Gauge Theories and the Standard Model

Problem Set 3

Due Tuesday, October 8, in class (BSP 727)

Problem: Massive vector propagators from the path integral

This problem is a warm up for the lecture on Spontaneous Symmetry Breaking, where we will understand the dynamical origin of theories that take this form.

Consider the theory of a massive vector, A^μ, with mass m coupled to a scalar, φ, via a kinetic mixing interaction. In generic ξ gauge the Lagrangian in Minkowski space reads:

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\partial^\mu\phi - mA^\mu\right)\left(\partial_\mu\phi - mA_\mu\right) + \mathscr{L}_{\text{gauge-fix}}\,, \quad \text{with}$$
$$\mathscr{L}_{\text{gauge-fix}} = -\frac{1}{2\xi}(\partial_\mu A^\mu + \tilde{m}\xi\phi)^2\,.$$

In this problem we will derive the Feynman propagator, $\Delta_F(x-y)$, for this theory using the path integral formalism.

(i) Compute the Euclidean action, $\mathcal{S}_E$, of this Lagrangian. Remember that to transform to Euclidean space both coordinates and vector field need to be transformed via

$$\begin{aligned}
x^0 &\to -ix^{E0}\,, & x_0 &\to -ix^E_0\,,\\
x^i &\to x^{Ei}\,, & x_i &\to -x^E_i\,,\\
\partial^0 &\to i\partial^{E0}\,, & \partial_0 &\to i\partial^E_0\,,\\
\partial^i &\to -\partial^{Ei}\,, & \partial_i &\to \partial^E_i\,,\\
A^0 &\to iA^{E0}\,, & A_0 &\to iA^E_0\,,\\
A^i &\to -A^{Ei}\,, & A_i &\to A^E_i\,.
\end{aligned}$$

Below, we do not keep track of all the labels "E" that indicate that the quantities are Euclidean.

(ii) Package all real fields into one field

$$\Phi(x) = \left(A^E_0(x), A^E_1(x), A^E_2(x), A^E_3(x), \varphi(x)\right)^T$$

and compute the kinetic matrix $\mathcal{K}_{\{M,x\};\{N,y\}}$, defined via

$$\mathcal{S}_E = \mathcal{S}^0_E = \frac{1}{2}\int\int d^4x d^4y \Phi_M(x)\mathcal{K}_{\{M,x\};\{N,y\}}\Phi_N(y)\,,$$

for this case.

(iii) Compute the matrix $\tilde{\mathcal{K}}(p)$ that is needed to obtain the Feynman propagator in fourier space as

$$\Delta^E_{MN}(x-y) = \int \frac{d^4p}{(2\pi^4)}e^{-ip(x-y)}(\tilde{\mathcal{K}}^{-1}(p))_{MN}\,.$$

(iv) For the specific case of $\tilde{m} = m$ invert $\tilde{\mathcal{K}}(p)$ to obtain the Euclidean propagator for the vector and scalar field. Continue your result back to Minkowski space and verify by choosing ξ and $\tilde{m}$ appropriately that you recover

- the photon propagator in generic ξ gauge and
- the propagator of a massive vector in unitary gauge.

(v) For generic $\tilde{m}$ the propagator does not split into a block-diagonal form. It is instead a 5×5 matrix in the space of fields. In this case, inverting the matrix $\tilde{\mathcal{K}}(p)$ is trickier. One way of doing this is to decompose $\tilde{\mathcal{K}}(p)$ into transverse and longitudinal parts. Namely

$$\tilde{\mathcal{K}}(p) = A\mathcal{P}_T + \mathcal{P}_i X^i{}_j \mathcal{P}^{j\dagger} \,,$$

with

$$\mathcal{P}_T = \begin{bmatrix} g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} & 0 \\ 0 & 0 \end{bmatrix} , \qquad \mathcal{P}_1 = \begin{bmatrix} i\frac{p_\mu}{|p|} \\ 0 \end{bmatrix} , \qquad \mathcal{P}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

Compute A and the 2×2 matrix X. Do you see why this decomposition is useful? (*Hint: show that* $\tilde{\mathcal{K}}^{-1}(p) = A^{-1}\mathcal{P}_T + \mathcal{P}_i (X^{-1})^i{}_j \mathcal{P}^{j\dagger}$)

(vi) Compute $\tilde{\mathcal{K}}^{-1}(p)$ and find the Feynman propagator in fourier space. Continue the result to Minkowski space and verify that you reproduce your previous computation when $\tilde{m} = m$.

(*Hint: after the rotation to Minkowski space you should find*

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \hat{K}^{-1}(p) \,,$$

with

$$\hat{K}^{-1}(p) = \begin{bmatrix} \frac{-i}{p^2-m^2}\left(g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2-\xi m\tilde{m}}\left(1+\xi\frac{(m-\tilde{m})}{p^2-\xi m\tilde{m}}\left(\frac{m-\tilde{m}}{1-\xi}+\tilde{m}\right)\right)\right) & -\xi\frac{(m-\tilde{m})p_\mu}{(p^2-\xi m\tilde{m})^2} \\ \xi\frac{(m-\tilde{m})p_\nu}{(p^2-\xi m\tilde{m})^2} & i\frac{p^2-\xi m^2}{(p^2-\xi m\tilde{m})^2} \end{bmatrix} .$$

You probably now realise why most people enjoy working in Feynman gauge, i.e., $\tilde{m} = m$ *and* $\xi = 1$.)