Introduction to holography - Lecture VIII

Last time : conformal transf, conformal group

- action on local operators $\mapsto$ primary ops $[K_\mu, O(0)] = 0$
  $[D, O(0)] = -i\Delta O(0)$
- their corr. f. were severely restricted by conformal symm.
  2 & 3 p.f. entirely det (Δ_i, C_{ijk}) , 4pf. f. of cross ratios

This time : CFTs in radial quantiz & state - operator corresp.

(1-1 corresp between states in rad. quantiz /cylinder & local ops/

- the OPE , which has finite radius of convergence , & thus allows writing n-pct. corr. f. in terms of lower -pct ones
- conformal blocks , conformal bootstrap

Radial quantization & the state-operator correspondence

- so far, wrote various prop. of corr. f. of primary ops w/o specifying a quantiz.

- we consider the euclidean th. We choose a foliation of the spt. by spatial slices (w/ an associated Hilbert sp) $\ni$ $\exists$ a unitary op. that repr. time evolution

- the euclidean corr. can then be interpreted as time-ordered product of ops.

$$\langle O_1(x_1) \dots O_n(x_n)\rangle = \langle 0 | T\{ \hat{O}_1(\tau_1, \vec{x}_1) \dots \hat{O}_n(\tau_n, \vec{x}_n)\} | 0 \rangle$$

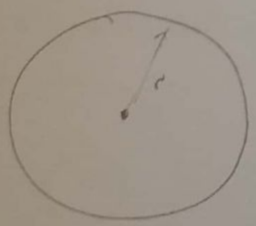
$$\cdot\, \theta(\tau_1 - \tau_2) \dots \theta(\tau_{n-1} - \tau_n) \quad \langle 0_1 \dots 0_n \rangle + \text{other orderings}$$

$O(\tau) = e^{\tau H} O(0) e^{-\tau H}$

(! $\neq$ Hilbert sp for $\neq$ foliations)

- will assume (chosen) time transl symm $\rightarrow$ $\hat{O}(\tau, \vec{x}) = e^{\tau \hat{H}} O(0, \vec{x}) e^{-\tau \hat{H}}$

- in radial quantiz, we foliate $\mathbb{R}^d$ w/ spheres S^{d-1} centred around the origin (arbitrarily pick.)

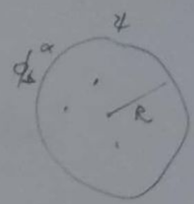


- can move b/w slices w/ the dilatation op.
- time ordering corresp. to radial ordering
- the state @ some (radial) time corresponds to some wavefn'l of the field config $\phi(\theta_i)$ on the corresp. sphere

- this wavef. is obtained by the path. int over int. of S^{d-1} (say, of rad R) w/ prescribed bnd cond ϕ_b

$$\Psi[\phi_b] = \int_{\phi_\alpha|_{S^{d-1}} = \phi_b} \mathcal{D}\phi_\alpha \underbrace{O_i(x_i)}_{\substack{r_i < R \\ \text{insertions}}} e^{-S[\phi_\alpha]}$$

(ϕ_α: fields in the theory)



$\langle \phi | \psi \rangle$

- the state $\leftrightarrow$ operator corresp.

- consider the state in radial quantiz defined on the unit sphere.

- there is an obvious map b/w operators inserted @ the origin & states on the unit S^{d-1}

$$\Psi_{\mathcal{O}}[\phi_b^a] = \int_{\phi_a|_{S^{d-1}} = \phi_a^b} \mathcal{D}\phi_a \; \mathcal{O}(0) \; e^{-S[\phi_a]} \equiv |\mathcal{O}\rangle \text{ or } \mathcal{O}(0)|0\rangle$$

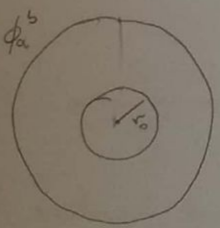
- if $\mathcal{O}(0)$ is a primary op $\Rightarrow$ $|\mathcal{O}\rangle$ is a primary state $\quad D|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle$, $K_\mu|\mathcal{O}\rangle = 0$.
  - use rep. of conformal gen. as $\int_{S^{d-1}}$

- the state $\mathcal{O}(x)|0\rangle =$ 

is not primary since $\mathcal{O}(x)|0\rangle = e^{iP_\mu x^\mu}\mathcal{O}(0)|0\rangle$ has many descendant contributions (also not eigenst. of D, as dilatations move insertion pt.)

- the vacuum $|0\rangle =$  $\leftrightarrow$ identity op.

- to go from states $\to$ operators, consider a state on the unit circle as a path. int over an annulus $1 \geq r > r_0$



$$\int_{\substack{\phi_{S^{d-1}} = \phi_b \\ \phi(r_0) = \phi_b'}} \mathcal{D}\phi \, \mathcal{D}\phi_b' \; e^{-S[\phi]} \; r_0^{D} \, \Psi_{r_0}[\phi_b']$$

equiv. repr. of $\Psi[\phi_b]$

$$= \int \langle \phi_b | \underbrace{e^{D \ln r_0}}_{r_0^{-D}} |\phi_b'\rangle r_0^{D} \, \Psi_{r_0}[\phi_b'] \, \mathcal{D}\phi_b'$$

$$= \Psi[\phi_b]$$

taking $r_0 \to 0$ $\quad \lim \Psi_{r_0}[\phi_b'] \to$ bnd. cond @ $r_0 = 0$ = insertion of local op.

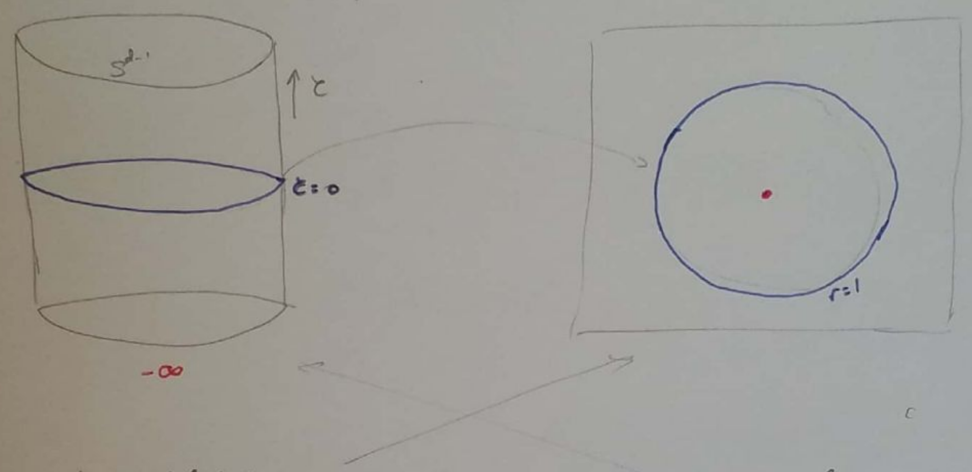
- local ops (@ 0) may be defined as eigenstates of D in radial quantization (not necess. primary)
  - if primary, these eigenstates are additionally annih. by K_μ. $\quad K_\mu|\Delta\rangle = 0$, $D|\Delta\rangle = \Delta|\Delta\rangle$.
  - descendants are built by acting w/ P_μ. $\quad P_\mu|\Delta\rangle$, $P_\mu P_\nu|\Delta\rangle$ etc.

- Another way to underst. radial quantiz

$\exists$ a conformal map from the cylinder $\mathbb{R}\times S^{d-1}$ to the plane $\mathbb{R}^d$

$ds^2_{cyl} = d\tau^2 + d\Omega^2_{d-1}$ $\qquad$ $ds^2_{pl} = dr^2 + r^2 d\Omega^2_{d-1} = r^2\Big(\underbrace{\frac{dr^2}{r^2}}_{d\tau^2} + d\Omega^2_{d-1}\Big)$

$r = e^{+\tau}$ $\qquad$ rel by Weyl resc.



- under it, dilatations $r\partial_r \leftrightarrow \partial_\tau$ τ translations

radial quantiz $\leftrightarrow$ standard quantiz (on $\tau = $ const^t. slices)

$\leftrightarrow$ const r slices

- under a Weyl transf, CFT corr. (f. of local ops) transform as.

$$\langle O_1(x_1)\dots O_n(x_n)\rangle_g = \prod_i \Omega(x_i)^{\Delta_i} \langle O_1(x_1)\dots O_n(x_n)\rangle_{\Omega^2 g}$$

norm by $\langle 1\rangle_g$ $\rightarrow$ $\qquad$ $/\langle 1\rangle_{\Omega^2 g}$

(careful Weyl anomaly in corr w/ stress!)

- natural to define cylinder ops. $O_{cyl}(\tau, \hat{\theta}_i) = e^{\Delta\tau} O_{flat}(x = e^\tau \hat{\theta}_i)$

- reflection positivity : unitary th. in Lorentz signature Wick rotate to ..
  - in part, H should be hermitean
  - for a hermitean local operator $O(x) = e^{iHt - i\vec{P}\vec{x}} O(0) e^{-iHt + i\vec{P}\vec{x}}$ $\quad$ $t \to -i\tau_E$

$$= e^{H\tau_E - i\vec{P}\vec{x}} O(0) e^{-H\tau_E + i\vec{P}\vec{x}}$$

$O^\dagger(x, \tau_E) = O(x, -\tau_E)$. from $O_E(\tau_E, \vec{x}) = O_L(-i\tau_E, \vec{x})$

refl. pos

$|\psi\rangle = O(-\tau_{E_1})\dots O(-\tau_{E_n})|0\rangle$

(for ops. w/ spin, add i's on time comp $\qquad$ $O^0_E(\tau_E, \vec{x}) = -iO^0_L(-i\tau_E, \vec{x})$

$O^i_E(\tau_E, \vec{x}) = O^i_L(-i\tau_E, \vec{x})$

$\langle\psi|\psi\rangle \geq 0$

equiv. to norm. pos. in lor

in euclidean, how Hermitean conj. acts dep. on time dir! $\left(O_E^{\mu_1\dots\mu_n}(\tau_E, \vec{x})\right)^\dagger = \Theta^{\mu_1}{}_{\nu_1}\dots\Theta^{\mu_n}{}_{\nu_n}(-\tau_E, \vec{x})$

• what does hermitean conjugation on the cylinder $O_{cyl}(\tau_E, \vec{n})^\dagger = O_{cyl}(-\tau_E, \vec{n})$ translate to on the plane, given $O_{cyl}(\tau_E, \vec{n}) = e^{\Delta \tau_E} O_{pl}(e^{\tau_E} \vec{n})$

$$e^{\Delta \tau_E} O_{pl}^{\dagger_{rad}}(x) = e^{-\Delta \tau_E} O_{pl}(e^{-\tau_E} \vec{n})$$

where "rad" means plane in radial quantiz

$$\text{or} \quad \boxed{O_{pl}^{\dagger_{rad}}(x) = \frac{1}{|x|^{2\Delta}} O_{pl}\left(\frac{x^\mu}{x^2}\right)}$$

- is true for the chosen quantization
- on the plane in radial quantiz, hermitean conj $\leftrightarrow$ inversion

(for ops. w/ spin $[O^{\mu_1 \dots \mu_\ell}(x)]^{\dagger} = I^{\mu_1}{}_{\nu_1}(x) \dots I^{\mu_\ell}{}_{\nu_\ell}(x) O^{\nu_1 \dots \nu_\ell}\left(\frac{x^\mu}{x^2}\right) \frac{1}{x^{2\Delta}}$

• in part, this maps $[T^{\mu\nu}(x)]^{\dagger_{rad}} = I^\mu{}_\rho(x) I^\nu{}_\sigma(x) T^{\rho\sigma}\left(\frac{x^\mu}{x^2}\right) \frac{1}{x^{2d}}$

– sign from $n_\mu \propto \frac{x_\mu}{r^2}$ $\quad n_\mu I^\mu{}_\nu = -n_\nu$

$$G_\xi^\dagger = \int_{\substack{S^{d-1} \\ r=1}} d\Omega_{d-1} [T^{\mu\nu}]^\dagger \xi_\nu = \int \tilde{I}^\mu{}_\rho T^{\rho\sigma} I^\nu{}_\sigma \xi_\nu = \int T^{\rho\sigma} I^\nu{}_\sigma \xi_\nu = G_{I\xi}.$$

$$\Rightarrow \quad P_\mu^\dagger = K_\mu \qquad D^\dagger = +D \qquad M_{\mu\nu}^\dagger = -M_{\mu\nu}$$

(conv. in Simmons-Duffin ≠ Shiraz '97).

(note $[D, P_\mu] = P_\mu$, $[D, K_\mu] = -K_\mu$, $[K_\mu, P_\nu] = 2\delta_{\mu\nu} D - 2 i M_{\mu\nu}$, other comm. have i's)

– same as the hermiticity rels obtained by choosing $SO(d+1,1)$ gen.

the Euclidean th. w/ such hermiticity cond (non-unitary) $\leftrightarrow$ unitary Lorentzian th.

• note $[D, \phi(0)] = \Delta \phi(0)$

• remember $|O\rangle = O(0)|0\rangle$ $\qquad (|O\rangle)^\dagger = \lim_{x \to \infty} x^{2\Delta} \langle 0| O(x) \equiv \langle O|$

in part $(D|O\rangle)^\dagger = \lim_{x \to \infty} x^{2\Delta} \langle 0| O(x) \underbrace{D^\dagger}_{-D} = \Delta \langle O|$ ✓

$(K|O\rangle)^\dagger = \langle O| P = 0$ for primary

• we can underst. the 2pf. as an overlap of in & out st. $\quad \left(y'^\mu = \frac{y^\mu}{y^2}\right)$

$$\langle O(y) O(x) \rangle = \langle 0 | (y^{-2\Delta} O(\tilde{y}))^\dagger | O(x) \rangle = \frac{1}{y^{2\Delta}} \langle 0 | (e^{-iP\tilde{y}} O(0) e^{+iP\tilde{y}})^\dagger O(x) \rangle$$

$-2y_\mu x_\nu D$

≈ leading term in the exp $\frac{1}{(y-x)^{2\Delta}} \approx \frac{1}{(y^2 - 2x \cdot y)^{2\Delta}} \sim 2\Delta \frac{x \cdot y}{y^{2\Delta+2}}$ $\quad \approx \langle O | e^{iP_{\tilde{y}}^\dagger} e^{iPx} | O \rangle \approx \langle O | K_\mu P_\nu | O \rangle \tilde{y}$

- requiring that the norms of states in radial quantization be positive
  → unitarity bounds on CFT operator dimensions

$[K_\mu, P_\nu] = 2\eta_{\mu\nu}D - 2i M_{\mu\nu}$

- e.g. $\|P_\mu|\mathcal{O}\rangle\|^2 = \langle\mathcal{O}|\underbrace{P_\mu^\dagger}_{K_\mu} P_\mu|\mathcal{O}\rangle = 2\Delta \underbrace{\eta_{\mu\mu}}_{\substack{\text{Eucl.} \\ \text{(not summed)}}} \langle\mathcal{O}|\mathcal{O}\rangle \geq 0 \Rightarrow \Delta \geq 0$.

- stronger bound (scalars) consider norm of $P_\mu P^\mu|\mathcal{O}\rangle$

$$\langle\mathcal{O}|\underbrace{K_\mu K^\mu} P_\nu P^\nu|\mathcal{O}\rangle = \langle\mathcal{O}|K_\mu(2\eta^{\mu\nu}D - 2iM^{\mu\nu})P_\nu|\mathcal{O}\rangle + \langle\mathcal{O}|K_\mu(P_\nu K^\mu)P^\nu|\mathcal{O}\rangle$$

$$= 2(\Delta+1)\langle\mathcal{O}|K_\mu P^\mu|\mathcal{O}\rangle - (2i)(-i)(d-1)\langle\mathcal{O}|K_\mu P^\mu|\mathcal{O}\rangle + 2\Delta\langle\mathcal{O}|K_\mu P^\mu|\mathcal{O}\rangle = 2(2\Delta-(d-2))\|P_\mu\mathcal{O}\|^2$$

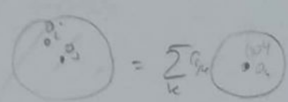
$$\Rightarrow \quad \Delta \geq \frac{d-2}{2}, \quad w/ \quad \Box\mathcal{O} = 0 \text{ if equal} \qquad P^2|\mathcal{O}\rangle \text{ null state}$$

- for ops. transf. in traceless symm. spin l reps, $\Delta \geq l + d - 2$ (cons. current if saturated).

The operator product expansion

- we have previously discussed how to produce a state in radial quantiz. by inserting an operator in the path int. over the unit disk

- if we insert two operators, $\mathcal{O}_i(x)\mathcal{O}_j(0)$ & perform the path int ⇒ some state. This state can be expanded in a basis of energy (dilatation) eigenst. as a linear combo of primary & descendant states, so we can write
  + state-op. corresp



$$\mathcal{O}_i(x)\,\mathcal{O}_j(0)|0\rangle = \sum_k C_{ijk}(x, P_\mu)\,\mathcal{O}_k(0)|0\rangle$$

↑ packages together primaries (k) & their descendants

- this decomp. is called the operator product expansion (OPE)

- it is an operator eqn (valid inside ≠ corr. f. as long as the other op. insertions are outside the ball of rad $|x|$ surrounding O_j)

- in CFT, the OPE has finite radius of convergence (radius sphere that contains the 2 ops. & not the others)

- the expansion can be performed around ≠ points (the coeff. functions $C'_{ijk}(x_{13}, x_{23}, \partial_3)\, O_k(x_3)$ will be ≠)

- the form of the OPE is highly restricted by conformal invar
  - e.g. for 3 scalars

$$O_i(x_1)\, O_j(x_2) = \sum_k \underbrace{C_{ijk}(x_{12}, \partial_2)}_{\frac{1}{x_{12}^{\Delta_i+\Delta_j-\Delta_k}}\left(1 + \#\, x_{12}^{\mu}\, \partial_{2\mu} + \#\, x_{12}^{\mu} x_{12}^{\nu}\, \partial_\mu \partial_\nu + \dots\right)} O_k(x_2)$$

fixed by acting w/ K_μ

  - the full form can be det by taking the exp. value w/ $O_k(x_3)$

$$\underbrace{\langle O_i(x_1)\, O_j(x_2)\, O_k(x_3)\rangle}_{\frac{C_{ijk}\ \}\ \#}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}\, |x_{23}|^{\Delta_j+\Delta_k-\Delta_i}\, |x_{13}|^{\Delta_i+\Delta_k-\Delta_j}}} = \sum_k C_{ijk}(x_{12}, \partial_2)\, \underbrace{\langle O_k(x_2)\, O_k(x_3)\rangle}_{\frac{1}{|x_{23}|^{2\Delta_k}}}$$

- note the OPE can be used to reduce an n-point f. to an $(n-1)$-pt. f.

- the OPE coefficient functions are entirely det. by the primary 3 p.f. coeff = #! (also structure const. of the operator algebra) (a.k.a OPE coefficients) & the dimensions of the ops involved, the descendant contr. being fully fixed by conformal invar
(also, only $\Delta_{ij} = \Delta_i - \Delta_j$ enters.)

=> the CFT corr. f are entirely determined by Δ_i, C_{ijk} CFT data
(∞)

An example

- note the OPE sum is over an ∞ set of operators. Useful to see this used in an example

- consider a 4pf of identical scalars, w/ dim Δ_ϕ. Conf. invar $\Rightarrow$ it must take the form

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}} \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

the OPE takes the form (more gen, $\frac{1}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}}\left|\frac{x_{24}}{x_{14}}\right|^{\Delta_{12}}\left|\frac{x_{14}}{x_{13}}\right|^{\Delta_{34}} g(u,v)$)

$$\phi(x_1)\phi(x_2) = \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}\, C_a(x_{12},\partial_2)\, \mathcal{O}^{a}(x_2)$$

spin

- assuming the points are such that the 12 & 34 OPEs converge, we have

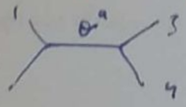
$$\langle \phi(x_1)\phi(x_2)\,\phi(x_3)\phi(x_4)\rangle = \sum_{\mathcal{O},\mathcal{O}'} C_{\phi\phi\mathcal{O}}\, C_{\phi\phi\mathcal{O}'}\, C_a(x_{12},\partial_2)\, C_b(x_{34},\partial_4)\langle \mathcal{O}^a(x_2)\mathcal{O}'^b(x_4)\rangle$$

orthonormal basis

$$= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\, C_a(x_{12},\partial_2)\, C_b(x_{34},\partial_4)\, \frac{I^{ab}(x_{24})}{|x_{24}|^{2\Delta_{\mathcal{O}}}} \quad \{ I^{\mu_1\ldots}{}_{\nu_1\ldots}(x)\; I^{\mu\ldots}{}_{\nu\ldots}(x) - \text{traces} \}$$

$$= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\, \underbrace{\mathcal{G}^{(12)(34)}_{\Delta,\ell}}\,(x_1;x_2,x_3,x_4)$$

conformal partial waves $\rightarrow$ entirely kinematic objects

schematically: $= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}$ 

$\rightarrow$ encode contr. of descendants within a given conformal family to the 4pf.

- more precisely, defining the projection onto the conf. multiplet of $\mathcal{O}$ (primary) by

$$\mathbb{P}_{\mathcal{O}} = \sum_{\alpha,\beta=\mathcal{O},P\mathcal{O},PP\mathcal{O}\ldots} |\alpha\rangle\, \mathcal{N}^{-1}_{\alpha\beta}\langle\beta| \qquad \mathcal{N}_{\alpha\beta} \equiv \langle\alpha|\beta\rangle$$

we have the resolution of the identity $\mathbb{1} = \sum_{\mathcal{O}\text{ primary}} \mathbb{P}_{\mathcal{O}}$

to see this

- inserting this into a 4pf in radial quantiz, w/ $|x_{34}| \geq |x_{12}|$ we have

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle 0|\, \mathcal{R}\{\phi(x_3)\phi(x_4)\}\ \mathcal{R}\{\phi(x_1)\phi(x_2)\}\rangle$$

$$= \sum_{\mathcal{O}} \langle 0|\, \mathcal{R}\{\phi(x_3)\phi(x_4)\}\ \mathbb{P}_{\mathcal{O}}\ \mathcal{R}\{\phi(x_1)\phi(x_2)\}\rangle$$

$$= \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\ \underbrace{\mathcal{G}^{(12,34)}_{\Delta,\ell}(x_1,\dots x_4)}$$

$$\text{CPW} = \frac{1}{x_{12}^{2\Delta_\phi}\, x_{34}^{2\Delta_\phi}}\ \underbrace{g_{\Delta,\ell}}_{\text{conformal block}}(u,v)$$

- conformal blocks satisfy differential eqns : $\forall$ state in the conformal family of $\mathcal{O}$

$$\mathcal{C}\,|\mathcal{O}\rangle = \lambda_{\Delta,\ell}\,|\mathcal{O}\rangle$$

conformal Casimir $\qquad \mathcal{C} = D^2 - \frac{1}{2}\left(K_\mu P^\mu + P_\mu K^\mu\right) + \frac{1}{2} M_{\mu\nu}M^{\mu\nu} = -\frac{1}{2} J_{AB}J^{AB}$

signs?

w/ $\lambda_{\Delta,\ell} = \Delta(\Delta-d) + \ell(\ell+d-2)$

- obviously $\quad \mathcal{C}\,\mathbb{P}_{\mathcal{O}} = \mathbb{P}_{\mathcal{O}}\,\mathcal{C} = \lambda_{\Delta,\ell}\,\mathbb{P}_{\mathcal{O}}$

- $J_{AB}\,\phi(x_1)\phi(x_2)|0\rangle = \left([J_{AB},\phi(x_1)]\,\phi(x_2) + \phi(x_1)\,[J_{AB},\phi(x_2)]\right)|0\rangle =$

$$= \left(\mathcal{L}_{AB,1} + \mathcal{L}_{AB,2}\right)\left(\phi(x_1)\phi(x_2)|0\rangle\right)$$

differential ops. @ pts x_1, x_2 (given in prev. lecture).

- it follows that $\quad \mathcal{C}\,\phi(x_1)\phi(x_2)|0\rangle = \underbrace{-\frac{1}{2}\left(\mathcal{L}_1^{AB} + \mathcal{L}_2^{AB}\right)\left(\mathcal{L}_{1,AB} + \mathcal{L}_{2,AB}\right)}_{\mathcal{D}_{1,2}}\phi(x_1)\phi(x_2)|0\rangle$

- then $\quad \mathcal{D}_{1,2}\,\langle 0|\,\mathcal{R}\{\phi(x_3),\phi(x_4)\}\ \mathbb{P}_{\mathcal{O}}\ \mathcal{R}\{\phi(x_1),\phi(x_2)\}|0\rangle =$

$$= \langle 0|\,\mathcal{R}\{\phi(x_3)\phi(x_4)\}\ \mathbb{P}_{\mathcal{O}}\ \mathcal{C}\,\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle$$

$$= \lambda_{\Delta,\ell}\ \langle 0|\,\mathcal{R}\{\phi(x_3)\phi(x_4)\}\ \mathbb{P}_{\mathcal{O}}\ \mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle .$$

- Plugging in the rel b/w the conformal partial wave & the conformal block

$$\mathcal{G}_{\Delta,\ell}(x_i) = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2}\,|x_{23}|^{\Delta_3+\Delta_4}}\left|\frac{x_{24}}{x_{14}}\right|^{\Delta_{12}}\left|\frac{x_{14}}{x_{13}}\right|^{\Delta_{34}} g_{\Delta,\ell}(\underbrace{u,v}_{z,\bar z}) \qquad u = z\bar z \qquad v = (1-z)(1-\bar z)$$

find $\ \mathcal{D}\, g_{\Delta,\ell}(z,\bar z) = \frac{1}{2}\lambda_{\Delta,\ell}\, g_{\Delta\ell}(z,\bar z) \qquad \mathcal{D} = \underbrace{\mathcal{D}_z}_{z^2(1-z)\partial_z^2 - \dots} + \mathcal{D}_{\bar z} + (d-2)\frac{z\bar z}{z-\bar z}\left((1-z)\partial_z - (1-\bar z)\partial_{\bar z}\right)$

for $\Delta_1 = \Delta_2$, $\Delta_3 = \Delta_4$

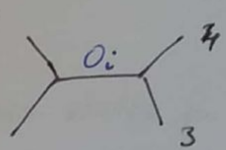
The conformal bootstrap

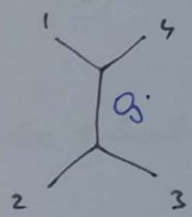
- we already saw how the OPE can be used to reduce n-point functions to 2 point-functions $\Rightarrow$ they are entirely det. by the CFT data Δ_i, C_{ijk}

- $\exists$ any constraints on these data?

- OPE associativity $\quad O_1 O_2 O_3 = O_1 O_2 O_3$

- by inserting another op. O_4 $\Rightarrow$ crossing eqn. for the 4-pct. f.

$$\langle O_1 O_2 O_3 O_4 \rangle = \sum_i C_{12i} C_{i34}$$


$$= \sum_j C_{14j} C_{23j}$$


crossing symm all 4pf $\Rightarrow$ crossing n-pct. f

- note a sol'n to the crossing eqn $\Rightarrow$ non-pert. def^n of all flat-sp. corr. of local ops, w/ no need for a Lagrangian

- we may take d set $\{\Delta_i, C_{ijk}\}$ to be the def^n of a CFT; however, additional (indep.) constraints may $\exists$ (e.g. modular invar in 2d)