Gauge Theories and the Standard Model

Problem Set 10

Due Tuesday, December 3, in class (BSP 727)

Problem: Symmetries of the linear σ-model

Consider the theory of QCD with 2 families of light quarks. This theory has a so-called isospin global symmetry, which refers to the approximate unbroken $SU(2)$ global symmetry of the theory. The Lagrangian for the associated linear σ-model is given by

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\left[(\partial_\mu \boldsymbol{\pi})^2 + (\partial_\mu \sigma)^2\right] + \bar{N} i \gamma^\mu \partial_\mu N \\ &+ g\bar{N}(\sigma + i\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi})N + \mu^2(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2+\boldsymbol{\pi}^2)^2\end{aligned}$$

where $N = \begin{pmatrix} p \\ n \end{pmatrix}$ is an isospin-$\frac{1}{2}$ nucleon field, $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ an isospin one pion field, and σ and isospin zero scalar field. It is convenient to use a 2×2 matrix to represent the spin zero fields collectively:

$$\Sigma = \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}. \tag{1}$$

(i) Show that the Lagrangian is invariant under the isospin transformations. Find the corresponding conserved isospin vector current V^i_μ, with $i = 1, 2, 3$.

(ii) Show that the Lagrangian is invariant under the axial isospin transformations

$$N \to N' = \exp\left(i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\beta}}{2}\gamma_5\right), \qquad \Sigma \to \Sigma' = V^\dagger \Sigma V^\dagger, \tag{2}$$

where $V = \exp\left(\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\beta}\right)$ is an arbitrary 2×2 unitary matrix with $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ a set of real constants. Find the corresponding axial-vector currents A^i_μ.

(iii) Calculate the charge commutators $[Q^i, Q^j]$, $[Q^i_5, Q^j_5]$ and $[Q^i, Q^j_5]$, where $Q^i = \int d^3x V^i_0(x)$ and $Q^i_5 = \int d^3x A^i_0(x)$.

(iv) Calculate the commutator of particle fields with the vector and axial-vector charges.

(v) What is the interpretation of these commutators?