Problem set 1 :

1°) Suppose $|\Phi^+\rangle$ is separable : Take the most general product state of a two-dimensional Hilbert space. $H_A \otimes H_B$ with $H_A = \langle\{|0\rangle, |1\rangle\}\rangle = H_B$.

$$|\Phi^+\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

$$= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

identifying the coefficient with $|\Phi^+\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$:

$$\begin{cases} \alpha\gamma = \frac{1}{\sqrt{2}} \Rightarrow \alpha \neq 0, \gamma \neq 0 \\ \alpha\delta = 0 \\ \beta\gamma = 0 \\ \beta\delta = \frac{1}{\sqrt{2}} \Rightarrow \beta \neq 0, \delta \neq 0 \end{cases}$$

, so this system is incompatible because $\alpha\delta = 0$ with $\alpha \neq 0, \gamma \neq 0$, $\beta \neq 0, \delta \neq 0$ is impossible.

So $|\Phi^+\rangle$ is not separable and thus entangled.

2°) Let's write $|\Psi\rangle \in H_A \otimes H_B$ with H_A, H_B Hilbert space of dimensions N and M respectively, we suppose $N \leqslant M$.

$\mathcal{B}_A = \{|i\rangle, i \in [\![1; N]\!]\}$
$\mathcal{B}_B = \{|j\rangle, j \in [\![1; M]\!]\}$ orthonormal basis.

In all generality, $|\Psi\rangle \in H_A \otimes H_B$ can be written

$$|\Psi\rangle = \sum_{ij} N_{ij} |i\rangle \otimes |j\rangle \quad , N_{ij} \in \mathbb{R}$$

$$\equiv \begin{pmatrix} N_{11} & \cdots & N_{1M} \\ \vdots & \ddots & \vdots \\ N_{N1} & \cdots & N_{NM} \end{pmatrix} \in M_{N\times M}(\mathbb{R})$$

$\Rightarrow$ We can apply the Singular Value Decomposition theorem :

$\exists P, Q^{-1} \in \mathbb{U}_{N\times N}(\mathbb{C}); \mathbb{U}_{M\times M}(\mathbb{C})$ (unitary matrix) such that :

$$P\begin{pmatrix} \nu_{11} & \cdots & \nu_{1M} \\ \vdots & \ddots & \vdots \\ \nu_{N1} & \cdots & \nu_{NM}\end{pmatrix} Q^{-1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \lambda_N & 0 & \cdots & 0 \end{pmatrix} \equiv \sum_{i=1}^{N} \lambda_i |\varphi_i\rangle \otimes |\phi_i\rangle$$

(and $\lambda_i \geq 0 \ \forall i \in [\![1;N]\!]$, ($\lambda_i \in \mathbb{R}^+$)) $\Rightarrow$ In SVD theorem.

With $\mathcal{B}'_A = \{ |\varphi_i\rangle, i \in [\![1;N]\!]\}$ and $\mathcal{B}'_B = \{ |\phi_j\rangle, j \in [\![1;M]\!]\}$ orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$ ($\mathcal{B}_A \xrightarrow{P} \mathcal{B}'_A$ with P unitary and we know that $\mathcal{B}_A$ is an orthonormal basis).

Moreover $\langle \Psi | \Psi \rangle = \sum_{i=1}^{N} \lambda_i^2 = 1$.

We write $\hat{\rho} = |\Psi\rangle\langle\Psi| = \sum_{i,j} \lambda_i \lambda_j |\varphi_i \phi_i\rangle\langle \varphi_j \phi_j|$

Then we compute $\hat{\rho}_A = \mathrm{Tr}_B(\hat{\rho}) = \sum_{a,b,c} \langle \varphi_a \phi_c | \hat{\rho} | \varphi_b \phi_c \rangle |\varphi_a\rangle\langle\varphi_b|$

$$= \sum_{\substack{a,b,c \\ i,j}} \lambda_i \lambda_j \underbrace{\langle \varphi_a \phi_c | \varphi_i \phi_i \rangle}_{= \mathbb{1}\{i = a = c\}} \underbrace{\langle \varphi_j \phi_j | \varphi_b \phi_c \rangle}_{= \mathbb{1}\{j = b = c\}} |\varphi_a\rangle\langle\varphi_b|$$

$\Rightarrow i = j, \ c = a = b$

$$= \sum_i \lambda_i^2 \, |\varphi_i\rangle\langle\varphi_i|$$

We can compute the purity using the Von Neumann entropy:

$$H(\hat{\rho}_A) = -\mathrm{Tr}(\hat{\rho}_A \ln_2(\hat{\rho}_A)) = -\sum_i (\lambda_i)^2 \ln_2(\lambda_i^2)$$

In the case of $|\phi^+\rangle$, we have already its Schmidt-decomposition:

$\lambda_1 = \frac{1}{\sqrt{2}}, \ \lambda_2 = \frac{1}{\sqrt{2}} \Rightarrow H(\hat{\rho}_A) = 1$.

3a) Let's expand the formula (2) and use the definition of the partial transpose:

$$\hat{\rho} = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B = \sum_{i,a,b,c,d} \lambda_i \nu^i_{ab} \xi^i_{cd} |ac\rangle\langle bd|$$

$$\Rightarrow \hat{\rho}^{T_B} = \sum_{i,a,b,c,d} \lambda_i \nu^i_{a,b} \xi^i_{c,d} |ad\rangle\langle bc|$$

$$= \sum_i \lambda_i \left(\sum_{a,b} \nu^i_{a,b} |a\rangle\langle b| \right) \otimes \underbrace{\left(\sum_{c,d} \xi^i_{c,d} |d\rangle\langle c| \right)}_{= (\rho_i^B)^T}$$

$$= \sum_i \lambda_i \rho_i^A \otimes (\rho_i^B)^T$$

We use the property that if H is an Hilbert space, $A \in \mathcal{L}(A)$ a linear operator hermitian operator:

$$A \text{ is positive } \left(\text{All eigenvalues } \geq 0\right) \Leftrightarrow \forall x \in H \quad \langle x | A x \rangle \geq 0$$

Let's take $|x\rangle \otimes |y\rangle \in H_A \otimes H_B$.

$$\langle x y | \rho_i^A \otimes (\rho_i^B)^T | x y \rangle = \langle x | \rho_i^A | x \rangle \langle y | (\rho_i^B)^T | y \rangle$$

We know that $\rho_i^A \geq 0$ because it is a density operator.

Moreover $(\rho_i^B)^T$ has the same eigenvalues as ρ_i^B (easily seen on finite dimension looking at the characteristic polynom: $\chi_{\rho_i^B}(\lambda) = \det(\rho_i^B - \lambda \mathbb{I}d)$

$$\chi_{(\rho_i^B)^T}(\lambda) = \det((\rho_i^B)^T - \lambda \mathbb{I}d) = \det((\rho_i^B - \lambda \mathbb{I}d)^T) = \det(\rho_i^B - \lambda \mathbb{I}d) \;).$$

So $\rho_i^A \geq 0$, $(\rho_i^B)^T \geq 0 \Rightarrow \langle x y | \rho_i^A \otimes (\rho_i^B)^T | x y \rangle \geq 0$

We conclude that: $\hat{\rho}$ separable $\Rightarrow \hat{\rho}^{T_B} \geq 0$.

contrapositive: ρ^{T_B} is not positive (at least eigenvalue < 0) $\Rightarrow \hat{\rho}$ not separable (entangled).

4)

1. $\frac{1}{4}\mathbb{I}_4 = \frac{1}{4}\mathbb{I}_2\otimes\mathbb{I}_2 = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$

So $\frac{1}{4}\mathbb{I}_4$ is separable.

2. $$\rho_w = \underbrace{\frac{w}{2}}_{=a}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix} + \underbrace{\frac{(1-w)}{4}}_{=b}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}a+b&0&0&a\\0&b&0&0\\0&0&b&0\\a&0&0&a+b\end{pmatrix}$$

$$\rho_w^{T_B} = \begin{pmatrix}a+b&0&0&0\\0&b&a&0\\0&a&b&0\\0&0&0&a+b\end{pmatrix}$$

We compute the characteristic polynom of $\rho_w^{T_B}$:

$$\chi_{\rho_w^{T_B}}(\lambda) = \det\left(\rho_w^{T_B} - \lambda\,\mathbb{I}d\right) = \begin{vmatrix}a+b-\lambda&0&0&0\\0&b-\lambda&a&0\\0&a&b-\lambda&0\\0&0&0&a+b-\lambda\end{vmatrix}$$

$$= (a+b-\lambda)^2\left((b-\lambda)^2 - a^2\right)$$

$$= (a+b-\lambda)^3(b-a-\lambda)$$

With $a+b = \frac{1+w}{2}$, $b-a = \frac{1-3w}{4}$.

So $\rho_w^{T_B}$ has one negative eigenvalue when $\frac{1-3w}{4} < 0 \Leftrightarrow w > \frac{1}{3}$.

So $w_{lim} = \frac{1}{3}$.

We conclude that ρ_w is entangled for $w \in]\frac{1}{3};1]$.

3.

a) $\hat{O} = \begin{pmatrix} O_2 & \sigma_x \\ \sigma_x & O_2 \end{pmatrix} + \begin{pmatrix} \sigma_z & O_2 \\ O_2 & -\sigma_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

b) We can write $\hat{\rho} = \frac{1}{2}(\mathbb{I}d + \vec{a}\cdot\vec{\sigma})$, for it to be physical $\hat{\rho}$ has to be positive semi definite $\Leftrightarrow \|a\|^2 \leqslant 1$.

c) $\langle \hat{O} \rangle_{\hat{\rho}_1 \otimes \hat{\rho}_2} = Tr(\hat{O}\hat{\rho}_1 \otimes \hat{\rho}_2)$

$$= Tr\left((\sigma_{x_1} \otimes \sigma_{x_2} + \sigma_{z_1} \otimes \sigma_{z_2}) \hat{\rho}_1 \otimes \hat{\rho}_2 \right)$$

$$\overset{\text{linearity of Trace}}{=} Tr(\sigma_{x_1} \otimes \sigma_{x_2} \hat{\rho}_1 \otimes \hat{\rho}_2) + Tr(\sigma_{z_1} \otimes \sigma_{z_2} \hat{\rho}_1 \otimes \hat{\rho}_2)$$

$$= Tr(\rho_1 \sigma_{x_1} \otimes \rho_2 \sigma_{x_2}) + Tr(\rho_1 \sigma_{z_1} \otimes \rho_2 \sigma_{z_2})$$

$$\overset{Tr(A\otimes B) = Tr(A)Tr(B)}{=} Tr(\rho_1 \sigma_{x_1}) Tr(\rho_2 \sigma_{x_2}) + Tr(\rho_1 \sigma_{z_1}) Tr(\rho_2 \sigma_{z_2})$$

We can write $\hat{\rho} = \frac{1}{2}(\mathbb{I}d + \vec{a}\cdot\vec{\sigma})$

So $Tr(\hat{\rho}\sigma_x) = \frac{1}{2} Tr\left(\sigma_x + a_x(\sigma_x)^2 + a_y \overbrace{\sigma_x\sigma_y}^{\propto \sigma_z} + a_z \overbrace{\sigma_x \sigma_z}^{\propto \sigma_y} \right)$

and we know that $\forall i \in \{x, y, z\}$, $Tr(\sigma_i) = 0$ and $Tr(\underbrace{\sigma_i^2}_{=\mathbb{I}d}) = 2$

$\Rightarrow Tr(\hat{\rho}\sigma_x) = a_x$ (same for σ_z: $Tr(\hat{\rho}\sigma_z) = a_z$)

So $\langle \hat{O} \rangle_{\rho_1 \otimes \rho_2} = a_{1x}a_{2x} + a_{1z}a_{2z} = \begin{pmatrix} a_{1x} \\ a_{1z} \end{pmatrix} \cdot \begin{pmatrix} a_{2x} \\ a_{2z} \end{pmatrix}$

It is a scalar product on $\mathbb{R}^2$, we can apply the Cauchy-Schwarz inequality:

$|\langle a | b \rangle| \leqslant \|a\| \|b\| \Rightarrow \langle a | b \rangle \leqslant \|a\| \|b\|$

So $\langle \hat{O} \rangle_{\hat{\rho}_1 \otimes \hat{\rho}_2} \leqslant \underbrace{(|a_{1x}|^2 + |a_{1z}|^2)^{1/2}}_{\leqslant (|a_{1x}|^2 + |a_{1y}|^2 + |a_{1z}|^2)^{1/2} = \|a\|^2 \leqslant 1} (|a_{2x}|^2 + |a_{2z}|^2)^{1/2} \leqslant 1$

$\Rightarrow \langle \hat{O} \rangle_{\hat{\rho}_1 \otimes \hat{\rho}_2} \leqslant 1 = \beta$. If $\hat{\rho} = \frac{1}{4}(\mathbb{I}d + \sigma_x) \otimes (\mathbb{I}d + \sigma_x)$ $\begin{pmatrix} a_{1x} = a_{2x} = 1 \\ a_{1y} = a_{2y} = 0 \\ a_{1z} = a_{2z} = 0 \end{pmatrix}$

We have $\hat{\rho}\hat{O} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 2 \end{pmatrix} \Rightarrow Tr(\hat{\rho}\hat{O}) = 1$, the bound is reached!

d) In the case of separable states,

$$\hat{\rho} = \sum_i \lambda_i \, \rho_i^A \otimes \rho_i^B \quad \text{with} \quad \sum_i \lambda_i = 1 \quad \text{and} \quad \lambda_i > 0$$

$$\langle \hat{O} \rangle_{\hat{\rho}} = \mathrm{Tr}(\hat{O}\hat{\rho}) \overset{\text{linearity of Trace}}{=} \sum_i \lambda_i \underbrace{\langle \hat{O} \rangle_{\rho_i^A \otimes \rho_i^B}}_{\leq 1} \leq \sum_i \lambda_i = 1$$

So separable states have the same bound $\beta = 1$.

Moreover a product state is a separable state, so the bound is also reached.

e) We just have to compute $\langle O \rangle_{\rho_W} = \mathrm{Tr}(O \rho_W)$

$$\rho_W O = \begin{pmatrix} a+b & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ a & 0 & 0 & a+b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2a+b & & & \\ & -b & & \\ & & -b & \\ & & & 2a+b \end{pmatrix}$$

$\Rightarrow \mathrm{Tr}(\rho_W O) = 4a = 2W$

so if $2W > 1$, the entanglement witness inequality is not verified $\Rightarrow$ the state is entangled for $W > \frac{1}{2}$

$(W \in]\frac{1}{2}; 1])$