How to exploit Bell non-locality to advance communication technologies?
Problem Set 3 Oct 24/25, 2022

We here focus again on Werner states to show that up to a certain point, its correlations can be reproduced by a local model. Putting together the results of problem sets 1 and 3 will convince us that there exists entangled states that are local, i.e. their correlation can be reproduced by locally causal theories.

Non-locality content of a Werner state

We consider a physical implementation of a Bell test in which a pair of particles is shared between Alice and Bob to be measured. Alice and Bob are asked to choose a setting, labelled A and B and receive a measurement result for each setting choice, labelled a and b. The protocol is repeated many times until Alice and Bob can access the statistics $p(ab|AB)$ of joint measurement results conditioned on their setting choice. We consider the case where the particles are described by qubit states. The measurement A is a projective qubit measurement of the form $(+1).\Pi_+^A$ + -(1).Π_-^A where $\Pi_\pm^A$ are projectors associated to eigenvalues ± 1. This is also true for B.

Note that a pure qubit state can be written in the form $|\psi\rangle = \cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle$. The parameters θ and ϕ re-interpreted in spherical coordinates as respectively the co-latitude with respect to the z-axis and the longitude with respect to the x-axis, specify a point $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ on the unit sphere in $\mathcal{R}^3$. The qubit state $|\psi\rangle$ can thus be seen as a vector – the Bloch vector – starting from the center of the sphere and pointing to this point. thecoordinates of this vector $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ can be found by computing $(\mathrm{Tr}(\sigma_x\,|\psi\rangle\langle\psi|)\,,\mathrm{Tr}(\sigma_y\,|\psi\rangle\langle\psi|)\,,\mathrm{Tr}(\sigma_z\,|\psi\rangle\langle\psi|))$ where σ_i, $i = x, y, z$ are the Pauli matrices. This also applies to mixed qubit states and in this case, the length of the Bloch vector is smaller than 1. We label $\vec{A}$ and $\vec{B}$ the Bloch vectors associated to the projectors Π_+^A and Π_+^B. Given that $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ from a basis for 2×2 operators, we can write Π_+^A and Π_+^B is this basis. Specifically, $\Pi_+^A = \frac{1}{2}\left(\mathbb{1} + \vec{A}.\vec{\sigma}\right)$ where $\vec{\sigma}$ is a vector having the three Pauli operators as components.

Given that $\vec{A}$ and $\vec{B}$ fully described the measurements, we define the correlation functions $E\left(\vec{A}\right) = \sum_a a\,p\left(a|\vec{A}\right)$, $E\left(\vec{B}\right) = \sum_b b\,p\left(b|\vec{B}\right)$ and $E\left(\vec{A},\vec{B}\right) = \sum_{a,b} ab\,p\left(a,b|\vec{A},\vec{B}\right)$ where $p\left(a|\vec{A}\right)$ for example is the probability of having the result a given the measurement characterized by $\vec{A}$.

When correlations are observed, we might be interested in finding an explanation according to the principle of local causality, i.e. we look for a set of past factors, described by some variables λ, having a joint influence on Alice and Bob outcomes and fully accounting for the dependence between a and b. Concretely, we say that a locally causal (or local) model can explain Alice and Bob's correlations if one can find a classical variable λ with probability density $\rho\left(\vec{\lambda}\right)$ such that

$$p\left(a,b|\vec{X},\vec{Y}\right)=\int p\left(a|\vec{X},\vec{\lambda}\right)p\left(b|\vec{Y},\vec{\lambda}\right)\rho\left(\vec{\lambda}\right)\mathrm{d}\vec{\lambda} \tag{1}$$

Our aim is to build a local model reproducing the statistics that one can observe from a Werner state.

A) Local statistics in terms of correlation functions

1. Express each correlation function $E\left(\vec{A}\right)=\sum_a a\,p\left(a|\vec{A}\right)$, $E\left(\vec{B}\right)=\sum_b b\,p\left(b|\vec{B}\right)$ and $E\left(\vec{A},\vec{B}\right)=\sum_{a,b} ab\,p\left(a,b|\vec{A},\vec{B}\right)$ in terms of the joint probabilities $p\left(a,b|\vec{A},\vec{B}\right)$, $(a,b)\in\{1,-1\}^2$. We here assume that the result on one side is independent of the choice of measurement setting on the other side, i.e. $p(a|AB)=p(a|A)$ and $p(b|AB)=p(b|B)$.
2. What additional general condition must the joint probabilities satisfy? Conclude that the correlators form another complete description of the two-party measurement.
3. Consider the case where one finds a set of past factors $\vec{\lambda}$ distributed according to $\rho(\vec{\lambda})$ such that the observed correlations can be written as

$$E\left(\vec{A},\vec{B}\right)=\int E\left(\vec{A},\vec{\lambda}\right)E\left(\vec{B},\vec{\lambda}\right)\rho\left(\vec{\lambda}\right)\mathrm{d}\vec{\lambda} \tag{2}$$

$$E\left(\vec{A}\right)=\int E\left(\vec{A},\vec{\lambda}\right)\rho\left(\vec{\lambda}\right)\mathrm{d}\vec{\lambda} \tag{3}$$

$$E\left(\vec{B}\right)=\int E\left(\vec{B},\vec{\lambda}\right)\rho\left(\vec{\lambda}\right)\mathrm{d}\vec{\lambda} \tag{4}$$

   with $E\left(\vec{A},\vec{\lambda}\right)=\sum_a a\,p(a|\vec{A}\lambda)$ and $E\left(\vec{B},\vec{\lambda}\right)=\sum_b b\,p(b|\vec{B}\lambda)$, what can we say about $p(ab|\vec{A}\vec{B})$? Conclude.

B) Local strategy

1. Give Alice's observable A as a function of the Bloch vector $\vec{A}$.

2. Alice and Bob share a vector $\vec{\lambda}$ with spherical coordinates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi[$ (respectively the colatitude with respect to the z-axis and the longitude with respect to the x-axis) which is uniformly distributed in the unit sphere. How does this translate in term of the probability density $\rho\left(\vec{\lambda}\right) = \rho(\theta, \phi)$?

3. Alice uses the shared variable $\vec{\lambda}$ to define her qubit's Bloch vector which she measures along the direction $\vec{A}$. What are the probabilities that she gets +1. Same question for -1. Compute $E\left(\vec{A}, \vec{\lambda}\right) = \sum_a aP\left(a \middle| \vec{A}, \vec{\lambda}\right)$. What is the physical significance of $E\left(\vec{A}, \vec{\lambda}\right)$?

4. Bob will directly compute the scalar product between $\vec{B}$ and $\vec{\lambda}$ and will define his outcome as -sign$\left(\vec{B}.\vec{\lambda}\right)$. By distinguishing cases find a compact expression for $E\left(\vec{B}, \vec{\lambda}\right) = \sum_b bP\left(b \middle| \vec{B}, \vec{\lambda}\right)$.

5. Compute $E\left(\vec{A}\right)$ and $E\left(\vec{B}\right)$. Comment the fact that neither of these depend on the measurement direction.
   Compute $E\left(\vec{A}, \vec{B}\right)$.

C) Werner state correlations

Now we drop the local variable and simply consider the case in which Alice and Bob share a Werner state $\rho_w = w \left|\psi_-\right\rangle \left\langle\psi_-\right| + \frac{(1-w)}{4}\mathbb{1}_4$ and measure it along the direction $\vec{A}$ for Alice and $\vec{B}$ for Bob. Note that $\left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(\left|01\right\rangle - \left|10\right\rangle)$ is invariant under local unitaries, i.e. $U \otimes U \left|\psi^-\right\rangle = \left|\psi^-\right\rangle$.

1. Show that $\left|\psi_-\right\rangle \left\langle\psi_-\right| = \frac{1}{4}\left(\mathbb{1} - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right)$.
2. What is the probability that they get the same result ? Different results ?
3. Compute $E\left(\vec{A}\right)$, $E\left(\vec{B}\right)$ and $E\left(\vec{A}, \vec{B}\right)$ in this scenario.
4. Deduce a $w = w_{loc}$ at which you can certify that the results obtained with the Werner state can be described by a local model.
5. Recall the separability bound of Werner states from a previous exercise sheet. Is the state with $w = w_{loc}$ entangled ? What can we conclude about the link between the set of entangled states and the set of non-local states ?