How to exploit Bell non-locality to advance communication technologies?
Problem Set 1 Oct 17/18, 2022

Reminder:

- A (pure) bipartite state $|\psi\rangle_{AB}$ is a product state if it is of the form
$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B \,. \tag{1}$$
A pure bipartite state $|\psi\rangle_{AB}$ which is not a product state is entangled. A bipartite state is separable if and only if it can be written in the form
$$\rho = \sum_i \lambda_i \rho_i^A \otimes \rho_i^B, \text{ with } \lambda_i > 0 \text{ and } \sum_i \lambda_i = 1 \tag{2}$$
If a state is not separable then it is entangled. By definition, the set of separable states is convex and the extremal points are the pure product states.

- For a bipartite state $\rho_{AB} = \sum_{ijkl} c_{ijkl} |i_A j_B\rangle \langle k_A l_B|$, the partial transpose with respect to Bob is given by $\rho_{AB} = \sum_{ijkl} c_{ijkl} |i_A l_B\rangle \langle k_A j_B|$. In the particular case of a product state $\rho_{prod} = \rho_A \otimes \rho_B$, the partial transpose with respect to Bob is defined as $\rho_{prod}^{T_B} = \rho_A \otimes (\rho_B)^T$, where $(\rho_B)^T = (\sum_{i,j} p_{i,j} |i\rangle \langle j|)^T = \sum_{i,j} p_{i,j} |j\rangle \langle i|$. Note that the eigenvalues of ρ_B^T and ρ_B are the same.

1. Explicite decomposition

Look for an explicite decomposition of $|\phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ as a tensor product of two pure states. Conclude.

2. Partial trace

Consider an arbitrary pure bipartite state $|\psi\rangle_{AB}$. Show that such a state can be written as $|\psi\rangle_{AB} = \sum_i \lambda_i |\varphi_a^i\rangle |\phi_b^i\rangle$ where λ_i are real and positive coefficient satisfying $\sum_i \lambda_i^2 = 1$ and $\{|\varphi_a^i\rangle\}$ and $\{|\phi_b^i\rangle\}$ are orthonormal basis for A and B. Give the reduced state of Alice and compute its purity. What can we conclude for the state $|\phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$?

3. PPT Criterion

Consider a state of the form (2). Write down its partial transpose ρ^{T_B} with respect to Bob. What is the sign of the eigenvalues of ρ^{T_B}? What can you conclude about a state for which the partial transpose has negative eigenvalues? This result is known as the PPT criterion.

4. Werner state

The Werner state is a two-qubit state defined as

$$\rho_w = w \left|\phi^+\right\rangle \left\langle\phi^+\right| + \frac{(1-w)}{4}\mathbf{1}_4, \tag{3}$$

where $|\phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and $0 \leq w \leq 1$.

1. Prove that $\frac{1}{4}\mathbf{1}_4$ is a separable state.
2. Give an interval for w for which the Werner state is entangled.
3. Experimentally, the full knowledge of the density matrix is rarely accessible. In practice, we have access to expectation values of some observables.
   Consider the observable $O = (\sigma_x\sigma_x + \sigma_z\sigma_z)$.
   a. Write the observable O in matrix form.
   b. Let $\rho_p = \rho_1 \otimes \rho_2$ a product state. ρ_1 and ρ_2 are qubit states and thus can be represented by a Bloch vector. What is the condition on this vector so that ρ_1 and ρ_2 are physical state?
   c. Find the maximum value β, of the expectation value of $\langle O\rangle_{\rho_p}$ a product state ρ_p can achieve.
   d. What is the value of β when considering separable states?
   e. The inequality $\langle O\rangle_\rho \leq \beta$ is called an entanglement witness. This is, if the inequality is not satisfied for a state ρ, one can conclude that this state is entangled.
   Compute O on the above defined Werner state. What is the minimum value w for which the witness based on O detect entanglement?