Introduction to holography - Lecture XI

Last time: showed that the isometries of empty AdS_{d+1} match the symmetries of the CFT_d vacuum state

discussed

· note, however, that symm. of the vacuum usually $\neq$ symm. theory

(e.g. $Virasoro^2$ vs $SO(2,2)$ etc)

· in gravity (or, more generally, $\forall$ gauge th.) the symm of the theory are asymptotic symmetries encoded @ the boundary of the spt. where the theory lives

This time: 1) Conserved charges in gauge th. ($\subset$ gravity)

2.) Examples: Maxwell in AdS box, ASG of AdS_3 gravity

3.) The FG expansion & Weyl anomaly in AdS/CFT

1. Conserved charges in gauge th

– Noether current vanishes on-shell, up to a boundary contr. ($\partial_\nu K^{\mu\nu}$)

assoc. w/ a local symm (w/ param $\varepsilon(x)$)

– remember 1st the construction of the Noether current assoc. w/ a global symm., char. by a symm. param. $\varepsilon =$ const.

• under it, $\delta_\varepsilon S = \int d^d x \, \partial_\mu M^\mu(\varepsilon)$ action invar. up to bnd. terms

where $S = \int d^d x \, \mathcal{L}(\phi)$

generic. fields

- under a general variation $\delta\phi$: $\delta S = \int d^dx \, [\underbrace{E\,\delta\phi}_{\text{e.o.m}} + \partial_\mu \underbrace{\Theta^\mu(\phi, \delta\phi)}_{\text{presymplectic potential}}] \quad \forall \delta\phi$

(entirely det. by the Lagr.)

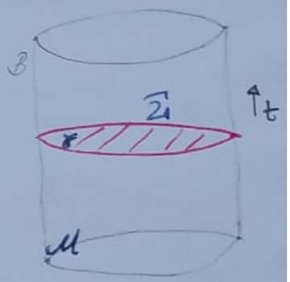
- then, the current $\boxed{J^\mu = \Theta^\mu(\delta_\varepsilon\phi) - M^\mu(\varepsilon)}$ Noether current is conserved on-shell

- Proof : $E\,\delta_\varepsilon\phi = \partial_\mu M^\mu(\varepsilon) - \partial_\mu \Theta^\mu(\delta_\varepsilon\phi) \underset{\text{on-shell}}{\overset{?}{=}} 0 \implies \partial_\mu J^\mu = 0$ on-shell

- Ex : complex scalar $S = -\int d^dx \, \partial_\mu\phi\, \partial^\mu\phi^*$ w/ $\phi \to e^{i\varepsilon}\phi \implies J^\mu = -i\varepsilon(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi)$

- the associated conserved charge

$$Q = \int_\Sigma d^{d-1}x \, \sqrt{\gamma}\, n^\mu J_\mu$$



- conserved (if no flux through $\mathcal{B}$), indep. choice Σ

Local symmetries

- same setup as before, but now symm. param is an arbitrary function $\varepsilon(x)$

- all we previously said still holds, but more is true

- for simplicity, assume that $\delta_\varepsilon\phi = f(\phi)\,\varepsilon(x) + f^\mu(\phi)\,\partial_\mu\varepsilon(x)$

e.g. E&M $\delta_\varepsilon A_\mu = \partial_\mu\varepsilon \implies f^\mu_\nu = \delta^\mu_\nu$

- symm $\implies$ $\delta_\varepsilon S = \int d^dx \left(E\, \underbrace{\delta_\varepsilon\phi}_{f\varepsilon + f^\mu\partial_\mu\varepsilon} + \partial_\mu \Theta^\mu(\delta_\varepsilon\phi) \right) = 0 \quad \forall \varepsilon(x)$ off-shell

$$= \int d^dx \, \Big[\varepsilon(x) \underbrace{\big(E f - \partial_\mu(f^\mu_{(\phi)} E)\big)}_{0} + \underbrace{\partial_\mu(\Theta^\mu + \varepsilon f^\mu E)}_{0 \text{ if } \varepsilon \text{ has compact supp.}} \Big] = 0$$

e.g. $\partial_\mu(\partial_\nu F^{\mu\nu}) = 0$ — Noether identities

$\leftrightarrow$ off-shell rels b/w. e.o.m. in th. w/ local symm.

- for $\varepsilon(x)$ arbitrary, $E\,\delta_\varepsilon \phi = E\left(f(\phi)\,\varepsilon(x) + f^\mu(\phi)\,\partial_\mu \varepsilon(x)\right) = \partial_\mu\Big[\underbrace{\varepsilon(x) f^\mu E}_{S^\mu}\Big] + \varepsilon(x)\Big(\underbrace{fE - \partial_\mu(f^\mu E)}_{0 \text{ by N.I}}\Big)$

  on-shell vanishing Noether current.

- however, from our previous defn.

$$E\,\delta_\varepsilon \phi = -\partial_\mu J^\mu_\varepsilon = \partial_\mu S^\mu \Rightarrow \boxed{J^\mu = -S^\mu - \partial_\nu Q^{\mu\nu}}$$

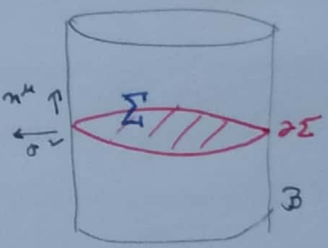
anti-symm.

$\Rightarrow$ on-shell, the Noether current J^μ assoc w/ a local symm. is a pure bnd. term

- general fact about gauge th: $\nexists$ a local cons. current assoc. w/ the "symm"

- however, $Q^{\mu\nu}$ is in gen non-trivial, & contributes to the charge

$$Q = \int_\Sigma d^{d-1}x \sqrt{\gamma}\, n_\mu J^\mu = \int_{\partial\Sigma} d^{d-2}x \sqrt{\sigma}\, n_\mu \sigma_\nu Q^{\mu\nu}$$



- consequently:
  - all gauge transf. w/ param $\varepsilon(x^\mu) \ni \varepsilon(x^\mu)\big|_{\mathcal{B}} = 0$ are unphysical, as they do not affect Q $\Rightarrow$ trivial gauge transf.
  - gauge transf. whose param $\varepsilon(x)|_{\mathcal{B}} \neq 0$ are physical symm. (do affect Q)
    $\Rightarrow$ large gauge transf.

$\Rightarrow$ in gauge th. (including gravity), the symm. are asymptotic b/c the cons. charges are $Q_\varepsilon = \int_{\partial\Sigma} d^{d-2}x \sqrt{\sigma}\, n_\mu \sigma_\nu Q^{\mu\nu}_\varepsilon \sim \varepsilon$

- constructed in an algorithmic fashion.

e.g. E&M $\quad S = \int d^dx \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \quad \delta S = \int -F^{\mu\nu}\partial_\mu \delta A_\nu = \partial_\mu\big(\underbrace{-F^{\mu\nu}\delta A_\nu}_{\Theta^\mu(\delta A)}\big) + \text{e.o.m}$

$$J^\mu = \Theta^\mu(\underbrace{\delta_\varepsilon A_\lambda}_{\partial\varepsilon}) = -F^{\mu\nu}\partial_\nu \varepsilon = -\partial_\nu\big(\underbrace{\varepsilon(x) F^{\mu\nu}}_{+Q^{\mu\nu}}\big) + \underbrace{\varepsilon\, \partial_\nu F^{\mu\nu}}_{\text{e.o.m}}$$

· of course, ∀ theory is defined by an action & a set of bnd. cond. $\mathcal{C}$.
on a manif w/ a bnd.

- an allowed diffeomorphism is such that $\underbrace{\mathcal{L}_\xi g_{\mu\nu}}_{\mathcal{C}} \in \mathcal{C}$.
  gauge transf
- a trivial diffeo is such that $Q_\xi \equiv 0$ ∀ $g_{\mu\nu} \subset \mathcal{C}$ b/c
  ξ falls off too fast near infinity

The asympt symm of the theory are def as $\dfrac{\text{Allowed diffeos}}{\text{Trivial diffeos}}$

& they corresp to true symm. of the theory

redundancies of descr.

Example 1. Maxwell field in AdS_{d+1}

- bnd. cond : A_μ fixed as $z=0$ ($r \to \infty$) ($A_\mu = 0$) as $z \to 0$
- allowed gauge transf : $\varepsilon(x^\mu, z) \ni \partial_\mu \varepsilon = 0$ @ z.

$$\varepsilon(x^\mu, z) = \varepsilon_0 + \underbrace{z^* \varepsilon^{(1)}(x^\mu) + \dots}_{O(z)\ \text{gt}}$$

trivial

⇒ asympt sym : constant $\qquad Q_\xi \simeq \int \varepsilon(x,z) F^{rt}$

phase rot on the ∂AdS ⇒ global symm!

dict. : A_μ in bulk ⟷ J^μ on the bn
gauge field $\qquad$ CFT current → global symm.

Example 2 : asympt symm. of AdS_3 $\qquad x^\pm = \varphi \pm t$

- go to FG. gauge $\quad ds^2 = \underbrace{\ell^2}_{AdS_3\ \text{radius}} \underbrace{\frac{dz^2}{z^2} + \frac{\overbrace{d\varphi^2 - dt^2}^{dx^+ dx^-}}{z^2}}_{\text{bnd. cond}} + \underbrace{\mathcal{L}(x^+) dx_+^2 + \bar{\mathcal{L}}(x^-) dx_-^2}_{\text{universal}}$

+ subleading (non-universal) (dep. on matter fields)

· the bulk diffeos that preserve this form of the metric must asympt become CKV of the bnd metric, $\gamma_{\alpha\beta}$

$\chi_L(x^+)$, $\chi_R(x^-)$

asympt AdS_3 spt.

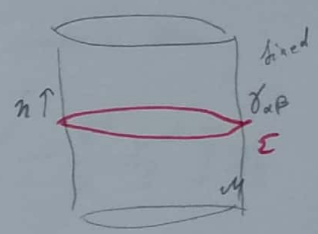
- thus, the allowed diffeos take the form

$$\xi_L = \chi_L(x^+)\partial_+ + \frac{1}{2}\chi_L'(x^+)\, z\partial_z - \frac{\ell^2 z^2}{2}\chi_L''(x^+)\partial_- + \dots$$

$$\xi_R = \chi_R(x^-)\partial_- + \frac{1}{2}\chi_R'(x^-)\, z\partial_z - \frac{\ell^2 z^2}{2}\chi_R''(x^-)\partial_+ + \dots$$

- now, to decide what are trivial & non-triv. diffeos, need to compute cons. charges. A convenient formula is based on the Brown-York stress tensor

$$T_{\alpha\beta} = -\frac{2}{\sqrt{\gamma}}\frac{\delta S^{\text{on-shell}}[\gamma]}{\delta\gamma^{\alpha\beta}} \qquad S = \underbrace{S_{EH}}_{\int R\sqrt{G}} + \underbrace{S_{GH}}_{\frac{1}{8\pi G}\int_{\partial M} K} + S_{ct}[\gamma]$$



$$= -\frac{1}{8\pi G}\big(\underbrace{K_{\alpha\beta}}_{\frac{1}{2}n^\lambda\partial_\lambda\gamma_{\alpha\beta}} - K\gamma_{\alpha\beta}\big) + ct$$

$$Q = \int_{\partial\Sigma} d^{d-1}x\,\sqrt{\sigma}\, n^\alpha T_{\alpha\beta}\xi^\beta$$

- for the aAdS$_3$ metric above

$$K_{\alpha\beta} = -\frac{z}{2\ell}\partial_z\gamma_{\alpha\beta} = \frac{1}{\ell}\left(\frac{\eta_{\alpha\beta}}{z^2} + O(z^2)\right)$$

$$K = \gamma^{\alpha\beta}K_{\alpha\beta} = z^2\Big(\overbrace{\eta^{\alpha\beta}}^{\begin{pmatrix}0&1\\1&0\end{pmatrix}} - z^2\begin{pmatrix}\bar{\mathcal{L}} & 0\\ 0 & \mathcal{L}\end{pmatrix}\Big)\left(\frac{\eta_{\alpha\beta}}{z^2} + O(z^2)\right) = \frac{2}{\ell} + O(z^4)$$

$$T_{\alpha\beta} = -\frac{1}{8\pi G\ell}\left(\frac{\eta_{\alpha\beta}}{z^2} - 2\left(\frac{\eta_{\alpha\beta}}{z^2} + \begin{pmatrix}\mathcal{L} & \\ & \bar{\mathcal{L}}\end{pmatrix}\right)\right) + O(z^2) = \frac{1}{8\pi G\ell}\begin{pmatrix}2\mathcal{L} & \frac{1}{z^2}\\ \frac{1}{z^2} & 2\bar{\mathcal{L}}\end{pmatrix} + O(z^2)$$

divergent!

$$Q \sim \int_{z\sim\varepsilon} d\varphi\; z\, T_{t\beta}\,\xi^\beta$$

need to add counterterm to make it finite $\quad S_{ct} = -\frac{1}{8\pi G\ell}\int d^2x\sqrt{\gamma}$

$$\Delta T_{\alpha\beta} = \frac{-2}{\sqrt{\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{\alpha\beta}} = -\frac{1}{8\pi G\ell}\gamma_{\alpha\beta}$$

$$T_{\alpha\beta}^{\text{ren}} = \frac{1}{8\pi G\ell}\begin{pmatrix}\mathcal{L} & 0\\ 0 & \bar{\mathcal{L}}\end{pmatrix}$$