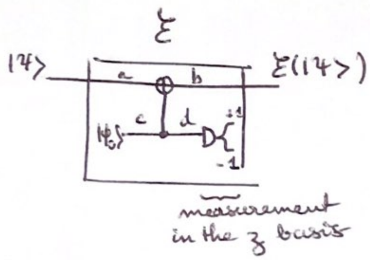
PROBLEM SET 2



$$|\varphi_0\rangle = \sqrt{t}\,|0\rangle + \sqrt{1-t}\,|1\rangle$$

① $\mathcal{H}_{qb} \otimes \mathcal{H}_{anc}$

$\sigma_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|00\rangle \mapsto |00\rangle$
$|10\rangle \mapsto |10\rangle$
$|01\rangle \mapsto |11\rangle$
$|11\rangle \mapsto |01\rangle$

$$\begin{array}{c c} & \begin{matrix} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{matrix} \\ \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} & \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix} \end{array} \quad \text{OR} \quad \begin{array}{c c} & \begin{matrix} |00\rangle & |10\rangle & |01\rangle & |11\rangle \end{matrix} \\ \begin{matrix} |00\rangle \\ |10\rangle \\ |01\rangle \\ |11\rangle \end{matrix} & \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right) \end{array}$$

Squared $\downarrow$ (Blocks)

$$\left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

Real & symmetric: $\mathcal{U}_{CNOT}^{+} = \mathcal{U}_{CNOT}^{t} = \mathcal{U}_{CNOT}$

~~And~~ $\mathcal{U}_{CNOT}^{2} = \mathbb{1}_4$

Thus CNOT is a unitary operation.

$$CNOT = \mathbb{1}_2 \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1|$$

② a/ If the measurement outcome is unknown ?

Before the measurement:

$$|\psi\rangle \otimes |\varphi_0\rangle = |\psi\rangle \otimes (\sqrt{t}\,|0\rangle + \sqrt{1-t}\,|1\rangle)$$

$$\downarrow$$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$\sqrt{t}\,|\psi\rangle \otimes |0\rangle + \sqrt{1-t}\,(\sigma_x|\psi\rangle) \otimes |1\rangle$$

$$= \sqrt{t}\,(\alpha|00\rangle + \beta|10\rangle) + \sqrt{1-t}\,(\beta|01\rangle + \alpha|11\rangle)$$

$|\alpha|^2 + |\beta|^2 = 1$

We write the density matrix in the 00 01 10 11 basis (convenient for partial trace over the ancillary).

$$\rho_{qb\otimes anc} = \begin{array}{c c} \left(\begin{array}{cc|cc} \bullet & & \bullet & \\ & \bullet & & \bullet \\ \hline \bullet & & \bullet & \\ & \bullet & & \bullet \end{array}\right) & \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} \\ \begin{matrix} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{matrix} & \end{array}$$

$$t\left(|\alpha|^2 |00\rangle\langle 00| + |\beta|^2 |10\rangle\langle 10| + \alpha\beta^* |00\rangle\langle 10| + \beta\alpha^* |10\rangle\langle 00|\right) + (1-t)\left(|\beta|^2 |01\rangle\langle 01| + |\alpha|^2 |11\rangle\langle 11| + \beta\alpha^* |01\rangle\langle 11| + \beta^*\alpha |11\rangle\langle 01|\right)$$

$$+ \sqrt{t(1-t)}\left(\alpha\beta^* |00\rangle\langle 01| + |\alpha|^2 |00\rangle\langle 11| + |\beta|^2 |10\rangle\langle 01| + \beta\alpha^* |10\rangle\langle 11| + \beta\alpha^* |01\rangle\langle 00| + |\beta|^2 |01\rangle\langle 10| + |\alpha|^2 |11\rangle\langle 00| + \alpha\beta^* |11\rangle\langle 10|\right)$$

$$\mathcal{E}(|\psi\rangle) = \mathrm{Tr}_{anc}(\rho_{qb\otimes anc}) = \begin{pmatrix} t|\alpha|^2 + (1-t)|\beta|^2 & t\alpha\beta^* + (1-t)\beta\alpha^* \\ t\beta\alpha^* + (1-t)\beta^*\alpha & t|\beta|^2 + (1-t)|\alpha|^2 \end{pmatrix}$$

Check that $\mathrm{Tr}(\mathcal{E}(|\psi\rangle)) = 1$:

$$\mathrm{Tr}(\mathcal{E}(|\psi\rangle)) = t(|\alpha|^2+|\beta|^2) + (1-t)(|\alpha|^2+|\beta|^2)$$

$$= 1$$

OK.

2b To see whether it is a pure state, we compute $\mathrm{Tr}(\mathcal{E}(|\psi\rangle)^2)$

$$\mathrm{Tr}(\mathcal{E}(|\psi\rangle)^2) = (t|\alpha|^2+(1-t)|\beta|^2)^2 + \overbrace{\underbrace{(t\alpha\beta^*+(1-t)\beta\alpha^*)(t\beta\alpha^*+(1-t)\beta^*\alpha)}_{= t^2|\alpha|^2|\beta|^2 + t(1-t)\alpha^2\beta^{*2} + t(1-t)\beta^2\alpha^{*2} + (1-t)^2|\beta|^2|\alpha|^2}}^{|\cdot|^2} \times 2$$

$$+ (t|\beta|^2+(1-t)|\alpha|^2)^2$$

$$= \underbrace{(t(|\alpha|^2+|\beta|^2)+(1-t)(|\alpha|^2+|\beta|^2))^2}_{=1} + 2t(1-t)(\alpha^2\beta^{*2}+\beta^2\alpha^{*2} - |\alpha|^4 - |\beta|^4)$$

$$= 1+2t(1-t)(\alpha^2(\beta^{*2}-\alpha^{*2}) + \beta^2(\alpha^{*2}-\beta^{*2}))$$

$$= 1+2t(1-t)(\alpha^{*2}-\beta^{*2})(\beta^2-\alpha^2)$$

$$= 1-2t(1-t)\underbrace{|\alpha^2-\beta^2|^2}_{\geq 0}$$

Thus: If $t \in]0,1[$, $\mathcal{E}(|\psi\rangle)$ is pure iff $\alpha = \pm\beta = \pm\frac{1}{\sqrt{2}}$.

$\Leftrightarrow |\psi\rangle = |\pm\rangle_x$

2c Starting from the pure state $|\psi\rangle$, the subsystem ends up in a mixed state $\mathcal{E}(|\psi\rangle)$. That is, its evolution is non unitary.

3 (a) <u>Kraus operators</u> :

The full density matrix writes:

$$\left(\sqrt{t}\,|\psi\rangle\otimes|0\rangle + \sqrt{1-t}\,\sigma_x|\psi\rangle\otimes|1\rangle\right)\left(\sqrt{t}\,\langle\psi|\otimes\langle 0| + \sqrt{1-t}\,(\sigma_x\langle\psi|\otimes\langle 1|)\right)$$

abusive notation

$$= t\,|\psi 0\rangle\langle\psi 0| + (1-t)\,|\sigma_x\psi\,.\,1\rangle\langle\sigma_x\psi\,1| + \sqrt{t(1-t)}\left(|\psi 0\rangle\langle\sigma_x\psi\,.\,1| + |\sigma_x\psi\,1\rangle\langle\psi 0|\right)$$

<u>Partial trace</u> :

<u>ie</u> $\mathcal{E}(|\psi\rangle) = \mathrm{Tr}_{anc}(\rho_{tot}) = t\,|\psi\rangle\langle\psi| + (1-t)\,|\sigma_x\psi\rangle\langle\sigma_x\psi|$

$$= t\,\mathbb{1}_2|\psi\rangle\langle\psi|\mathbb{1}_2^\dagger + (1-t)\,\sigma_x|\psi\rangle\langle\psi|\sigma_x^\dagger$$

Note that this is consistent with the previous result (q°2) and a faster way to obtain <u>it</u>.

↳ Kraus operators $\underline{\sqrt{t}\,\mathbb{1}_2}$ and $\underline{\sqrt{1-t}\,\sigma_x}$

Then, we get that:

$$\mathrm{Tr}(\mathcal{E}|\psi\rangle^2) = t^2 + (1-t)^2 + 2t(1-t)\underbrace{\mathrm{Tr}(\rho\sigma_x\rho\sigma_x)}_{\mathrm{Tr}(|\psi\rangle\langle\psi|\sigma_x|\psi\rangle\langle\psi|\sigma_x)}$$

$$= 2t^2 + 1 - 2t + 2t(1-t)\,\langle\sigma_x\rangle_\psi^2$$

$$= 1 + 2t(1-t)\,(\underbrace{\langle\sigma_x\rangle_\psi^2 - 1}_{\leq 0}) \leq 1$$

<u>thus</u> : $\mathcal{E}|\psi\rangle$ is pure $\iff \langle\sigma_x\rangle_\psi^2 = 1$

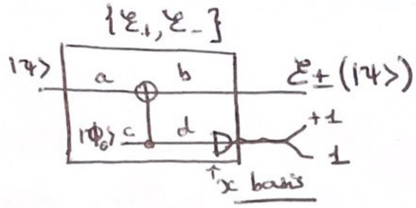
$$\iff \langle\sigma_x\rangle_\psi = \pm 1$$

$$\iff \psi = |\pm\rangle_x = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \text{ up to}$$

consistent with the results on $\underline{\alpha, \beta}$.

II Generalised measurement :

1 a We get access to the measurement outcome



$|+\rangle_x = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|-\rangle_x = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$|0\rangle_z = \frac{|+\rangle_x + |-\rangle_x}{\sqrt{2}}$

$|1\rangle_z = \frac{|+\rangle_x - |-\rangle_x}{\sqrt{2}}$

As before; the state before the measurement writes

$$|\varphi_{tot}\rangle = \sqrt{t}\,|\psi\rangle \otimes \underbrace{|0\rangle}_{z} + \sqrt{1-t}\,(\sigma_x|\psi\rangle) \otimes \underbrace{|1\rangle}_{z}$$

$$= \sqrt{t}\,|\psi\rangle \otimes \left(\frac{|+\rangle_x + |-\rangle_x}{\sqrt{2}}\right) + \sqrt{1-t}\,(\sigma_x|\psi\rangle) \otimes \left(\frac{|+\rangle_x - |-\rangle_x}{\sqrt{2}}\right)$$

$$= \Big(\underbrace{\sqrt{\tfrac{t}{2}}\,|\psi\rangle + \sqrt{\tfrac{1-t}{2}}\,\sigma_x|\psi\rangle}_{\mathcal{E}_+(|\psi\rangle)}\Big) \otimes |+\rangle_x + \Big(\underbrace{\sqrt{\tfrac{t}{2}}\,|\psi\rangle - \sqrt{\tfrac{1-t}{2}}\,\sigma_x|\psi\rangle}_{\mathcal{E}_-(|\psi\rangle)}\Big) \otimes |-\rangle_x$$

For the σ_x measurement :

Proba to get +1 :

$$p_+ := \| \mathbb{1}_2 \otimes |+\rangle_x\langle +|_x\, |\varphi_{tot}\rangle \|^2 = \frac{t}{2} + \frac{1-t}{2} + 2\sqrt{\frac{t(1-t)}{4}}\,\langle\psi|\sigma_x|\psi\rangle$$

$$= \frac{1}{2} + \sqrt{t(1-t)}\,\langle\sigma_x\rangle_\psi$$

Proba to get −1 :

$$p_- := \| \mathbb{1}_2 \otimes |-\rangle_x\langle -|_x\, |\varphi_{tot}\rangle \|^2 = \frac{1}{2} - \sqrt{t(1-t)}\,\langle\sigma_x\rangle_\psi$$

Rq: $\underline{p_+ + p_- = 1}$.

1 b We read :

$$\mathcal{E}_+(|\psi\rangle) = \left(\sqrt{\tfrac{t}{2}}\,|\psi\rangle + \sqrt{\tfrac{1-t}{2}}\,\sigma_x|\psi\rangle\right) * \frac{1}{\sqrt{p_+}} \quad \text{(to be renormalised)}$$

$$\mathcal{E}_-(|\psi\rangle) = \left(\sqrt{\tfrac{t}{2}}\,|\psi\rangle - \sqrt{\tfrac{1-t}{2}}\,\sigma_x|\psi\rangle\right) * \frac{1}{\sqrt{p_-}}$$

$\boxed{2a}$ To look for Kraus operators, remind that any 2x2 matrix can be expanded into the $(\mathbb{1}_2, \sigma_x, \sigma_y, \sigma_z)$.

- Case: +1 outcome

  For $\mathcal{E}_+(|\psi\rangle) = \sqrt{\frac{t}{2}}\,|\psi\rangle + \sqrt{\frac{1-t}{2}}\,\sigma_x|\psi\rangle$, the transformation for the initial density matrix $\rho = |\psi\rangle\langle\psi|$ writes:

  $$\mathcal{E}_+(\rho) = \frac{t}{2}\rho + \frac{1-t}{2}\sigma_x\rho\sigma_x + \sqrt{\frac{t(1-t)}{4}}\,(\rho\sigma_x + \sigma_x\rho)$$

  We can thus try to guess some Kraus operator with coordinates over $\mathbb{1}_2$ and σ_x.

  It turns out that

  $$K_+ := \sqrt{\frac{t}{2}}\,\mathbb{1}_2 + \sqrt{\frac{1-t}{2}}\,\sigma_x$$

  works.

- Case: -1 outcome

  Similarly, for $\mathcal{E}_-(\rho) = \sqrt{\frac{t}{2}}\,|\psi\rangle - \sqrt{\frac{1-t}{2}}\,\sigma_x|\psi\rangle$,

  a suitable guess is:

  $$K_- := \sqrt{\frac{t}{2}}\,\mathbb{1}_2 - \sqrt{\frac{1-t}{2}}\,\sigma_x$$

However, the problem set asks us to ~~first~~ ~~write~~ write the Kraus operators and then use them to compute the post measurement states.

$\boxed{2a}$ The auxiliary qubit starts in the pure state

$$|\phi_0\rangle\langle\phi_0|$$

We choose the $\{|0\rangle_{z\,anc}, |1\rangle_{z\,anc}\}$ basis for the ancilla space, and $\{|0\rangle_z, |1\rangle_z\}$ for the qubit space.

Unitary evolution $\mathcal{U}$ writes:

$$\begin{array}{c} \quad |00\rangle|10\rangle\;\; |01\rangle|11\rangle \\ \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right) \end{array} = \left(\begin{array}{c|c} \mathbb{1}_2 & \\ \hline & \sigma_x \end{array}\right)$$

and the projectors associated with the x measurement for the ancilla are:

$$\mathbb{1}_2 \otimes \begin{cases} |+\rangle_x\langle +|_x & \leftarrow P_+ \\ |-\rangle_x\langle -|_x & \leftarrow P_- \end{cases}$$

We know that:

$$\mathcal{E}_+(\rho) = \mathrm{Tr}_{anc}\left(P_+ U (\rho \otimes |\phi_0\rangle\langle\phi_0|) U^+ P_+\right)$$

$$= \mathrm{Tr}_{anc}\left(|0\rangle\langle 0| P_+ U (\rho \otimes |\phi_0\rangle\langle\phi_0|) U^+ P_+ |0\rangle\langle 0|\right)$$
$$+ \mathrm{Tr}_{anc}\left(|1\rangle\langle 1| P_+ U (\rho \otimes |\phi_0\rangle\langle\phi_0|) U^+ P_+ |1\rangle\langle 1|\right)$$

$$= \langle 0| P_+ U |\phi_0\rangle \, \rho \, \langle\phi_0| U^+ P_+ |0\rangle$$
$$+ \langle 1| P_+ U |\phi_0\rangle \, \rho \, \langle\phi_0| U^+ P_+ |1\rangle$$

The latter equation gives us candidates for Kraus operators, with

$$\langle 0| P_+ U |\phi_0\rangle = \underbrace{\langle 0|+\rangle_x}\, {}_x\langle +| U |\phi_0\rangle$$
$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \left(\langle 0| + \langle 1|\right) U \left(\sqrt{t}\,|0\rangle + \sqrt{1-t}\,|1\rangle\right)$$
$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \left(\sqrt{t}\,(\mathbb{1}_2 + 0_2) + \sqrt{1-t}\,(0_2 + \sigma_x)\right)$$
$$= \frac{1}{2}\left(\sqrt{t}\,\mathbb{1}_2 + \sqrt{1-t}\,\sigma_x\right)$$

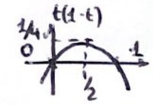
$$\langle 1| P_+ U |\phi_0\rangle = \langle 1|+\rangle_x \, {}_x\langle +| U |\phi_0\rangle$$
$$= \frac{1}{2}\left(\sqrt{t}\,\mathbb{1}_2 + \sqrt{1-t}\,\sigma_x\right)$$

So that we can consider:

$$\boxed{K_+ = \sqrt{\frac{t}{2}}\,\mathbb{1}_2 + \sqrt{\frac{1-t}{2}}\,\sigma_x}$$

where we check that $K_+^+ K_+ = \left(\frac{t}{2} + \frac{1-t}{2}\right)\mathbb{1}_2 + \sqrt{t(1-t)}\,\sigma_x = \frac{1}{2}\mathbb{1}_2 + \sqrt{t(1-t)}\,\sigma_x$

is such that $\mathbb{1}_2 - K_+^+ K_+$ has positive eigenvalues (



)

Similarly, we take

$$\boxed{K_- = \sqrt{\frac{t}{2}}\,\mathbb{1}_2 - \sqrt{\frac{1-t}{2}}\,\sigma_x}$$

2b2c

Case + :

Output: $\dfrac{K_+ \rho K_+^\dagger}{p_+}$ with probability $p_+ = \mathrm{Tr}(K_+ \rho K_+^\dagger)$

$$K_+ \rho K_+^\dagger = \frac{t}{2}\rho + \frac{(1-t)}{2} \sigma_x \rho \sigma_x + \sqrt{\frac{t(1-t)}{4}} (\sigma_x \rho + \rho \sigma_x)$$

and : $\mathrm{Tr}(K_+ \rho K_+^\dagger) = \frac{t}{2} \mathrm{Tr}(\rho) + \frac{1-t}{2} \mathrm{Tr}(\sigma_x^2 \rho) + \sqrt{\frac{t(1-t)}{4}} \times 2 \mathrm{Tr}(\sigma_x \rho)$

ie $$p_+ = \frac{1}{2} + \sqrt{t(1-t)} \langle \sigma_x \rangle_\rho$$

It is also quite straightforward to see that $K_+ \rho K_+^\dagger$ is a pure state, with

$$\mathcal{E}_+(|\psi\rangle) = \frac{1}{\sqrt{p_+}} \left(\sqrt{\frac{t}{2}} |\psi\rangle + \sqrt{\frac{1-t}{2}} \sigma_x |\psi\rangle \right)$$

Case − : Similarly :

$$\mathcal{E}_-(|\psi\rangle\langle\psi|) = \frac{1}{p_-} \left(\frac{t}{2}\rho + \frac{1-t}{2} \sigma_x \rho \sigma_x - \frac{1}{2}\sqrt{t(1-t)} (\sigma_x \rho + \rho \sigma_x) \right)$$

with $$p_- = \frac{1}{2} - \sqrt{t(1-t)} \langle \sigma_x \rangle_\rho$$

and as a pure state :

$$\mathcal{E}_-(|\psi\rangle) = \frac{1}{\sqrt{p_-}} \left(\sqrt{\frac{t}{2}} |\psi\rangle - \sqrt{\frac{1-t}{2}} \sigma_x |\psi\rangle \right)$$

$\boxtimes$