collection of measurement operators $\{M_m\}$, called Kraus operators, satisfying

$$M_m^\dagger M_m \geq 0, \tag{43}$$

$$\sum_m M_m^\dagger M_m = \mathbb{1}. \tag{44}$$

[4]A bipartite state is a product state if there is a decomposition of the form $|w\rangle = |j\rangle \otimes |k\rangle$, i.e. $a_{jk} = 1$ for one pair jk only. As λ_i are the eigenvalues of $aa^\dagger$, only one $\lambda_i \neq 0$ for product states.

The index m refers to the measurement outcome. If the quantum system is in a state $|\psi\rangle$ immediately before the measurement, then the probability of observing the result m is given by

$$p_m = \langle\psi|M_m^\dagger M_m|\psi\rangle, \tag{45}$$

which is guaranteed to be positive because $M_m^\dagger M_m \geq 0$. Moreover, $\sum_m M_m^\dagger M_m = \mathbb{1}$ ensures that $\sum_m p_m = 1$. The state of the system immediately after the measurement is given by

$$\frac{M_m|\psi\rangle}{\langle\psi|M_m^\dagger M_m|\psi\rangle}. \tag{46}$$

Such a general quantum measurement is called positive-operator-value-measure (POVM), the operators $E_m = M_m^\dagger M_m$ are called POVM elements and $\{M_m\}$ is a set of Kraus operators.

5 General description of evolution

It is the case that when the evolution of a system uses a unitary transformation involving an auxiliary system which is then traced out, the evolution cannot be described by a unitary. In full generality, any such transformation corresponds to a (linear) map

$$\mathcal{E} : \rho \mapsto \mathcal{E}[\rho] \tag{47}$$

satisfying

1. Trace reserving: $\mathrm{tr}\mathcal{E}[\rho] = 1$ if $\mathrm{tr}\rho = 1$
2. Completely positive: $(\mathcal{E}_A \otimes \mathbb{1}_B)[|\Psi\rangle\langle\Psi|] \geq 0$ for any state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

Such a map is called completely positive trace preserving (CPTP).

Note that for any CPTP map $\mathcal{E}$, there is a family of Kraus operators $\{K_i\}_{i=1}^n$ satisfying $K_i^\dagger K_i \geq 0$ and $\sum_i K_i^\dagger K_i = \mathbb{1}$ such that

$$\mathcal{E}[\rho] = \sum_{i=1}^n K_i \rho K_i^\dagger. \tag{48}$$

The decomposition of a CPTP map in Kraus operators is not unique. Finally, it is worth mentioning that any set of operators $\{K_i\}_{i=1}^n$ satisfying $K_i^\dagger K_i \geq 0$

and $\sum K_i^\dagger K_i = \mathbb{1}$ defines a valid CPTP map.

Three equivalent representations are possible when describing the evolution of a subsystem using either

- a CPTP map,
- a unitary acting on an extended system, where the auxilary system is traced out,
- a set Kraus operators $\{K_i\}_{i=1}^n$ satisfying $\sum K_i^\dagger K_i = \mathbb{1}$.