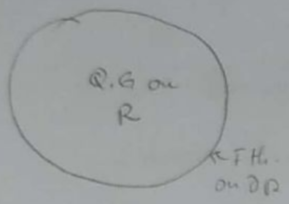
Introduction to holography - Lecture IV

Last time: showed b.h. do emit thermal rad. @ $T_H = \frac{\kappa \hbar}{2\pi}$

very-very strongly suggests $\Rightarrow$ the laws of b.h. mechanics = laws of thermodyn. applied to b.h.

& $S_{BH} = \frac{\mathcal{A}_H}{4G\hbar}$ is the entropy of the b.h. (v. large, but invisible)

– argued $S_{BH}(R)$ is the max. entropy one can have in a spt. region of size R $\Rightarrow$ holography ('t Hooft '93)



Gravitational th on R $\leftrightarrow$ non-grav. th. (FT) on ∂R

This time: will present a $\neq$ arg, also due to 't Hooft, that a large N gauge th (or, more generally, a th. w/ matrix d.o.f.) is connected to a string th. (usually contains gravity) ≈ emergent gravity descr.

– start discussing pert. strings prop. on flat space

The large N limit of gauge th.

- consider $SU(N)$ Yang-Mills th, w/ N = # of colours
- the gauge fields (N^2-1) transform in the adjoint rep. of the gauge gp. A_μ^a
- these can be repr. as hermitean $N \times N$ matrices $(A_\mu)^i{}_j = \sum_{a=1}^{N^2-1} A_\mu^a \underset{\text{gen. of the fund rep}}{(T_a)^i{}_j}$

where the 1st (upper) index transf. under the fund. & the 2nd (lower) -ıı- anti-fund.

$\delta_a A^b = -f_{ac}{}^b A^c \qquad \delta_a A^i{}_j = (\delta_a A^b)(T_b)^i{}_j = -f_{ac}{}^b A^c (T_b)^i{}_j \overset{\text{alg}}{=} i[T_a, T_c]^i{}_j A^c$

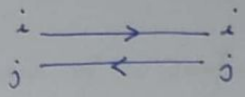
$[T_a, T_b] = i f_{ab}{}^c T_c \qquad = i \underbrace{T_a{}^i{}_k}_{\text{fund}} (A)^k{}_j - i (A)^i{}_k (T_a)^k{}_j \quad (\bar{T}_a)_j{}^k$

- the Lagr. is $\mathcal{L} = -\frac{1}{4} \operatorname{tr} F^2$ $\quad (F_{\mu\nu})^i{}_j = \partial_\mu A_\nu{}^i{}_j - \partial_\nu A_\mu{}^i{}_j + g[A_\mu, A_\nu]^i{}_j$

- will be useful to work in terms of $A' \sim gA \Rightarrow F' = dA' + [A', A'] = gF$
  & $\mathcal{L} = -\frac{1}{4g^2} \operatorname{tr} F'^2$

- Feynman rules $\quad {}^i{}_j \sim\!\sim\!\sim {}^k{}_l \; \propto g^2 \delta^{ik} \delta_{jl}$
  (ignoring all Lorentz & momentum indices, we are only interested in dependence on N & g)

- useful to introduce the double line notation
  (upper index: outgoing ; lower: incoming)

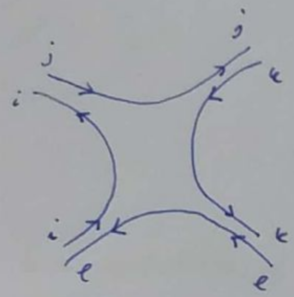


$\times g^2$

propagator

- interactions

$\frac{1}{g^2}$



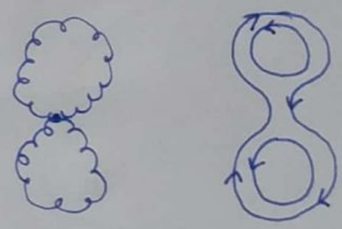
$\times \frac{1}{g^2}$

+ 3-pt. vertex

- each $\boxed{\text{index}}$ loop $\to \sum_i \to N$
  (not # loops in Feynman diag!)

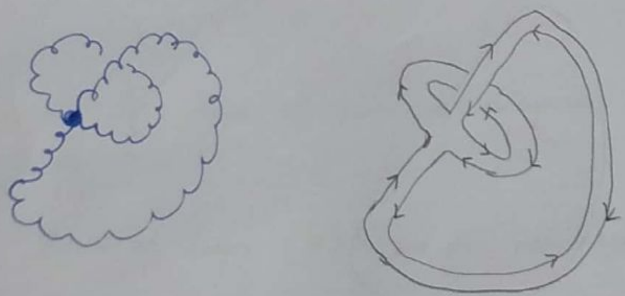
- Feynman diag. will look like

1.



$\frac{1}{g^2}(g^2)^2 N^3$
$\sim N^3 g^2$

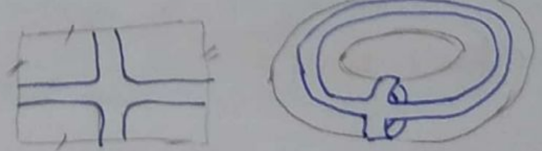
2.



$\frac{1}{g^2}(g^2)^2 N \;\sim N g^2$

- the 1^{st} diag. is called planar (can be drawn on a plane)

- the 2^{nd} cannot be drawn on a plane, but can be straightened out on a torus ($\ni$ it doesn't self-cross)



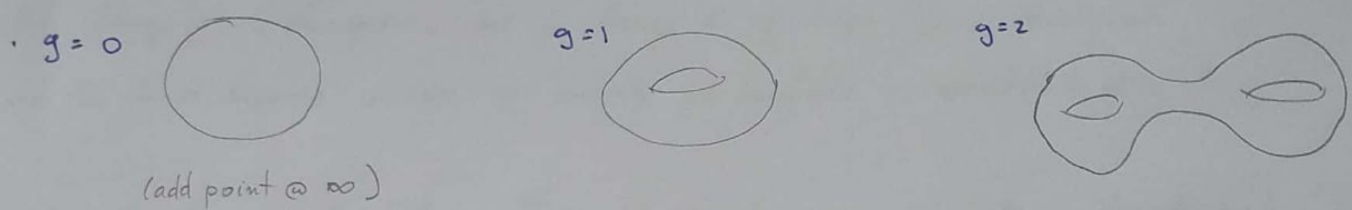
note that power of N = # faces after straightening

- what is the general N-counting / classification of diagrams?

- A Feynman diag. w/ E propagators, V vertices and F loops can be straightened out on a (compact) 2d surface of genus g (# holes in the surface), w/

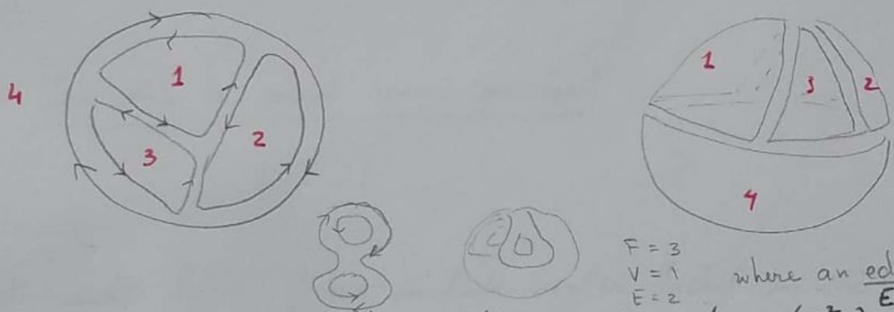
$$2-2g \equiv \underset{\sim}{\chi} = F+V-E$$

Euler characteristic (topological invar. of 2d surfaces $\sim \int R\sqrt{g}$)



- the Feynman diagram corresp. to a partition of the surface that separates it into polygons (≈ " triangulation)

e.g.



$F = 4$
$V = 4$
$E = 6$

$\chi = 2 \Rightarrow g = 0$

$F = 3$
$V = 1$
$E = 2$

where an edge is defined to connect b/w 2 vertices (or 1 vertex w/ itself)

- the g, N dependence of such a diagram is $A \sim (g^2)^{E-V} N^F$

Note though that ○ △
$E=1$ $V=1$ $E=3$ $V=3$ } same $E-V$

- since the # of loops (F) can be arbitrarily large, it does not look like ∃ good large N expansion

- however, 't Hooft noticed that taking N large w/ $\lambda \equiv g^2 N$ fixed
't Hooft limit

then $A \sim (g^2N)^{E-V} N^{\underbrace{F+V-E}_{\chi = 2-2g \leq 2}}$ does have one

$F+2g-2$

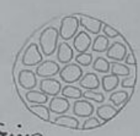
- the expansion is in terms of the topology of the Riemann surf. assoc. to the Feynman diagram, & takes the general form

$$\mathcal{A} = \sum_{g=0}^{\infty} N^{2-2g} \sum_{n=0}^{\infty} c_{g,n} \lambda^n = N^2 f_0(\lambda) + N^0 f_1(\lambda) + \frac{1}{N^2} f_2(\lambda) + \dots$$

- good large N expansion (low genus dominates)

- as $N \to \infty$, only planar diagrams survive (large simplification, but still an ∞ # of them!)

- for large λ, diagrams w/ a large # of loops (faces) dominate as the lines become dense $\to$ smooth 2d surface $\approx$ worldsheet of a string



- interestingly, perturbative string th. is also organised into a topological expansion involving 2d surfaces, & string. th. is known to contain gravity

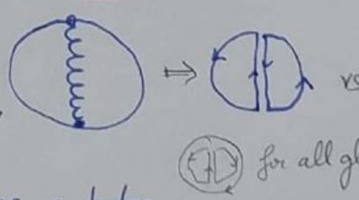
large N gauge th ('t Hooft limit) $\approx\!>$ strings? $\approx\!>$ gravity?

Q: Can this be made more precise?

- similar results hold if we include matter transf. in the fundamental repr. (quarks)
  - same counting for vertices & prop. after rescaling
  - quark / anti-quark : single line : $\longrightarrow$:
  - for the same "triangulation", one factor of N less for each quark loop. $\Rightarrow$ $\mathcal{A} \sim N^{2-2g-L}(g^2N)^{E-V}$

quark loops = holes

Ex:



vs

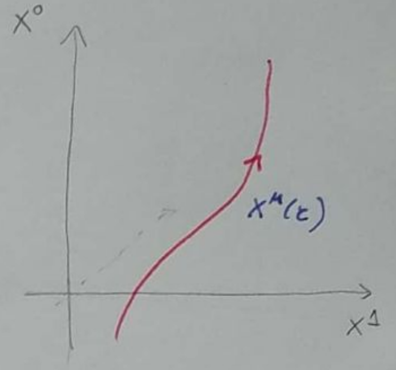
for all gluons

- this large N counting applies to $\forall$ model w/ matrix-valued fields

String theory

- starting assumption: basic objects are strings, rather than point part.*

- their action can be understood via an analogy w/ the relativistic point part. we have already discussed



  - relativistic point part. travelling in some D-dim'l spt. w/ coord X^μ $\quad \mu = \{0, \dots, D-1\}$

  - traces worldline $X^\mu(\underset{\uparrow}{\tau})$
    arbitrary worldline param.

  - action : length of the space-time interval $\quad ds^2 = \underbrace{G_{\mu\nu}(x)}_{\eta_{\mu\nu} \text{ for now}} \dot X^\mu \dot X^\nu \qquad \dot{} = \frac{d}{d\tau}$

$$S = -m \int d\tau \sqrt{-\dot X^\mu \dot X_\mu}$$

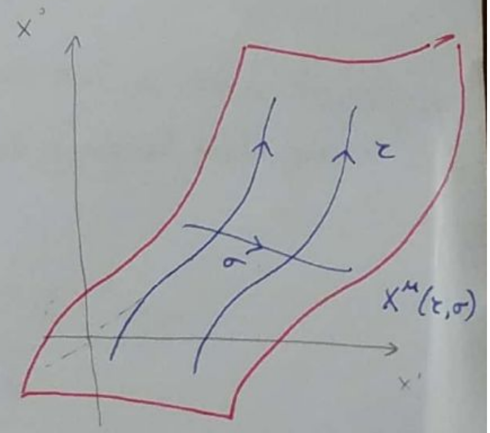
classical traj : geodesics

  - manifest target space Lorentz invar

  - reparametrization invariant $\quad \tau \to \tau' = f(\tau) \quad \Rightarrow$ constraints

$$p^\mu = \frac{m \dot X^\mu}{\sqrt{-\dot X^2}} \text{ sat. } \underbrace{p^2 + m^2}_{\propto H} = 0$$

- a string is an object w/ 1 spatial dim. ⇒ traces a 2d worldsheet as it moves throughout spt. (target sp.)



$\underbrace{X^\mu(\tau,\sigma)}$ $\qquad \mu = \{0, \dots, D-1\}$
arbitrary w-sheet param

$\tau \in \mathbb{R} \quad , \quad \sigma \in (0, \pi)$ open string

$\sigma \in (0, 2\pi)$ closed string

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$$

target sp. metric $\eta_{\mu\nu}$

- induced metric on the string = pullback of the target sp. metric

$$h_{ab} = \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \qquad \sigma^a = \{\tau, \sigma\} \quad a = \{0,1\} \qquad h_{ab} = \begin{pmatrix} \dot{X}_\mu \dot{X}^\mu & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'\cdot X' \end{pmatrix}$$

$\dot{} = \frac{d}{d\tau}$ $\quad {}' = \frac{d}{d\sigma}$

- bosonic string action : area of the string w-sheet (pt. part. analogy)

$$\underbrace{S_{NG}}_{\text{Nambu-Goto}} [X^\mu] = - \underbrace{\frac{1}{2\pi\alpha'}}_{\text{string tension}} \int d^2\sigma \sqrt{-\det h_{ab}}$$

reparam. invar.
Lorentz invar

string tension $\quad [T] = (\text{length})^{-2}$

string length : l_s $\quad$ w/ $\quad \alpha' = l_s^2$

- the string tension can be interpreted as its energy / unit length

  - note NG action takes the form $\quad S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 (X')^2}$

  - in static gauge , $\tau = X^0/R \leftarrow$ dimensionful ct. (drops out)

  - consider snapshot of string when $\frac{d\vec{X}}{dt} = 0$ (no kinetic eng). Then $\dot{X}^\mu = \{R, 0, 0, \dots\}$ &

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma \cdot \frac{dt}{R} \sqrt{\left|\frac{d\vec{X}}{d\sigma}\right|^2 R^2} = -\frac{1}{2\pi\alpha'} \int dt \cdot \text{length of string}$$

$\Rightarrow$ potential energy $\quad V = \frac{1}{2\pi\alpha'} \times$ length string

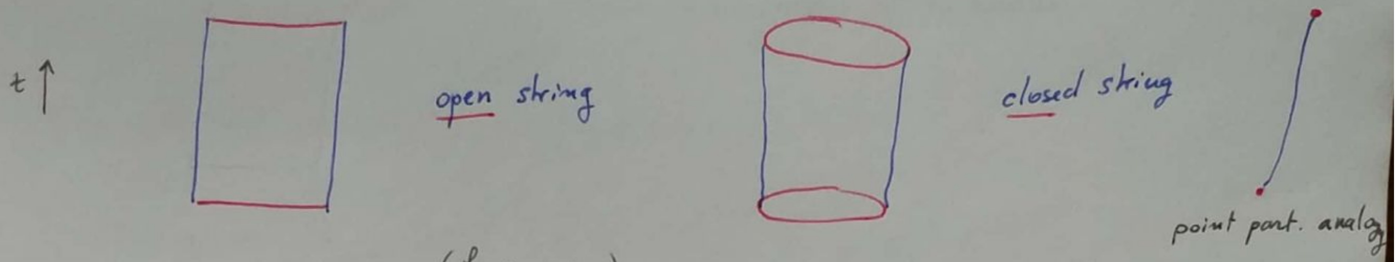
- $\exists$ another, classically equivalent, form of the action that is more useful when attempting to quantize this system using path integral techniques : Polyakov action

$$S_P [X^\mu, \underbrace{\gamma_{ab}}_{\text{worldsheet metric}}] = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\, \gamma^{ab} \underbrace{\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}}_{h_{ab}}$$

- γ_{ab} e.o.m $\quad T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \gamma_{ab} \partial^c X^\mu \partial_c X_\mu = 0 \quad \rightarrow$ solve for γ_{ab} & plug into S_P to get S_{NG}.

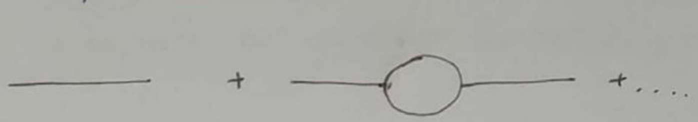
- before getting into the details of quantization, let us discuss the structure of string perturbation th.

- the amplitude for a string to propagate from some initial to some final state is given by the path integral $\int \mathcal{D}X^\mu \mathcal{D}\gamma \, e^{iS_P}$ w/ fixed initial/final curves

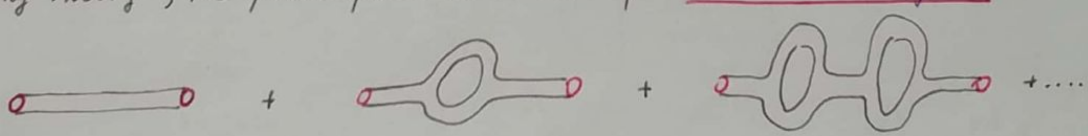


(free case)

- to include perturbative interactions in QFT we sum up Feynman diagrams

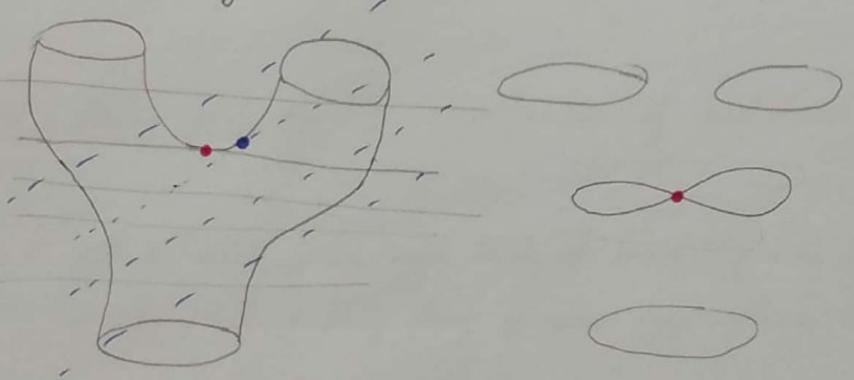


- in string theory, the prescription is to sum up smooth 2d surfaces



- locally, the worldsheet is the same as in the free case
- interactions only arise from the global topology of the w-sheet

- reason: in string th, the interaction does not happen @ a specified pt. in spt.



string splitting interaction.

$\neq$ Lorentz obs see int. happening @ $\neq$ points

(no Lorentz-invar contact interaction)

- smoothing out $\Rightarrow$ no UV divergences in str. th.

- one may add a weighing factor for $\neq$ topologies by letting

will discuss this in detail later

$$S_{string} = S_P + \lambda \chi \qquad \chi = \frac{1}{4\pi}\int_\Sigma d^2\sigma \sqrt{\gamma}\, R[\gamma] + \frac{1}{2\pi}\int_{\partial\Sigma} k\, ds$$

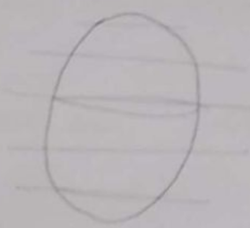
Euler characteristic of the surface

- then, for e.g. vacuum processes we have

$$Z_{string} = e^{-2\lambda} \cdot$$  $$+$$  $$+ \; e^{2\lambda} \cdot$$  $$+ \ldots$$

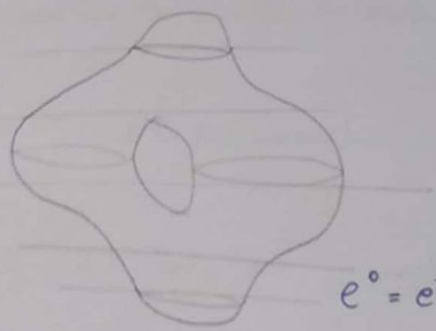
same genus expansion as that we found in large N gauge th!

- e^λ : interpreted as amplitude for emitting a closed string $\sim g_s$ string coupling



$e^{-2\lambda}$

string nucleates from & gets reabsorbed into vacuum



$e^0 = e^{-2\lambda} \cdot e^{2\lambda}$ $\Big\langle$ e^λ reabsorb, e^λ emit

string emerges from vacuum, spits out a string, reabsorbs it & disappears into vac

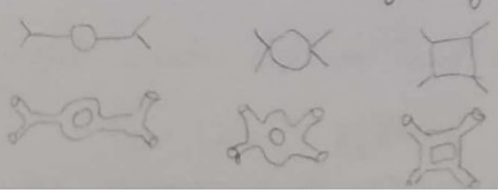
- similarly, for open strings, adding a hole $\Rightarrow \Delta\chi = -1$  $\sim e^\lambda$

$\Rightarrow$ amplitude for emitting / abs. an open string is $e^{\lambda/2}$

- thus, string pert. th $\rightarrow$ classification of euclidean 2d surfaces (genus g)

- in a certain limit, one may think of string "Feynman" diag. as just fattened QFT diag

note, however, that $\neq$ string diag @ given loop order $\ll$ $\neq$ QFT Feynman diag, e.g.



same topology, $\neq$ ε param

The bosonic string spectrum

- as stated, convenient to work w/ the Polyakov action

D scalars from w-sheet persp.

$$S_P[X^\mu, \gamma_{ab}] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\, \gamma^{ab}\, \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

- symmetries: Poincaré invar. $X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu$ global

  - reparametrization invar. (diffeos) $\sigma^a \to \tilde{\sigma}^a(\sigma^a)$ $\quad X^\mu(\sigma^a) \to \tilde{X}^\mu(\tilde{\sigma}^a) = X^\mu(\sigma^a)$

    $\gamma_{ab} \to \tilde{\gamma}_{ab} = \frac{\partial\sigma^c}{\partial\tilde{\sigma}^a}\frac{\partial\sigma^d}{\partial\tilde{\sigma}^b}\gamma_{cd}$

  - Weyl invar $\gamma_{ab}(\sigma^a) \to \Omega^2(\sigma^a)\gamma_{ab}(\sigma^a)$

    $X^\mu(\sigma) \to X^\mu(\sigma)$. gauge

    arbitrary rescalings of the metric keeping angles fixed.

- the Weyl + reparam. invar can be used to gauge-fix $\gamma_{ab} = \eta_{ab}$

  (2d metric $\leftrightarrow$ 3 indep. comp, can use the 2 diffeos to make $\gamma \propto \eta$, & Weyl to $\propto \to =$)

  * for $g=0$, Riemann thm, for $g>0$ this is only true locally, but not globally $\to$ moduli of the Riemann surf.

- the e.o.m are $\Box_\gamma X^\mu = 0$. (D free bosons)

  +

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu = 0 \qquad \text{constraints}$$

- choosing $\gamma_{ab} = \eta_{ab}$, the constraints are $\dot{X}^2 + X'^2 = 0 \qquad \dot{X}\cdot X' = 0$

- A feel for them: $\dot{X}\cdot X' = 0$ $\quad \underbrace{\text{lines of } \sigma = \text{const}}_{\text{normal } n_\sigma} \perp \underbrace{\text{lines of } \tau = \text{const}}_{\text{normal } n_\tau}$ $\quad n^{\sigma\tau} \sim \dot{X}\cdot X' = 0$

  - can use left-over gauge freedom ($\sigma^\pm \to f(\sigma^\pm)$) to fix static gauge $X^0 = R\tau = t$

    ($\Rightarrow X^{0\prime} = 0$, $\dot{X}^0 = R$). Then $X^\mu = (t, \vec{X})$ & the constraints are

    $\dot{X}\cdot X' = \dot{\vec{X}}\cdot\vec{X}'$ velocity of the string $\perp$ string $\longrightarrow$ transverse oscill. only

    $\dot{X}^2 + X'^2 = -R^2 + \dot{\vec{X}}^2 + \vec{X}'^2$ end pts of open string ($X' = 0$) move @ speed of light: $\left|\frac{d\vec{X}}{dt}\right| = 1$

- useful to intro lightcone coord on the w-sheet $\sigma^\pm = \tau \pm \sigma$ (LM / RM) $\Rightarrow$ constraints

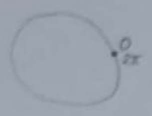
$$T_{++} = (\partial_+ X)^2 = \partial_+ X^\mu \partial_+ X_\mu = 0 \quad ; \quad T_{--} = \partial_- X^\mu \partial_- X_\mu = 0$$

- the e.o.m is simply $\partial_+ \partial_- X^\mu(\sigma^\pm) = 0$

- for the action to be minimized, we also need the fields to satisfy appropriate bnd. conditions. Several possibilities..

  - closed strings : periodic $X^\mu(\tau, 0) = X^\mu(\tau, 2\pi)$ τ

    continuous $\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, 2\pi)$.



  - open strings : Neumann bnd. cond $\partial_\sigma X^\mu(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0$

    endpoint can sit anywhere in target sp.

  - Dirichlet bnd. cond $\delta X^\mu(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0$

    endpoints are @ fixed positions in target sp

- the var. of the action is $$\delta S = \frac{1}{2\pi\alpha'}\left[\int d\tau d\sigma \left(\eta^{ab}\partial_a\partial_b X^\mu\right)\delta X_\mu - \int d\tau\, \partial_\sigma X_\mu \delta X^\mu \Big|_{\sigma=0}^{\overset{2\pi \text{ for closed}}{\pi}}\right]$$

- sols to the e.o.m. can be expanded in left/right-moving modes

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \qquad \sigma^\pm = \tau \pm \sigma$$

w/ $$X^\mu_{\substack{L\\R}}(\sigma^\pm) = \frac{1}{2}\underset{\text{CM. position}}{x^\mu} + \frac{1}{2}\alpha' \underset{\text{C.M. momentum}}{p^\mu}\sigma^\pm + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\underset{\tilde\alpha^\mu_n}{\alpha^\mu_n} e^{-in\sigma^\pm}$$ closed string

reality cond : $(\alpha^\mu_n)^* = \alpha^\mu_{-n}$

- note that $X^\mu_{L,R}$ are not individually periodic in σ, but their sum is ($\sigma^+ - \sigma^- = 2\sigma$)

- p^μ : cons. charge assoc. w/ $X^\mu \to X^\mu +$ const $\quad \int d\sigma \frac{\partial \mathcal{L}}{\partial \dot X_\mu} = \frac{1}{\pi\alpha'}\int d\sigma \dot X^\mu = p^\mu$

- in the open string, the LM & RM osc. get related via the bnd. cond, e.g. for Neumann $\alpha^\mu_n - \tilde\alpha^\mu_n = 0 = e^{-in\pi}\alpha^\mu_n - e^{in\pi}\tilde\alpha^\mu_n \quad \therefore \tilde\alpha^\mu_n = \alpha^\mu_n$

- then, we need to take into account the constraints $(\partial_+ X)^2 = (\partial_- X)^2 = 0$

$\Rightarrow$ non-linear constraints on p^μ, α_m^μ. We compute

$$\partial_+ X^\mu = \partial_+ X_L^\mu = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^+} \equiv \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in\sigma^+}$$

where we defined $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$

$$\partial_+ X^\mu \cdot \partial_+ X_\mu = \frac{\alpha'}{2} \sum_{m,p} \alpha_m^\mu \cdot (\alpha_p)_\mu e^{-i\underbrace{(m+p)}_{n}\sigma^+} = \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} e^{-in\sigma^+}$$

$$\equiv \alpha' \sum_n L_n e^{-in\sigma^+} = 0 \quad w/ \quad \boxed{L_n \equiv \frac{1}{2} \sum_m \alpha_m \cdot \alpha_{n-m}}$$

- similarly for the RM modes, we define $\tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$, & the constr. is

$$0 = \partial_- X^\mu \cdot \partial_- X_\mu = \alpha' \sum_n \tilde{L}_n e^{-in\sigma^-} \quad w/ \quad \underline{\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m}}$$

$\Rightarrow$ ∞ # of constraints : $L_n = \tilde{L}_n = 0$, $n \in \mathbb{Z}$

- note that $L_0, \tilde{L}_0$ contain $p^2 \propto \tilde{\alpha}_0^2 = \alpha_0^2$ $\rightarrow$ square of CM mom. $= -M^2$ rest mass of Lor. part.

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}$$

two different expr. for the mass $\simeq$ level matching