Exercice 3 $\quad$ 1) & 2)

A) Local statistics in terms of correlation functions

1. $E(\vec{A}\vec{B}) = \sum_{a,b} ab\, p(ab|\vec{A}\vec{B})$

$$E(\vec{A}) = \sum_a a\, p(a|\vec{A}) = \sum_{ab} a\, p(ab|\vec{A}\vec{B}) \quad \text{(locality assumption)}$$

$$E(\vec{B}) = \sum_b b\, p(b|\vec{B}) = \sum_{ab} b\, p(ab|\vec{A}\vec{B})$$

2. $\sum_{ab} p(ab|\vec{A}\vec{B}) = 1$

i.e. we have a linear set of 4 independent equations which uniquely define $p(ab|\vec{A}\vec{B})$

$$p(+1+1|\vec{A}\vec{B}) = \frac{1}{4}\left(1 + E(\vec{A}) + E(\vec{B}) + E(\vec{A}\vec{B})\right)$$
$$p(-1+1|\vec{A}\vec{B}) = \frac{1}{4}\left(1 - E(\vec{A}) + E(\vec{B}) - E(\vec{A}\vec{B})\right)$$
$$p(+1-1|\vec{A}\vec{B}) = \frac{1}{4}\left(1 + E(\vec{A}) - E(\vec{B}) - E(\vec{A}\vec{B})\right)$$
$$p(-1-1|\vec{A}\vec{B}) = \frac{1}{4}\left(1 - E(\vec{A}) - E(\vec{B}) + E(\vec{A}\vec{B})\right)$$

3. We have

$$E(\vec{A},\vec{\lambda}) = p(+1|\vec{A}\vec{\lambda}) - p(-1|\vec{A}\vec{\lambda})$$
$$= p^A_{+1}\left(p^B_{+1} + p^B_{-1}\right) - p^A_{-1}\left(p^B_{+1} + p^B_{-1}\right)$$
$$= \underset{\times}{\underline{p(+1|\vec{A}\vec{\lambda})\, p(+1|\vec{B}\vec{\lambda})}} + p(+1|\vec{A}\vec{\lambda})\, p(-1|\vec{B}\vec{\lambda})$$
$$- p(-1|\vec{A}\vec{\lambda})\, p(+1|\vec{B}\vec{\lambda}) - p(-1|\vec{A}\vec{\lambda})\, p(-1|\vec{B}\vec{\lambda})$$

$$E(\vec{B},\vec{\lambda}) = \underset{\times}{\underline{p(+1|\vec{A}\vec{\lambda})\, p(+1|\vec{B}\vec{\lambda})}} + p(-1|\vec{A}\vec{\lambda})\, p(+1|\vec{B}\vec{\lambda})$$
$$- p(+1|\vec{A}\vec{\lambda})\, p(-1|\vec{B}\vec{\lambda}) - p(-1|A\lambda)\, p(-1|\vec{B}\vec{\lambda})$$

$$E(A\lambda).E(B\lambda) = E(AB\lambda) = \left(p^A_{+1} - p^A_{-1}\right)\left(p^B_{+1} - p^B_{-1}\right)$$
$$= \underset{\times}{\underline{p_{+1}p_{+1}}} - p_{-1}p_{+1} - p_{+1}p_{-1} + p_{-1}p_{-1}$$
$$1 = p_{+1}p_{+1} + p_{+1}p_{-1} + p_{-1}p_{+1} + p_{-1}p_{-1}$$

$$p(+1+1|\vec{A}\vec{B}) = \frac{1}{4}\left(1 + E(\vec{A}) + E(\vec{B}) + E(\vec{A}\vec{B})\right)$$
$$= \frac{1}{4}\int d\vec{\lambda}\, p(\vec{\lambda})\left(1 + E(\vec{A},\vec{\lambda}) + E(\vec{B},\vec{\lambda}) + E(\vec{A}\vec{B}\vec{\lambda})\right)$$
$$= \int d\vec{\lambda}\, p(\vec{\lambda})\; p(+1+1|\vec{A}\vec{B}\vec{\lambda})$$

and similarly for $p(-1+1)$, $p(+1-1)$, $p(-1-1)$

$\Rightarrow$ local model can also be defined from correlation functions.

B) Local strategy:

$\cancel{A = A^+}$

$\hookrightarrow A = \sum \lambda \boxed{|\lambda\rangle\langle\lambda|}$

B.1. $A = +1\, \Pi_+^A - 1\, \Pi_-^A$

$= +1 \frac{1}{2}(\mathbb{1} + \vec{A}\cdot\vec{\sigma}) - 1 \cdot \frac{1}{2}(\mathbb{1} - \vec{A}\vec{\sigma})$

$= \underline{\vec{A}\cdot\vec{\sigma}}$

B.2. Uniform means $\rho(\vec{\lambda}) = \frac{1}{4\pi}$ or $\rho(\vec{\lambda}) = \rho(\theta, \phi) = \frac{\sin\theta}{4\pi}$

B.3 $p_{+1} = \mathrm{Tr}\, \Pi_+^A \rho_{\vec{\lambda}} = \mathrm{Tr}\left(\frac{1}{2}(\mathbb{1} + \vec{A}\vec{\sigma}) \cdot \frac{1}{2}(\mathbb{1} + \vec{\lambda}\cdot\vec{\sigma})\right)$

$= \frac{1}{4}(2 + 2 \cdot \vec{A}\cdot\vec{\lambda}) = \frac{1}{2}(1 + \vec{A}\cdot\vec{\lambda})$

$p_{-1} = 1 - p_{+1} = \frac{1}{2}(1 - \vec{A}\vec{\lambda})$

$E(\vec{A}, \vec{\lambda}) = \sum_{a=\pm 1} a\, p(a | \vec{A}\vec{\lambda}) = p_{+1} - p_{-1} = \vec{A}\cdot\vec{\lambda}$

B.4. Let $\vec{B} = (\sin\theta_B \cos\phi_B, \sin\theta_B \sin\phi_B, \cos\theta_B)$

$\vec{\lambda} = (\sin\theta_\lambda \cos\phi_\lambda, \sin\theta_\lambda \sin\phi_\lambda, \cos\theta_\lambda)$

$\vec{B}\vec{\lambda} = \sin\theta_B \cos\phi_B \sin\theta_\lambda \cos\phi + \sin\theta_B \sin\phi_B \sin\theta_\lambda \sin\phi_\lambda + \cos\theta_B \cos\theta_\lambda$

$:= \cos\Xi_{B\lambda}$ where $\Xi = \arccos(\vec{B}\vec{\lambda})$

$p_{+1} = 1 - p_{-1}$

$p_{-1} = H(\cos\Xi_{B\lambda})$ with H the heaviside function

$\Rightarrow E(\vec{B}\vec{\lambda}) = 1 - 2H(\cos\Xi_{B\lambda})$

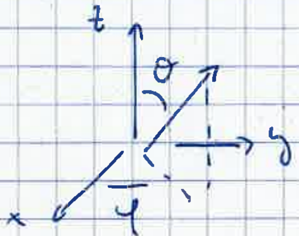
B.5. $E(\vec{A}) = \int d\vec{\lambda}\, \rho(\vec{\lambda})\, E(\vec{A}, \vec{\lambda}) = \int d\vec{\lambda} \frac{1}{4\pi} \vec{A}\cdot\vec{\lambda} = 0$ since $\vec{\lambda}$ is uniformly distributed.

Similarly $E(\vec{B}) = 0$

$$E(\vec{A},\vec{B}) = -\int d\vec{\lambda}\, \rho(\vec{\lambda})\, \mathrm{sign}(\vec{B}\cdot\vec{\lambda})\, \vec{A}\cdot\vec{\lambda}$$

$E(\vec{A},\vec{B})$ is invariant under any rotation of $\vec{A}$ and $\vec{B}$.

Let us choose $\vec{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{A} = \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}$, $\vec{\lambda} = \begin{pmatrix} \sin\theta_\lambda \cos\phi_\lambda \\ \sin\theta_\lambda \sin\phi_\lambda \\ \cos\theta_\lambda \end{pmatrix}$



$$\left.\begin{matrix} 0 \le \theta_\lambda \le \pi \\ 0 \le \varphi_\lambda \le 2\pi \end{matrix}\right) \quad \mathrm{sign}(\vec{B}\cdot\vec{\lambda}) \ge 0 \quad \text{if} \quad 0 \le \theta \le \frac{\pi}{2}$$

$$E(\vec{A},\vec{B}) = -\int_0^{\pi/2} d\theta_\lambda \int_0^{2\pi} d\varphi_\lambda \frac{\sin\theta_\lambda}{4\pi}\left(\sin\alpha \cancel{\sin\theta_\lambda \cos\phi_\lambda} + \cos\alpha\cos\theta_\lambda\right) \quad (1)$$

$$+ \int_{\pi/2}^{\pi} d\theta_\lambda \int_0^{2\pi} d\varphi_\lambda \frac{\sin\theta_\lambda}{4\pi}\left(\sin\alpha \cancel{\sin\theta_\lambda \cos\phi_\lambda} + \cos\alpha\cos\theta_\lambda\right) \quad (2)$$

(1) $$-\frac{2\pi}{4\pi}\int_0^{\pi/2} d\theta_\lambda \cos\alpha \left(\frac{1}{2}\sin 2\theta_\lambda\right)$$

$$= -\frac{1}{2}\cdot\frac{1}{2}\cos\alpha \underbrace{\int_0^{\pi/2} \sin 2\theta_\lambda\, d\theta_\lambda}_{\left[-\frac{1}{2}\cos(2\theta_\lambda)\right]_0^{\pi/2} = \frac{1}{2}+\frac{1}{2} = \frac{2}{2} = 1} = -\frac{1}{4}\cos\alpha$$

(2) $$\frac{2\pi}{4\pi}\int_{\pi/2}^{\pi} d\theta_\lambda \cos\alpha \left(\frac{1}{2}\sin 2\theta_\lambda\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cos\alpha \underbrace{\left[-\frac{1}{2}\cos 2\theta_\lambda\right]_{\pi/2}^{\pi}}_{= -\frac{1}{2}\left[1+1\right] = -1} = -\frac{1}{4}\cos\alpha$$

$$\Rightarrow E(\vec{A},\vec{B}) = -\frac{1}{2}\cos\alpha = -\frac{1}{2}\vec{A}\cdot\vec{B}$$

$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ (5)

$|00\rangle\ |10\rangle\ |01\rangle\ |11\rangle$

C.1.

$$|\psi^-\rangle\langle\psi^-| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Given that $\sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$$\sigma_y \otimes \sigma_y = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

$$\sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

$$\Rightarrow |\psi^-\rangle\langle\psi^-| = \frac{1}{4}\left(\mathbb{1} - \sigma_x\sigma_x - \sigma_y\sigma_y - \sigma_z\otimes\sigma_z\right)$$

C.2

$$p_{+1+1} = \mathrm{Tr}\left(\frac{\mathbb{1} + \vec{A}\cdot\vec{\sigma}}{2} \otimes \frac{\mathbb{1} + \vec{B}\cdot\vec{\sigma}}{2}\ |\psi^-\rangle\langle\psi^-|\right)$$

$$= \frac{1}{4} + \left(-\frac{1}{4}\vec{A}\cdot\vec{B}\right) \quad \text{since} \quad \mathrm{Tr}\,\sigma_i\sigma_j = 0 \text{ for } i \neq j$$

$$\mathrm{Tr}\,\sigma_i\sigma_i = 2$$

$$p_{-1-1} = \mathrm{Tr}\left(\frac{\mathbb{1} - \vec{A}\cdot\vec{\sigma}}{2} \otimes \frac{\mathbb{1} - \vec{B}\cdot\vec{\sigma}}{2}\ |\psi^-\rangle\langle\psi^-|\right)$$

$$= \frac{1}{4} + \left(-\frac{1}{4}\vec{A}\cdot\vec{B}\right)$$

$$p_{-1+1} = p_{+1-1} = \frac{1}{4} + \frac{1}{4}\vec{A}\cdot\vec{B}$$

for $|\psi^-\rangle\langle\psi^-|$ while for $\mathbb{1}/4$ $p_{++} = p_{--} = p_{-+} = p_{+-} = \frac{1}{4}$

C.3.

$$\Rightarrow E(\vec{A}, \vec{B}) = p(a=b|\vec{A}\vec{B}) - p(a\neq b|\vec{A}\cdot\vec{B}) = W\vec{A}\cdot\vec{B} = -W\ \vec{A}\cdot\vec{B}$$

$$E(\vec{A}) = (p_{+1+1} + p_{+1-1}) - (p_{-1-1} + p_{-1+1}) = 0$$

$$E(\vec{B}) = (p_{+1+1} + p_{-1+1}) - (p_{+1-1} + p_{-1-1}) = 0$$

C.4. The model provided in B) leads to

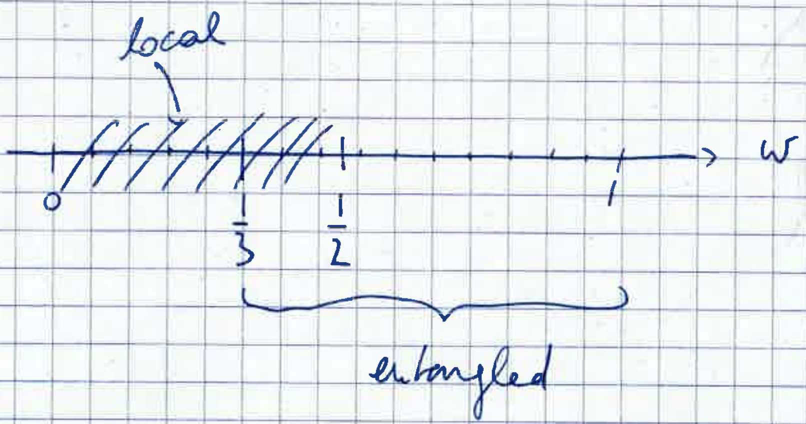
$$E(\vec{A},\vec{B}) = -\frac{1}{2}\,\vec{A}\cdot\vec{B}$$

$$E(\vec{A}) = 0 = E(\vec{B})$$

=> it can reproduce the correlation of the Werner state for $w = \frac{1}{2}$

A modification of the local model can also reproduce the correlation of the Werner state for $p < \frac{1}{2}$

C.5



=> This shows that there exist entangled states that are local

(They cannot be used to violate a Bell inequality)

The set of non local states is thus a subset of entangled states.