Gauge Theories and the Standard Model

Problem Set 11

Due Tuesday, December 10, in class (BSP 727)

Problem 1: Leptonic Meson Decays and the Cabbibo Angle

In this exercise sheet we will exploit the symmetries of QCD to extract the value of the Cabbibo angle ($\sin\theta_{\rm C}$) from the leptonic decays $\pi^+ \to \ell^+\nu$ and $K^+ \to \ell^+\nu$, with $\ell = e, \mu$.

(i) Within the Fermi theory, draw the Feynman diagrams for the above processes in the quark and the hadron picture, and write the corresponding amplitudes. Which QCD matrix-elements are needed for the computation? Use the matrix-element

$$\langle 0|\bar{q}\gamma_\mu\gamma_5 q'|M(p)\rangle = -ip_\mu f_M\,,$$

where M is a meson interpolated by $\bar{q}q'$, and f_M its decay constant, to compute the two-body widths

$$\Gamma(\pi^+ \to \ell^+\nu)\,, \qquad \text{and} \qquad \Gamma(K^+ \to \ell^+\nu)\,.$$

(ii) Use the lattice-QCD value $f_{\pi^+} = (130.2 \pm 1.7)\,\text{MeV}$ to extract the Cabbibo angle *and* its uncertainty. Take the ratio of the two widths, and use the Wolfenstein parametrisation ($\lambda \sim \sin\theta_{\rm C} \sim V_{us}$) to obtain $\sin\theta_{\rm C}$ as a function of the ratio of decay constants, f_{K^+}/f_{π^+}.

The decay constants are non-perturbative QCD quantities and must thus be either extracted from data or computed within a non-pertubative framework, e.g., lattice-QCD. However, we can use the (approximate) symmetries of QCD to relate f_{K^+} to f_{π^+}.

(iii) Use the pion matrix

$$|\pi_\alpha{}^\beta\rangle = \sum_{a=1}^{8} |\pi^a\rangle \frac{(\lambda^a)_\alpha{}^\beta}{2}$$

to find the explicit expression of the states $|\pi^+\rangle$ and $|K^+\rangle$ in terms of the $\text{SU}(3)_{\mathcal{G}}$ eigenstates $|\pi_a\rangle$ with $a = 1, \ldots, 8$.

(iv) As discussed in the lecture, the matrix elements of operators with definite transformation properties are *invariant* tensors. Here, we are interested in specific parts of the operator

$$\mathcal{O}_a^\mu = \bar{q}^\alpha(\lambda^a)_\alpha{}^\beta\gamma^\mu\gamma_5 q_\beta\,.$$

Under which representation of $\text{SU}(3)_{\mathcal{G}}$ does $\mathcal{O}_a^\mu$ transform? The matrix elements with the pion octet that are relevant for us then take the form

$$\langle 0|\mathcal{O}_a^\mu|\pi^b(p)\rangle = -ip^\mu \sum_k c_k (I_k)_a{}^b\,.$$

Find the invariant tensor(s) $(I_k)_a{}^b$.

(v) Compute the pieces of $\mathcal{O}^\mu_a$ that are relevant for the $\pi^+ \to \ell^+\nu$ and $K^+ \to \ell^+\nu$ decays, i.e., $\mathcal{O}^\mu_{\pi^+} \equiv \bar{d}\gamma^\mu\gamma_5 u$ and $\mathcal{O}^\mu_{K^+} \equiv \bar{s}\gamma^\mu\gamma_5 u$, in terms of $\mathcal{O}^\mu_a$.

Hint: the SU(3) *completeness relation can prove useful to extract the relevant parts:*

$$(\lambda^a)_\alpha{}^\beta(\lambda^a)_{\alpha'}{}^{\beta'} = 2\left(\delta^{\beta'}_\alpha\delta^\beta_{\alpha'} - \frac{1}{3}\delta^\beta_\alpha\delta^{\beta'}_{\alpha'}\right) .$$

(vi) Use the results of (iii)–(v) to relate the π^+ and K^+ matrix elements, i.e., relate f_{K^+} and f_{π^+}. Extract the Cabbibo angle from the ratio of the two-body widths that you computed.

(vii) Finally, use the quantity that is most precisely computed on the lattice, namely the ratio $f_{K^+}/f_{\pi^+} = 1.1928 \pm 0.0026$, to also extract the Cabbibo angle *and* its uncertainty. Compare it to your previous results and to the PDG value.