Introduction to holography - Lecture X

We are now ready to perform our 1^{st} few checks of the AdS/CFT corresp, following our derivation of an AdS_5/CFT_4 corresp. a few lectures back & the general intro to properties of CFTs.

The most basic check is that the symmetries on the two sides of the AdS_{d+1}/CFT_d correspondence are the same.

There are 2 levels @ which one may discuss the symm.

- symmetries of the vacuum state $\leftrightarrow$ isometries of empty AdS
- symmetries of the theory $\leftrightarrow$ asymptotic symm. of a AdS spt. (asympt.)

Anti - de Sitter spacetimes

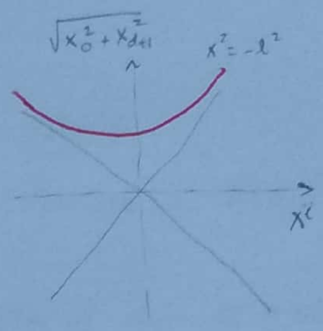
AdS_{d+1} : maximally symm. spt. of constant negative curvature
max. # Killing. vect. $\frac{(d+1)(d+2)}{2}$

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

- isometry gp $SO(d,2)$ (for Lorentzian AdS_{d+1}), as computed from K.V. alg.
same as "Lorentz" gp in $\mathbb{R}^{d,2}$ ($\cong$ d-dim'l conformal gp.)

- simplest construction: embedding into $\mathbb{R}^{d,2}$, w/ metric

$$ds^2 = \tilde{\eta}_{MN}\, dx^M dx^N \qquad \tilde{\eta}_{MN} = \begin{pmatrix} - & & & \\ & + & & \\ & & \ddots & \\ & & & + \\ & & & & - \end{pmatrix}$$



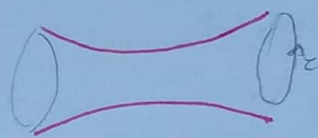
* restrict to hyperboloid $\tilde{\eta}_{MN} X^M X^N = -\ell^2$

 – manifestly $SO(d,2)$ preserving constr.

- a metric on AdS_{d+1} can be obtained by writing down an explicit coord. system that solves this constraint, e.g.

global coord : $\begin{cases} X^0 = \ell \cosh\rho \cos\tau & X^i = \ell \sinh\rho\, \Omega_i \\ X^{d+1} = \ell \cosh\rho \sin\tau & \text{w/ } \sum_{i=1}^{d} \Omega_i^2 = 1 \end{cases}$

$\rho \in (0,\infty) \quad \tau \in (0, 2\pi)$



$$\Rightarrow ds^2 = \ell^2 \big(-\underbrace{\cosh^2\rho\, d\tau^2}_{SO(2)\to\mathbb{R}} + d\rho^2 + \sinh^2\rho \underbrace{d\Omega^2_{d-1}}_{\substack{\text{unit } S^{d-1} \\ \text{manifest } SO(d)}} \big)$$

after decompactifying the τ coord.

later

- the isometry gen. are $J_{MN} = -i\left(X_M \frac{\partial}{\partial X^N} - X_N \frac{\partial}{\partial X^M}\right)$ & are identif. w/ the conformal gen. as $J_{0,d+1} = -D \quad J_{\mu\nu} = M_{\mu\nu} \quad J_{\mu 0} \cdot \frac{1}{2}(P_\mu + K_\mu) \quad J_{\mu,d+1} = \frac{i}{2}(P_\mu - K_\mu)$

- the quadratic Casimir $\frac{1}{2} J_{MN} J^{MN}$ $\leftrightarrow$ identified w/ the AdS Laplacian.
 (only isometry-invar 2nd order diff. op.)

$$\frac{1}{2} J_{AB} J^{AB} \underbrace{\phi}_{\text{scalar}} = -\big(X^2 \underbrace{\partial_X^2}_{\mathbb{R}^{d,2}\text{ Lapl.}} + X\cdot\partial_X\, (d + X\cdot\partial_X)\big]\phi$$

- $\mathbb{R}^{d,2}$ ($\text{Mink}^{d+1,1}$) can be foliated by (euclidean) AdS_{d+1} slices as

 $(X\partial_X = \ell\partial_\ell)$

$$d\tilde{s}^2 = -d\ell^2 + \ell^2 \overbrace{ds^2_{AdS_{d+1}}}^{\text{rescaled}}, \quad \text{w/ Laplacian } \partial_X^2 = -\partial_\ell^2 - \frac{d+1}{\ell}\partial_\ell + \underbrace{\Box_{AdS}}_{\text{not rescaled}}$$

thus $\frac{1}{2} J_{AB} J^{AB} \Phi = \big[-\cancel{\ell^2\partial_\ell^2} - \cancel{(d+1)\ell\partial_\ell} + \ell^2\Box_{AdS} + \cancel{(d+1)\ell\partial_\ell} + \cancel{\ell^2\partial_\ell^2} \big]\phi = \ell^2 \Box_{AdS}\phi$

Detour

· for a free scalar $(\Box_{AdS} - m^2)\phi = 0 \Rightarrow \frac{1}{2} J_{AB} J^{AB} \phi = \underset{\sim}{m^2}\ell^2\phi$

Casimir : $\Delta(\Delta - d)$

$\Rightarrow$ free scalar field in AdS $\subset$ conf. family of a primary operator of dim $\Delta \ni \Delta(\Delta-d)\ell^{-2} = m^2$

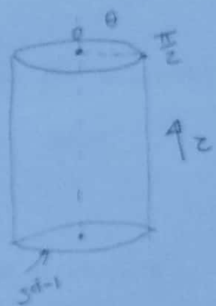
· AdS spt. is non-compact $\rightarrow$ how can we represent it?

tool : conformal compactif. : given (M, g) w/ M non compact, a conformal compactif. is a choice of metric, $\tilde{g}$ on M w/ $\tilde{g} = \Omega^2 g$ $\ni$ $(M, \tilde{g})$ can be embedded into a compact domain $\tilde{M}$ of some (other) manifold.

$\partial\tilde{M}$ = conformal ∞. / bnd. (want to bring ∞ to finite dist.)

· in practice, we choose coord on M w/ finite ranges $\cosh\rho = \frac{1}{\cos\theta}$ $\in (1, \infty)$ $\rightarrow \theta \in (0, \frac{\pi}{2})$

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(-d\tau^2 + \underbrace{d\theta^2 + \sin^2\theta\, d\Omega^2_{d-1}}_{\text{metric on } 1/2\ S^d \text{ (disk)}}\right)$$



$$d\tilde{s}^2 = \cos^2\theta\, ds^2 = \mathbb{R}_\tau \times \text{disk}.$$

· $\Rightarrow$ Penrose diagram of AdS_{d+1} = ∞ solid cylinder w/ boundary $S^{d-1} \times \mathbb{R}$

· since $\tilde{g} = \Omega^2 g$, the causal str. of $\tilde{M}$ is the same as that of M.

Exercise i): Show that lightrays shot radially outwards from the center of AdS take a finite coord. time to reach the bnd. Compute it.

- thus, if one puts reflecting bnd. cond. for lightrays @ the bnd, they come back in finite coord time $(\pi \cdot \ell)$

· note, however, that the proper dist. to the bnd along a spacelike geodesic is infinite $\int_0^{\pi/2} \frac{d\theta}{\cos\theta}$

Exercise ii) Show that massive particle's geodesics never reach the bnd (assume for simplicity that the motion is just in the radial plane $d\Omega = 0$)

- ∂_t = Killing vect $\Rightarrow$ cons. quantity $E = - \xi^\mu g_{\mu\nu} \frac{dx^\nu}{d\tau}$ ($\xi = \partial_t$; τ: proper time) $= \frac{\ell^2}{\cos^2\theta} \frac{dt}{d\tau}$ (drop for dim'l reasons)

E: eng. part. measured by obs @ $\theta = 0$

$$ds^2 = -d\tau^2 = \frac{\ell^2}{\cos^2\theta}(d\theta^2 - dt^2) \Rightarrow \frac{\ell^2}{\cos^2\theta}\left(\frac{d\theta}{d\tau}\right)^2 = \frac{\ell^2}{\cos^2\theta}\left(\frac{E^2\cos^2\theta}{}\right)^2 - 1$$

$$= \ell^2 E^2 \cos^2\theta \quad -1 > 0$$

$\Rightarrow$ $\cos\theta$ cannot become too small $(\cos\theta)_{min} = \frac{1}{\ell^2 E^2} \Rightarrow \theta_{max}$

$$= \frac{1}{\operatorname{ch}\rho_{max}}$$

$\Rightarrow$ AdS behaves as a confining box for massive part, indep. of initial energy.

w/o breaking $\forall$ symm!

- conseq, Dirichlet bnd. cond. @ the AdS bnd are natural for massive fields, whereas they can be imposed for massless ones.

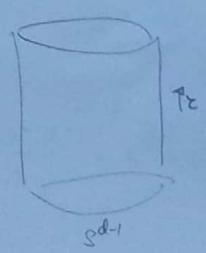
(in AdS, $\exists$ causal fields w/ slightly negative m^2 – more precisely, $\Delta \in (\underbrace{\tfrac{d}{2} - 1}_{\text{unit. bnd}}, \tfrac{d}{2})$ for which either Dirichlet or Neumann bnd. cond are possible.

BF bound $\to$ map to $-\psi'' + \left(\vec{k}^2 + \frac{1}{z^2}\left(\overbrace{m^2 - \frac{1-d^2}{4}}^{\alpha}\right)\right)\psi = \omega^2\psi$ + normalizable $\alpha < -\frac{1}{4}$ norm. neg-eng. states (i.e. $\omega^2 - k^2 < 0$ bad) $\Rightarrow m^2 > -\frac{d^2}{4}$

Let us no goback to discussing the (action of) AdS isometries ($\to$ pg 2).

- as noted the isometry gp of $AdS_{d+1} \cong$ conformal gp. d-dim.

- note in global coord, $D \leftrightarrow \frac{\partial}{\partial \tau}$ (extension into the bulk of time transl. on the cylinder).



$\Rightarrow$ energy $\leftrightarrow$ conformal dim.

spectrum (in global AdS!)

- try repr. the special conf. gen. K_μ in AdS & requiring that it annih. the wavef → build primary wavef. in AdS

- construct descendants by acting w/ P_μ ⇒ integer-spaced spectrum

- in fact, it is easy to see that the spectrum of (scalar, for simplicity) excit. in global AdS is discrete : the sols to the wave equ. are

$$\phi = e^{-i\omega t}\, Y^{(l)}_{l\{m\}}\, \chi(\rho) \qquad w/ \quad \Box_{S^{d-1}} Y_l = -l(l+d-2) Y_l$$

w/ $$\chi^{\pm}(\rho) = (\cos\theta)^{\Delta_\pm} (\sin\theta)^{l}\, {}_2F_1\left(\frac{\Delta_\pm}{2} + \frac{l+\omega}{2}, \frac{\Delta_\pm}{2} + \frac{l-\omega}{2}, \Delta_\pm + 1 - \frac{d}{2}, \cos^2\theta\right)$$

w/ $$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 l^2} \qquad \text{(generic)}$$

- for m large enough, only the Δ_+ soln is normalizable @ the AdS bnd.

- near $\theta \simeq 0$ ($z \simeq 1$), we use
$${}_2F_1(a,b,c,z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1(a,b,a+b+1-c,1-z)$$
$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b, 1+c-a-b, 1-z)$$

⇒ $\Gamma(b)$ must have a pole ⇒ $\omega = \Delta + l + 2n$

integer-spaced spectrum !
(in the free approx)

- ∃ other useful coord systems on AdS, e.g. Poincaré

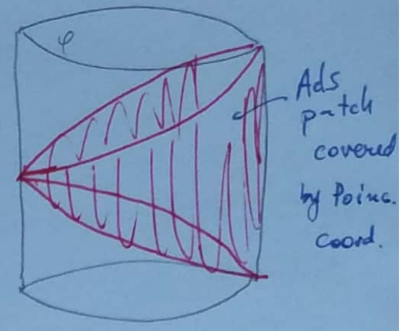
embedding : $$X^0 = \frac{l^2 + x_\mu x^\mu + z^2}{2z}, \quad X^i = l\frac{x^i}{z}, \quad X^d = \frac{l^2 - x_\mu x^\mu - z^2}{2z}, \quad X^{d+1} = l\frac{t}{z}$$
$i = 1, \dots, d-1$

$$\Rightarrow ds^2 = \frac{l^2}{z^2}\left(\eta_{\mu\nu}\, dx^\mu dx^\nu + dz^2\right)$$
conf. bnd @ $z = 0$

↖ makes Poincaré $\mathbb{R}^{d-1,1}$ & dilatations manifest.

- Poincaré coord only cover a patch of the AdS spt.

  e.g. for $d=2$



$$\begin{cases} X^0 = \ell \cos\tau \cosh\rho = \dfrac{\ell^2 + x^2 - t^2 + z^2}{2z} \\ X^1 = \ell \sinh\rho \sin\varphi = \dfrac{x}{z}\ell \\ X^2 = \ell \sinh\rho \cos\varphi = \dfrac{\ell^2 - x^2 + t^2 - z^2}{2z} \\ X^3 = \ell \sin\tau \cosh\rho = \dfrac{t}{z}\ell \end{cases}$$

$$\tan\tau = \frac{2t\ell}{\ell^2 + x^2 - t^2 + z^2} \qquad \tan\varphi = \frac{2x\ell}{\ell^2 - x^2 + t^2 - z^2} \qquad \text{as } z\to 0 \quad \tan\frac{\varphi\pm\tau}{2} = \frac{x\pm t}{\ell}$$

Minkowski diamond on bnd. cylinder $\Rightarrow$ range of $\underset{\text{null}}{\varphi\pm\tau}$ is compact. $(-\pi,\pi)$

- can also obtain euclidean global / Poincaré AdS via $\tau \to \tau_E = i\tau$, or the appropriate embedding

- note for Poincaré AdS (eucl), the map is exponential

$$X^0 = \ell \cosh\tau_E \cosh\rho = \frac{\ell^2 + x^2 + t_E^2 + z^2}{2z} \qquad X^1 = \ell \sinh\rho \sin\varphi = \frac{t_E}{z}\ell$$

$$X^3 = \ell \sinh\tau_E \cosh\rho = \frac{\ell^2 - x^2 - t_E^2 + z^2}{2z} \qquad X^2 = \ell \sinh\rho \cos\varphi = \frac{x}{z}\ell$$

(more symm. identif. b/c both X^3 & X^2 are spacelike now)

$$\ell e^{\tau_E} \cosh\rho = \frac{\ell^2}{z} \quad \gg \quad t_E = \ell e^{-\tau_E} \tanh\rho \sin\varphi$$

$$x = \ell e^{-\tau_E} \tanh\rho \cos\varphi$$

cover all.

• finally, we have the following coord system on eucl. AdS only

$$\left.\begin{aligned} X^0 &= l \cosh \rho \\ X^I &= l \sinh \rho\, \Omega_I \\ &\quad {\scriptstyle I = 1, \dots, d+1} \end{aligned}\right\} \Rightarrow ds^2 = l^2 \left(d\rho^2 + \sinh^2 \rho \underbrace{d\Omega_d^2}_{\text{unit } S^d} \right)$$

spt. looks like a hyperbolic ball ($SO(d+1)$ manifest)

• conformal compactif: let $r = \tanh \frac{\rho}{2} \in (0,1)$, then $ds^2 = \frac{4}{(1-r^2)^2} \underbrace{\left(dr^2 + r^2 d\Omega_d^2 \right)}_{\substack{\text{ball of radius 1} \\ \text{in flat. sp.}}}$

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