PROBLEM SET 4

Bipartite scenario

ALICE: $X_A = \{\underbrace{1, 2, 3}_{x}\}$ $\quad \mathbb{P}(X_A = 1) = \mathbb{P}(X_A = 2) = \frac{1}{3}$
System A

↳ Prepares $|\psi_x\rangle$ accordingly.

→ Sent

BOB: POVM E $\{E_y\}$ → Register Y

$$\rho_{XA} = \sum_x p(x) \overbrace{|x\rangle\langle x|}^{\text{ONB}} \otimes \rho(x)$$

Bob's measurement: $A \mapsto AY$

$$\rho(x) \mapsto \sum_y \pi(y) \rho(x) \pi(y)^\dagger \otimes |y\rangle\langle y| \quad \text{where } \pi(y)^\dagger \pi(y) = E(y)$$

So that: $\rho'_{XAY} = \sum_{x,y} p(x) |x\rangle\langle x| \otimes \pi(y) \rho(x) \pi(y)^\dagger \otimes |y\rangle\langle y|$

Strategy to get the greatest amount of information about Alice's register?

① Strategy using two orthogonal states:

$$\begin{cases} |\psi_1\rangle = |\psi_2\rangle = |\psi\rangle \\ |\psi_3\rangle = |\psi_\perp\rangle \end{cases}$$

ⓐ Using the projectors? (Alice destroys some information)

$E_1 = |\psi\rangle\langle\psi| \; (= E_2)$

$E_3 = |\psi_\perp\rangle\langle\psi_\perp|$

$$\mathbb{P}(y=1) = \mathrm{Tr}(E_1 \rho(x)) = \mathrm{Tr}\left(|\psi\rangle\langle\psi| \left(p_1 |\psi\rangle\langle\psi| + p_2 |\psi\rangle\langle\psi|\right)\right) = p_1 + p_2 = \frac{2}{3}$$

$$\mathbb{P}(y=3) = \mathrm{Tr}(E_3 \rho(x)) = p_3 = \frac{1}{3}$$

$$\left.\begin{array}{l} \mathbb{P}(y=1 \mid x=1) = 1 \\ \mathbb{P}(y=1 \mid x=2) = 1 \\ \mathbb{P}(y=3 \mid x=3) = 1 \end{array}\right\} \quad \begin{array}{l} \mathbb{P}(y=1) = \frac{1}{3} \times 2 \\ \mathbb{P}(y=3) = \frac{1}{3}. \end{array}$$

$$\mathbb{P}(y=1 \mid x=3) = 0$$

$$\mathbb{P}(y=3 \mid x=\underset{2}{1}) = 0$$

Mutual information:

$$I(X;Y) = \sum_{x=1}^{3} \sum_{y=1,3} \mathbb{P}(x,y) \log \frac{\mathbb{P}(x,y)}{\mathbb{P}(x)\mathbb{P}(y)}$$

$$= \left(\mathbb{P}(1,1) \log \frac{\mathbb{P}(1,1)}{\mathbb{P}_x(1)\mathbb{P}_y(1)} + \mathbb{P}(2,1) \log \frac{\mathbb{P}(2,1)}{\mathbb{P}_x(2)\mathbb{P}_y(1)} + \mathbb{P}(3,3) \log \frac{\mathbb{P}(3,3)}{\mathbb{P}_x(3)\mathbb{P}_y(3)} \right)$$

log → ln

$$= \frac{1}{3} \log\left(\frac{\cancel{1/3}}{\cancel{\frac{1}{3}} \times \frac{2}{3}} \right) + \frac{1}{3} \log\left(\frac{\cancel{1/3}}{\cancel{1/3} \times 2/3} \right) + \frac{1}{3} \log\left(\frac{\cancel{1/3}}{\cancel{1/3} \times \frac{1}{3}} \right)$$

$$= \frac{2}{3} \log\left(\frac{3}{2}\right) + \frac{1}{3} \log(3)$$

$$= \log(3) - \frac{2}{3} \log(2)$$

ⓒ For a density matrix:

$$\rho = \sum p_x \rho_x$$

Von Neumann entropy writes

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho)$$

Holevo Bound:

$$I(X;Y) \leq \chi_h = S(\rho) - \sum_x p_x S(\rho_x)$$

Here:

$$\rho = p_1 |\psi\rangle\langle\psi| + p_2 |\psi\rangle\langle\psi| + p_3 |\psi_\perp\rangle\langle\psi_\perp|$$

$$= \frac{2}{3} |\psi\rangle\langle\psi| + \frac{1}{3} |\psi_\perp\rangle\langle\psi_\perp|$$

$$S(\rho) = -\left(\frac{2}{3} \ln\left(\frac{2}{3}\right) \times 1 + \frac{1}{3} \ln\left(\frac{1}{3}\right) \times 1 \right)$$

$$= \frac{2}{3} \ln\left(\frac{3}{2}\right) + \frac{1}{3} \ln(3)$$

$$\sum p_x S(\rho_x) = 0$$

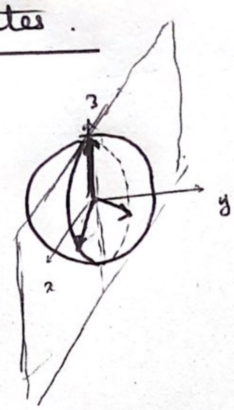
Thus $\chi = I(X;Y)$ here : Bob retrieved the maximal possible amount of information.

2 Strategy for non-orthogonal states.

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\psi_2\rangle = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$|\psi_3\rangle = \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}$$



Bob 3 POVM

$$E_a = \frac{2}{3}\left(\mathbb{1} - |\psi_a\rangle\langle\psi_a|\right) \qquad \text{where } a = 1, 2, 3.$$

Rq:
$$\sum_a E_a = 2\mathbb{1} - \frac{2}{3}\sum |\psi_a\rangle\langle\psi_a|$$
$$= 2\mathbb{1} - \frac{2}{3}\left(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{|\psi_1\rangle\langle\psi_1|} + \underbrace{\begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\frac{\sqrt{3}}{4} & 3/4 \end{pmatrix}}_{|\psi_2\rangle\langle\psi_2|} + \underbrace{\begin{pmatrix} 1/4 & +\sqrt{3}/4 \\ \frac{\sqrt{3}}{4} & 3/4 \end{pmatrix}}_{|\psi_3\rangle\langle\psi_3|} \right)$$
$$= 2\mathbb{1} - \frac{2}{3}\begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$
$$= \mathbb{1}.$$

$$\mathbb{P}(a=1|x=1) = \mathrm{Tr}\left(E_1 |\psi_1\rangle\langle\psi_1|\right)$$
$$= \mathrm{Tr}\left(\frac{2}{3}|\psi_1\rangle\langle\psi_1| - \frac{2}{3}|\psi_1\rangle\langle\psi_1|\right)$$
$$= 0$$

Similarly
$$\mathbb{P}(a=2|x=2) = 0$$
$$\mathbb{P}(a=3|x=3) = 0$$

$$\mathbb{P}(a=2|x=1) = \mathrm{Tr}\left(\frac{2}{3}\left(|\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|\psi_1\rangle\langle\psi_1|\right)\right)$$
$$= \frac{2}{3}\left(1 - |\langle\psi_2|\psi_1\rangle|^2\right)$$
$$= \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$$

Thus
$$\mathbb{P}(a=3|x=1) = \frac{1}{2}$$

$$\mathbb{P}(a=1|x=2) = \frac{2}{3}\left(1 - |\langle\psi_1|\psi_2\rangle|^2\right)$$
$$= \frac{1}{2}$$

Thus
$$\mathbb{P}(a=3|x=2) = \frac{1}{2}$$

$$\mathbb{P}(a=1|x=3) = \frac{2}{3}\left(1 - |\langle\psi_1|\psi_3\rangle|^2\right) = \frac{1}{2}$$

Thus $\mathbb{P}(a=2|x=2) = \frac{1}{2}$

$$\mathbb{P}(x|a) = \frac{\mathbb{P}(x,a)}{\mathbb{P}(a)} = \mathbb{P}(a|x)\,\frac{\mathbb{P}(x)}{\mathbb{P}(a)}$$

et $\mathbb{P}(a) = \sum_x \mathbb{P}(a,x) = \sum_x \mathbb{P}(a|x)\,\mathbb{P}(x)$

$\mathbb{P}(a=1) = \frac{1}{3} \times 1 = \frac{1}{3}$ $\qquad \mathbb{P}(a=2) = \mathbb{P}(a=3) = \frac{1}{3}$.

$\hookrightarrow$ Thus $\mathbb{P}(x|a) = \mathbb{P}(a|x)$.

$\mathbb{P}(x=1|a=1) = 0 = \mathbb{P}(x=2|a=2) = \mathbb{P}(x=3|a=3)$

$\mathbb{P}(x=2|a=1) = \frac{1}{2}$ $\quad$ etc.

b) $$I(A:B) = \sum_{x=1}^{3}\sum_{a=1}^{3} \mathbb{P}(a|x)\,\mathbb{P}(x)\,\log\left(\frac{\mathbb{P}(a|x)}{\mathbb{P}(a)}\right)$$

$$= 3 \times 2 \times \frac{1}{2} \times \frac{1}{3} \log\left(\frac{1/2}{1/3}\right)$$

$$\boxed{= \ln\left(\frac{3}{2}\right)}$$

c) $$\chi_h = S(\rho_{prep}) - \sum p_x \underbrace{S(\rho_x)}_{=0} \quad \text{(pure states)}$$

$$= S\left(\frac{1}{3} \times \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}\right)$$

$$= S\left(\frac{1}{2}\mathbb{1}\right) = -\mathrm{Tr}\left(\frac{1}{2}\mathbb{1}\ln\left(\frac{1}{2}\mathbb{1}\right)\right)$$

$$= -\frac{1}{2} \times (-2\ln 2)$$

ie $\boxed{\chi_h = \ln 2}$

Check that :

$I(A:B) \leq \chi_h$ $\qquad \boxed{\text{Ok}}$

:) $\ln(2) - \ln\left(\frac{3}{2}\right) = 2\ln 2 - \ln 3 = \ln(4) - \ln(3) > 0$.