How to exploit Bell non-locality to advance communication technologies?
Problem Set 4 Oct 31 / Nov 1, 2022

We consider a bipartite scenario where Alice randomly chooses an integer $x = 1, 2, 3$ according to the probability $\{p_x = \frac{1}{3}\}$, store the result in a classical register X and prepares a qubit state $|\psi_x\rangle$ accordingly. This quantum state is then send to Bob. Finally, Bob performs a POVM measurement E with elements $\{E_y\}$, store the result in a classical register Y. The aim for Bob is to gain information about X using Y. After Alice's preparation of her system A, the joint state of XA is given by

$$\rho_{XA} = \sum_x p(x)|x\rangle\langle x| \otimes \rho(x). \tag{1}$$

Bob's measurement is a mapping A to AY according to

$$\rho(x) \to \sum_y M(y)\rho(x)M(y)^\dagger \otimes |y\rangle\langle y|, \tag{2}$$

where $M(y)^\dagger M(y) = E(y)$, yielding the state for XAY

$$\rho'_{XAY} = \sum_{x,y} p(x)|x\rangle\langle x| \otimes M(y)\rho(x)M(y)^\dagger \otimes |y\rangle\langle y| \tag{3}$$

In this exercise, we study different implementations of this scenario. We aim at understanding what strategies are optimal for Alice and Bob so that Bob gets the greatest amount of information about Alice's register – i.e. correctly guess the state Alice has sent.

1. Strategy using two orthogonal states

Consider the case where the states available to Alice are

$$\begin{cases} |\psi_1\rangle = |\psi_2\rangle = |\psi\rangle\,, \\ |\psi_3\rangle = |\psi_\perp\rangle \end{cases} \tag{4}$$

a. What are the POVM elements Bob should use?

b. Compute Bob's information gain.

c. Give the Holevo bound for this information gain.

2. Strategy using non-orthogonal states

Now consider the case where Alice's possible states are fixed to be the three following non-orthogonal states

$$|\psi_1\rangle = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \tag{5}$$

$$|\psi_2\rangle = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \tag{6}$$

$$|\psi_3\rangle = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}. \tag{7}$$

These are states symmetrically distributed on the xz-plane – i.e. each pair shares the same overlap.
Bob uses a measurement with 3 POVM elements

$$E_a = \frac{2}{3}(\mathbb{1} - |\psi_a\rangle\langle\psi_a|), \quad \text{with } a = 1, 2, 3 \tag{8}$$

a. Compute the probability $p(x|a)$ of receiving the outcome a for each value of $a = 1, 2, 3$ and for all possible states $|\psi_x\rangle$, x=1,2,3.

b. Deduce Bob's information gain.

c. Compare Bob's information gain with the Holevo bound. Comment.