How to exploit Bell non-locality to advance communication technologies?
Problem Set 2 Oct 17/18, 2022

1. Evolution of an open quantum system

We consider the evolution of a qubit open quantum system. The qubit is prepared in the state $|\psi\rangle$ and the evolution is described by a quantum channel $\mathcal{E}$. This channel is realized as a unitary interaction U (which we will fixed to be the CNOT operation) between the system qubit and an auxiliary qubit prepared in the state $|\phi_0\rangle = \sqrt{t}\,|0\rangle + \sqrt{1-t}\,|1\rangle$ (also referred as ancillary system). After the interaction, the auxiliary qubit is measurement in the z basis. The outcome of the measurement are not revealed, hence we don't have access to this information. Such an open quantum system is represented in Fig. 1.

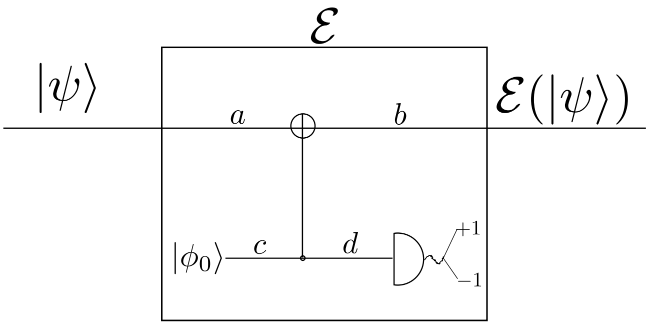

Figure 1: The open quantum system, with a CNOT and a σ_z measurement.

1. A CNOT is an operation which have the following behaviour: If $|\phi_0\rangle = |0\rangle$, the identity is applied to $|\psi\rangle$. If $|\phi_0\rangle = |1\rangle$, σ_x is performed on $|\psi\rangle$. Write down the CNOT in a matrix form. Check that this operation is a unitary operation.

2. a. Express $\mathcal{E}(|\psi\rangle)$, the state of the system after the channel $\mathcal{E}$ has been applied on its initial state $|\psi\rangle$, when the outcome of the measurement are unknown (auxiliary state traced out).
   b. Is it a pure state?
   c. What can you conclude about this system's evolution?

3. a. Give a Kraus operator representation of the channel $\mathcal{E}$.
   b. Using these Kraus operators, express the final state $\mathcal{E}(|\psi\rangle)$. Verify that this state is similar to the one obtain in question 2.a.

2. Generalized measurement

We now study a slightly modified version of the previous operation : A measurement is now performed in the x basis, and we have access to its classical outcome, see Fig. 2. This setup is similar to a measurement device with a quantum input state $|\psi\rangle$, a quantum output, and a classical outcome. The full outcomes are the quantum-classical pairs $\{\mathcal{E}_+(|\psi\rangle), +1\}$, or $\{\mathcal{E}_-(|\psi\rangle), -1\}$.

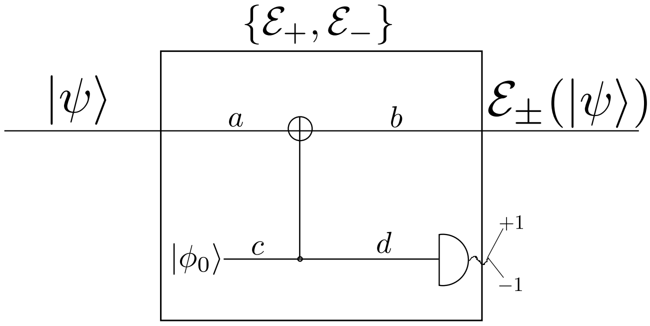

Figure 2: The open quantum system, with a CNOT and a σ_x measurement.

1. Give first the the global state before the measurement. From this state,
   a. give the probability to get the outcome +1. Same question for -1,
   b. give the two post measurement states conditioned on the classical outcome.
2. a. Write the Kraus operators associated with each of the outcomes.
   b. Using these Kraus operators, deduce the two post measurements states.
   c. In a similar manner, give the probabilities to get the outcome +1. Same question with -1.