Introduction to holography - Lecture III

Last time : grav. collapse & singularity thms.

- the laws of black hole mechanics $\approx$ laws of thermodyn.

0^{th} : $\kappa = \text{const.}$ on $\mathcal{H}$ (stationary $\rightarrow$ Killing hor.)

1^{st} : $\frac{\kappa}{8\pi G} \delta A = \delta M - \Omega \delta J + \ldots$

2^{nd} : $\delta A \geq 0$ classically , $3^{rd} \sim \checkmark$

As I already mentioned, these laws were viewed as a mere analogy b/w the mech. of b.h. & the laws of thermodyn, as the b.h. was believed to have exactly zero temperature : it absorbs everything & emits nothing

- true classically, but not QM : as Hawking famously showed, black holes do emit blackbody radiation at a temperature

$$\boxed{T_H = \frac{\hbar \kappa}{2\pi}} \qquad \text{Hawking temperature}$$

- thus $\kappa \Leftrightarrow$ real. temp. $\Rightarrow A \Leftrightarrow$ "true" entropy $\qquad S = \frac{A}{4G\hbar}$
- since this is a quantum effect, it necessitates a QFT treatment in the b.h. background

This time : study an analogous effect in flat space : Unruh effect

- study implications of $S = \frac{A}{4G\hbar}$ for quantum gravity
- but 1^{st}, let me comment a bit more on the 2^{nd} law

The 2nd law of b.h. mechanics (comments)

- b.h. uniqueness (b.h. equilibrium states completely specified by M, J, Q)
  $\Rightarrow$ the 2nd law of thermodyn becomes impossible to verify when matter is thrown into a b.h. by an outside observer

- Bekenstein ('72) proposed to assign an entropy $S_{BH} = \eta \frac{A_H}{\ell_P^2}$ (η: O(1) const)
  to the black hole (! note no micro. interpret. as state-counting!)
  to salvage the 2nd law: it is now the generalized entropy

$$\boxed{S_{gen} = S_{out} + \eta \frac{A_H}{\ell_P^2}}$$

that should increase in any physical process: generalized 2nd law (GSL)

- his proposal for S_{BH} was motivated by Hawking's area thm:

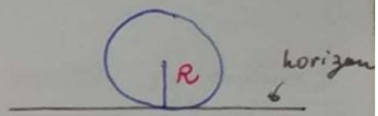
  - if matter obeys the null eng. cond ($T_{\mu\nu} k^\mu k^\nu \geq 0$, $\forall k^\mu$ lightlike & future-oriented) & cosmic censorship holds, then the area of the event horizon can never decrease*

- the fact that the area always increases can also be checked in processes where energy is extracted from a rotating black hole (Kerr) (Penrose process): the max. eng. that can be extracted as such is limited by the "irreducible mass" of the Kerr b.h. ($\propto A_H$)

- *note that the Hawking area thm is a classical thm, & can be violated QM.

- note that the validity of the GSL requires a connection b/w the increase in horizon area (det. by the mass of the object we throw in) & the entropy of the matter system, which a priori seem unrelated

- to see this, consider the following thought experiment:
- consider lowering a box of high-entropy matter into a b.h. very slowly, thus converting all of its eng. into work performed by the lab. lowering it; only then, drop it. Then, the increase in mass of the b.h. ≈ 0 (E_{box}, as seen from ∞, is ≈ 0 due to redshift) $\Rightarrow$ no increase in area; however, the entropy of the box would be lost to the external obs
- note, however, that since $\Delta S_{BH} = \frac{\Delta A_H}{\ell_P^2}$, even a tiny amount of eng. thrown in $\Rightarrow$ tiny ΔA_H can result into a finite ΔS_{BH}, which could then compensate the loss in S_{matter} $\rightarrow$ need to be more careful
- suppose the box has finite size R, which is the min. dist. from the horizon that we can lower it. This corresp. to a Schwarzschild coord dist $r(R)$ given by:



$\sqrt{r(r-2GM)} + GM\ln\left(2r-2GM+2\sqrt{r(r-2GM)}\right)$

$$R = \int_{r=2GM}^{r(R)} \frac{dr}{\sqrt{1-\frac{2GM}{r}}} \approx 2\sqrt{2GM\,\underbrace{(r(R)-2GM)}_{\text{small}}}$$

$$\Rightarrow \quad r(R) - 2GM \approx \frac{R^2}{8GM}$$

- the redshift factor @ $r(R)$ is $\sqrt{1-\frac{2GM}{r(R)}} \approx \frac{R}{4GM} \ll 1$
- thus, the eng. of the box as seen from ∞ is $E_\infty = \frac{ER}{4GM} = \Delta M_{BH}$ (rest frame eng: E)
- the increase in b.h. entropy $S_{BH} = \eta \frac{A_H}{G\hbar}$ is

$$\Delta S_{BH} = \eta \frac{8\pi \, 2GM \cdot 2\cancel{G}\Delta M_{BH}}{\cancel{G}\hbar} = \frac{8\pi\eta}{\hbar} ER \qquad \eta = \frac{1}{4}$$

- thus, if the entropy of the matter satisfies

Bekenstein bound
$$\boxed{S_{matter} \leq \frac{2\pi}{\hbar} ER}$$

then the GSL is saved

- ! note : G (Newton) does <u>not</u> appear in the bound $\Rightarrow$ pure QFT restriction

  – proper def^n in terms of quantum information theoretic quantities (Casini 2008).

- Bekenstein's proposal for b.h. entropy $S_{BH} = \frac{A_H}{4G\hbar}$ taken seriously after Hawking's discovery that b.h. carry a physical temperature T_H.

- comments : proving that $A_H/4G\hbar$ is the entropy of the b.h. had to wait for major developments in string th. First successful <u>microscopic</u> account of a b.h's entropy was in '96 (Strominger & Vafa) who reproduced it by counting states in a "dual" CFT description, in which the microstates can be easily described & counted (! they are invisible in the gravity picture

- the proper int. of Sout is as entanglement entropy of matter (quantum) fields outside the b.h. horizon. The quantity

$$S_{gen} = \frac{\langle A \rangle}{4G\hbar} + S_{out}^{\text{UV div.}}$$

  is better-defined than each term taken separately

- $S_{BH} \propto A$ holds for b.h. in Einstein gravity + matter. If higher curvature correction are present, then the formula for S_{BH} needs to be appropriately modified ($\ni$ the 1st law still holds) $\rightarrow$ Wald entropy formula $\oint_B$ local, geom. quantity)

Literature: Jacobson gr-qc/0308048
Carroll ch 9
Mukhanov & Winitzki Intro to quantum effects in gravity
Townsend 9707012

The Hawking effect

- black holes <u>emit</u> thermal radiation at a temp. $\boxed{T_H = \frac{\kappa \hbar}{2\pi}}$

  vacuum: annih by pos freq. modes $a_\omega|0\rangle = 0$
- main physics: $\neq$ observers have $\neq$ notions of time $\Rightarrow$ $\neq$ Hamiltonians
  what looks like the vacuum state to one obs $\to$ can contain part. for another

- to see the Hawking effect $\to$ QFT in a (weakly) curved spt.
  - sufficient to consider a free scalar field

- comments: in a general curved backgnd, no <u>preferred</u> notion of time
  - in a spt. w/ a <u>timelike isometry</u> $\to$ natural to consider Ham. assoc. w/ this isometry, but not compulsory (see Unruh effect we will discuss shortly).
  - the notion of "particle" does not make sense in a gen. spt; however for $\omega \gg \frac{1}{R_{curv.}}$, the spt. looks approx. flat, & we can import this notion from Minkowski sp.
  - if the spt. is asympt. flat, then notion makes sense for all frequencies. (can also call it "field excit").

- in the following, we will discuss the Unruh effect: accelerated obs. in Minkowski spt. sees a <u>thermal flux</u> of part. @ $T = \frac{a\hbar}{2\pi}$

- this is analogous to the Hawking effect, but simpler, a = surface gravity of Rindler horizon

- the general discussion $\to$ QFT in curved spt, particularized to flat space
  natural setting.

Free scalar in Minkowski spt

- consider a free real scalar field $(\Box - m^2)\varphi(\vec{x},t) = 0$

  $\Box$: Minkowski, but easily gen

- QM : $\phi(\vec{x},t) \to$ operator $\phi^\dagger = \phi$, satisfies KG. eqn & $[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\hbar\,\delta^{d-1}_{(\vec{x}-\vec{x}')}$

  equal-time comm. rel. $\pi = \dot\phi$

- expand $\phi(\vec{x},t)$ in a set of normalizable sols. to the KG eqn $f_k = \dfrac{e^{-i\omega_k t + i\vec{k}\vec{x}}}{\sqrt{2\omega_k}}$

$$\phi(\vec{x},t) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \left(a_{\vec{k}} \frac{e^{-i\omega_k t + i\vec{k}\vec{x}}}{\sqrt{2\omega_k}} + \underbrace{a^\dagger_{\vec{k}} \frac{e^{i\omega_k t - i\vec{k}\vec{x}}}{\sqrt{2\omega_k}}}_{h.c.} \right)$$

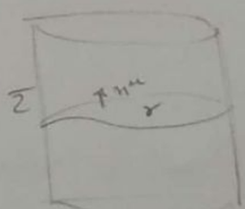
→ ∞ decoupled set of harmonic osc. w/ frequencies $\omega_k = \sqrt{|\vec{k}|^2 + m^2}$

– to see this, one can invert the rel. b/w $a_{\vec{k}}$ & ϕ, & derive the comm. rels of $a_k, a^\dagger_k$ from those of ϕ, $\pi = \dot\phi$.

- this relation can be inverted by introducing the Klein-Gordon inner product on the space of complex sols. to the wave eqn. We keep the discussion general, so as to be able to apply it to a free scalar in a general curved backgnd.

unit timelike normal

$$\langle f, g\rangle = \frac{i}{\hbar} \int_\Sigma d^{d-1}x \sqrt{\gamma}\, n_\mu \underbrace{g^{\mu\nu}(\bar{f}\partial_\nu g - \partial_\nu \bar{f} g)}_{\text{KG current } j^\mu}$$



- $\nabla_\mu j^\mu = 0$ if f, g are sols. to the wave eqn $(\Box - m^2) f = 0$.

  $\Rightarrow$ integral indep. of Σ & of time

- $\overline{\langle f, g\rangle} = \langle g, f\rangle = -\langle \bar{f}, \bar{g}\rangle$ $\qquad \langle f, \bar{f}\rangle = 0$

- not positive definite

- the Minkowski scalar field written above takes the form

$$\phi(\vec{x},t) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \left(f_{\vec{k}}(\vec{x},t)\, a_{\vec{k}} + \bar{f}_{\vec{k}}\, a^{+}_{\vec{k}} \right)$$

where $f_{\vec{k}}, \bar{f}_{\vec{k}}$ satisfy $\quad (g_{\mu\nu} = \eta_{\mu\nu})$ $\qquad f_{\vec{k}} = \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t + i\vec{k}\cdot\vec{x}}$

$$\langle f_{\vec{k}}, f_{\vec{k}'} \rangle = \frac{i}{\hbar} \int d^{d-1}x \left(\bar{f}_{\vec{k}} \underbrace{\dot{f}_{\vec{k}'}}_{-i\omega_{k'} f_{k'}} - \underbrace{\dot{\bar{f}}_{\vec{k}}}_{i\omega_k \bar{f}_{\vec{k}}} f_{\vec{k}'} \right) = \frac{1}{\hbar} \frac{(\omega_k + \omega_{k'})}{2\sqrt{\omega_k \omega_{k'}}} \overbrace{e^{-i(\omega_{k'}-\omega_k)t}} \underbrace{\int d^{d-1}x\, e^{i(\vec{k}'-\vec{k})\vec{x}}}_{(2\pi)^{d-1}\delta^{(d-1)}(k-k')}$$

$$= \frac{1}{\hbar} (2\pi)^{d-1} \delta^{(d-1)}(k-k')$$

$$\langle f_{\vec{k}}, \bar{f}_{\vec{k}'} \rangle = 0$$

- then $\quad a_{\vec{k}} = \langle f_{\vec{k}}, \phi \rangle \qquad a^{+}_{\vec{k}} = -\langle \bar{f}_{\vec{k}}, \phi \rangle \qquad$ time indep.

- using the canonical $\hat{}$ equal-time comm. rels. $\quad [\phi(\vec{x},t), \phi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$

$$[\phi(\vec{x},t), \underset{\dot{\phi}}{\pi(\vec{x}',t)}] = i\hbar\, \delta^{(d-1)}(\vec{x}-\vec{x}')$$

we find $\quad [a_{\vec{k}}, a_{\vec{k}'}] = -\langle f_k, \bar{f}_{k'} \rangle = 0 \quad , \quad [a^{+}_{k}, a^{+}_{k'}] = -\langle \bar{f}_k, f_{k'} \rangle = 0.$

$$[a_{\vec{k}}, a^{+}_{k'}] = \langle f_k, f_{k'} \rangle = (2\pi)^{d-1} \delta^{(d-1)}(k-k')$$

$\Rightarrow$ commut. relations of the a_k, a^{+}_k $\Leftrightarrow$ K.G. inner product of the mode functions. They corresp. to standard creation-annih. operators if $\langle f, f \rangle > 0$

- for $\dot{f} = \pm i\omega f \qquad \langle f, f \rangle = \mp \frac{2\omega}{\hbar} \int |f|^2 \qquad$ need to choose lower sign

$$f_{\vec{k}} = \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t + i\vec{k}\vec{x}}$$

= normalized positive frequency sol'n to the e.o.m.

- since the Hamiltonian is $\propto$ number operator $a^\dagger_{\vec{k}} a_{\vec{k}}$ (up to an ∞ ct. shift)

  & $[N, a^\dagger] = a^\dagger \qquad [N, a] = -a$

→ to build the Hilbert sp. of the th., intro the vacuum st

$$a_{\vec{k}} |0\rangle = 0 \qquad \forall \, \vec{k}$$

– Fock space of st. $\quad a^\dagger_{k_1} \dots a^\dagger_{k_n} |0\rangle \quad$ excited states w n particles

The free quantum scalar in curved spt

- the above disc. can be easily generalized to a free scalar prop. in a curved spt.

- $(\Box_g - m^2)\, \Phi = 0$

- as discussed, in curved spt $\nexists$ an a priori preferred notion of time & positive freq. modes w.r.t. it

- so, we will simply decompose the scalar field in a some basis of complex sol's to the e.o.m

$$\Phi(x) = \overset{\text{continuous}}{\sum_i} a_i f_i(x) + a_i^\dagger \bar{f}_i(x)$$

choice normaliz.

w/ the property that $\quad \langle f_i, f_j \rangle = \delta_{ij} \qquad \langle f_i, \bar{f}_j \rangle = 0 \quad \forall f_i, \bar{f}_i$

$(= -\langle \bar{f}_i, \bar{f}_j \rangle)$

≈ decomp. of the space of complex sols into a positive norm subsp. & its complex conjugate.

- given such a decomposition, we can simply define annihilation operators

$$a_i = \langle f_i, \phi \rangle \qquad a_i^+ = -\langle \bar{f}_i, \phi \rangle$$

(from which it follows ϕ can be decomp. as above)

- the norm. prop. of the f_i $\Rightarrow$ a_i, a_i^+ satisfy standard creation-annihilation comm. rels. $[a_i, a_j^+] = \langle f_i, f_j \rangle = \delta_{ij}$

- the "vacuum" state can be def. as $a_i |0_f\rangle = 0$

- "excited st" $a_i^+ \dots a_p^+ |0_f\rangle$ Fock sp. of states.

However, the basis f_i of sols to the KG eqn is by no means unique.

$\Rightarrow$ many $\neq$ notions of "vacua" possible (in general, no preferred one)

- if $\exists$ another basis of complex sols $g_i = \sum_j \underbrace{\alpha_{ij}}_{\langle g_i, f_j \rangle} f_j + \underbrace{\beta_{ij}}_{-\langle g_i, \bar{f}_j \rangle} \bar{f}_j$

then g_i has same norm. prop. if $\begin{cases} \sum_k \alpha_{ik} \alpha^*_{jk} - \beta_{ik} \beta^*_{jk} = \delta_{ij} \\ \sum_k \alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk} = 0. \end{cases}$

- the creation-annih. ops assoc. w/ the g_i basis b_i, b_i^+ can be defined as above, & we can decomp.

$$\phi = \sum_i b_i g_i + b_i^+ \bar{g}_i \qquad = \sum_i a_i f_i + a_i^+ \bar{f}_i$$

$\Rightarrow$ the two sets of creation-annih. ops. are rel. as

Bogolyubov transf.

$$a_i = \sum_j \alpha_{ji} b_j + \beta^*_{ji} b_j^+ \qquad b_i = \sum_j \alpha^*_{ij} a_j - \beta^*_{ij} a_j^+$$

? Bogolyubov coeff.

- the upshot of this discussion is that the exp. value of the # op. assoc. w/ the g_i modes in the vacuum defined w.r.t. the f_i ones is $\neq 0$

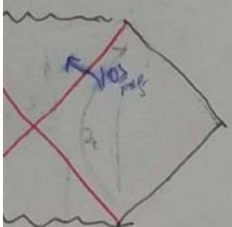
$$\langle 0_f | b_i^\dagger b_i | 0_f \rangle = \sum_{jk} \langle 0_f | (\alpha_{ij} a_j^\dagger - \beta_{ij} a_j)(\alpha_{ik}^* a_k - \beta_{ik}^* a_k^\dagger) | 0_f \rangle$$

$$= \sum_j \beta_{ij} \beta_{ij}^* \geq 0$$

$\Rightarrow$ if the $\beta_{ij} \neq 0$, then the $|0_f\rangle$ vacuum appears to carry field excit. from the p.d.v. of the g Fock space.

Back to the Hawking effect

- what lies @ the heart of the Hawking effect is that the state of the quantum fields on a b.h. backgnd should be $\ni$ an infalling observer through the event horizon feels nothing part. happening ($\sim |0\rangle_{infalling}$, in part no high. freq. modes in this frame should be present). This state looks like a thermal state to the observer using Schwarzschild time to define his +/- frequency modes (accelerated observers)



This main physical effect can be illustrated via a simpler computation in Rindler space (patch of 2d Minkowski seen by accelerated observers). To see why, expand the Schwarzschild metric near the horizon

$$ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + dr_*^2) + r^2 d\Omega_2^2 \qquad dr_* = \frac{dr}{1 - \frac{2M}{r}}$$

$$r = 2M + \rho^2 \quad \Rightarrow \quad ds^2 = \frac{\rho^2}{2M}\left(-dt^2 + \left(2\rho d\rho \cdot \frac{2M}{\rho^2}\right)^2\right)$$

Rindler space.

$$= 8M\, d\rho^2 - \frac{\rho^2 dt^2}{2M} = 8M\left(d\rho^2 - \rho^2 \frac{dt^2}{16M^2}\right)$$

- also, a free scalar in the Schw. backgnd. can be decomp. in spherical harmonics

$$\Phi(t,r,\theta,\varphi) = \sum_{\ell,m} \frac{\varphi_{\ell m}(t,r)}{r} Y_{\ell m}(\theta,\varphi)$$

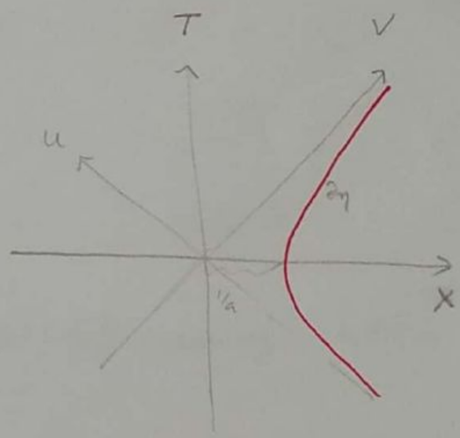
$$(\Box - m^2)\Phi = 0 \quad \Rightarrow \quad (\partial_t^2 - \partial_{r_*}^2 + \underbrace{V_{\ell m}}_{(1-\frac{2M}{r})\left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2} + m^2\right)}) \varphi_{\ell m} = 0$$

$\to 0$ near the ho[rizon]

$\approx$ massless scalar in 2d.

- so, we can consider a toy model = massless scalar in 2d Rindler sp.

The Unruh effect



- remember Rindler space is a patch of 2d Minkowski, as seen by an unif. accelerated obs (a)

- his worldline is $T(\tau) = \frac{1}{a}\sinh a\tau$

$$X(\tau) = \frac{1}{a}\cosh a\tau$$

& one can compute $a^\mu = \frac{D^2x^\mu}{d\tau^2} = \frac{d^2x^\mu}{d\tau^2} = (a\sinh a\tau, a\cosh a\tau)$

$\Rightarrow a^\mu a_\mu = a^2 = \text{const.}$

$$X^2 = T^2 + \frac{1}{a^2}$$

- can choose coord. on Minkowski sp. $|T| < X$ adapted to accelerated motion

$$T = \frac{1}{a} e^{a\xi} \sinh(a\eta) \qquad X = \frac{1}{a} e^{a\xi}\cosh(a\eta) \quad , \eta, \xi \in (-\infty, \infty)$$

(previous obs has $\eta = \tau$ & $\xi = 0$)

obs w/ ≠ acc α have $\tau_\alpha = \frac{a}{\alpha}\eta$ & $\xi_\alpha = \frac{1}{a}\ln\frac{a}{\alpha}$

- in these coord, the Minkowski metric takes the form

$$ds^2 = e^{2a\xi}\left(-d\eta^2 + d\xi^2\right) = \frac{\left(d\left(e^{a\xi}\right)\right)^2 - a^2 e^{2a\xi} d\eta^2}{a^2}$$

- the surface gravity of the assoc. Rindler horizon is $\kappa = a$

- we would now like to show that the Minkowski vacuum looks like an excited state from the p.d.v. of the accelerated observer

- as advertised, we will look at a massless 2d scalar field & compute the corresp. Bogoliubov coeff.

- intro null Minkowski coord $U, V = T \mp X$
  Rindler coord $\tilde{u}, \tilde{v} = \eta \mp \xi$

$$\left\} \quad U = -\frac{1}{a} e^{-\tilde{u}a} < 0 \qquad V = \frac{1}{a} e^{\tilde{v}a} > 0\right.$$

- the wave eqn. in these coord. are $\partial_U \partial_V \phi = 0$; $\partial_{\tilde{u}} \partial_{\tilde{v}} \phi = 0$,
  w/ sols $\underset{RM}{f(U)} + \underset{LM}{g(V)}$, $\underset{RM}{f(\tilde{u})} + \underset{LM}{g(\tilde{v})}$ right / left - moving $h \doteq \pm\omega$

- in the $X > |T|$ patch, the scalar field can be written as

$$\phi = \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{\sqrt{2\omega}} \left(e^{-i\omega U} a_\omega + e^{i\omega U} a_\omega^+\right) + LM \qquad \text{Minkowski mods.}$$

or

$$\phi = \int_0^\infty \frac{d\Omega}{2\pi} \frac{1}{\sqrt{2\Omega}} \left(e^{-i\Omega\tilde{u}} b_\Omega + e^{i\Omega\tilde{u}} b_\Omega^+\right) + LM \qquad \text{Rindler mods.}$$

w/ both sets a_ω, b_Ω satisfying the standard comm. rels.

$$[a_\omega, a^+_{\omega'}] = \delta(\omega - \omega') \qquad [b_\Omega, b^+_{\Omega'}] = \delta(\Omega - \Omega')$$

- the Minkowski vacuum is defined as

$$a_\omega |0_M\rangle = 0 \qquad + LM \qquad \text{global vac. st. in Minkowski}$$

- the Rindler vacuum corresp. to $b_\Omega |0_R\rangle = 0$. no part. detected by acc. obs. (but singular state)

- the two sets of ops. are related via a Bogolyubov transf

$$b_\Omega = \int d\omega \left(\alpha^*_{\Omega\omega} a_\omega - \beta^*_{\Omega\omega} a^\dagger_\omega \right)$$

w/ the normaliz. cond. $\int d\omega \left(\alpha_{\Omega\omega} \alpha^*_{\Omega'\omega} - \beta_{\Omega\omega} \beta^*_{\Omega'\omega} \right) = \delta(\Omega - \Omega')$

- not possible to invert b/c Rindler modes are defined only in 1/4 Mink. sp.

- plugging this into the scalar field, we obtain

$$\phi_{I} = \int_0^\infty \frac{d\Omega}{2\pi} \frac{1}{\sqrt{2\Omega}} \left(e^{-i\Omega\tilde{u}} \int d\omega \left(\alpha^*_{\Omega\omega} a_\omega - \beta^*_{\Omega\omega} a^\dagger_\omega \right) + e^{i\Omega\tilde{u}} \int d\omega \left(\alpha_{\Omega\omega} a^\dagger_\omega - \beta_{\Omega\omega} a_\omega \right) \right)$$

reg. I

$$= \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{\sqrt{2\omega}} \left(e^{-i\omega u} a_\omega + e^{i\omega u} a^\dagger_\omega \right)$$

$$\int_{-\infty}^{\infty} d\tilde{u}\, e^{i(\Omega - \Omega')\tilde{u}} = 2\pi\delta(\Omega - \Omega')$$

$$\Rightarrow \quad \frac{1}{\sqrt{\omega}} e^{-i\omega u} = \int \frac{d\Omega'}{\sqrt{\Omega'}} \left(\alpha^*_{\Omega'\omega} e^{-i\Omega'\tilde{u}} - \beta_{\Omega'\omega} e^{i\Omega'\tilde{u}} \right) \qquad \times e^{\pm i\Omega\tilde{u}}$$

& int over $\tilde{u} \in (-\infty, \infty)$

$$2\pi\, \alpha^*_{\Omega\omega} = \frac{\sqrt{\Omega}}{\sqrt{\omega}} \int_{-\infty}^{\infty} d\tilde{u}\, e^{i\Omega\tilde{u} - i\omega u} = \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{0} (-) \frac{du}{au} (-au)^{-\frac{i\Omega}{a}} e^{-i\omega u}$$

$$2\pi\, \beta_{\Omega\omega} = -\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} d\tilde{u}\, e^{-i\Omega\tilde{u} - i\omega u} = -\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{0} du\, (-au)^{\frac{i\Omega}{a} - 1} e^{-i\omega u}$$

$z = -au$

- These int. are of the form $\int_0^\infty dz\, z^{\frac{i\Omega}{a} - 1} e^{\mp i\omega z/a}$ (for α, β).

similar to Γ function $b^{-s}\Gamma(s) = \int_0^\infty dz\, z^{s-1} e^{-bz}$ w/ $s = \frac{i\Omega}{a}$, $b = \pm\frac{i\omega}{a}$

need $\mathrm{Re}\, b = \varepsilon > 0$ for convergence. $b^{-s} = e^{-s \ln b} = e^{-\frac{i\Omega}{a} \underbrace{\ln\left(\varepsilon \pm \frac{i\omega}{a}\right)}_{\ln\left|\frac{\omega}{a}\right| \pm \frac{i\pi}{2}}}$

$$b^{-s} = e^{-\frac{i\Omega}{a} \ln\left|\frac{\omega}{a}\right| \pm \frac{\pi\Omega}{2a}}$$

- thus, we find that $|\alpha_{\Omega\omega}|^2 = e^{\frac{2\pi\Omega}{a}} |\beta_{\Omega\omega}|^2$

- sanity check: $|\beta_{\Omega\omega}|^2 \to 0$ as $a \to 0$

- according to our previous considerations, the exp. value of the b-particle # op. (Rindler) in the a-vacuum state (Minkowski) is

$$\langle 0_M | N_\Omega | 0_M \rangle = \int_0^\infty d\omega \, |\beta_{\omega\Omega}|^2$$

mean # of part. of freq. Ω found by the accelerated observer

- using the norm. cond for $\Omega' = \Omega$

$$\int d\omega \left(|\alpha_{\Omega\omega}|^2 - |\beta_{\Omega\omega}|^2 \right) = \delta(\Omega - \Omega) = \text{Vol. spt} \cdot V$$

$$= \int d\omega \left(e^{\frac{2\pi\Omega}{a}} - 1 \right) |\beta_{\Omega\omega}|^2 = V$$

$$\Rightarrow \langle 0_M | N_\Omega | 0_M \rangle = \frac{V}{e^{\frac{2\pi\Omega}{a}} - 1}$$

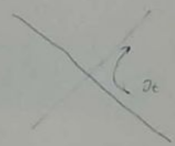
$\Rightarrow$ mean density of. part. w/ freq Ω is

$$\boxed{n_\Omega = \frac{\langle N_\Omega \rangle}{V} = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1}}$$

- thus, the acc. obs. sees a Bose-Einstein distr. w/ $\boxed{T = \frac{a\hbar}{2\pi}}$

thermal bath of particles @ $T = \frac{a\hbar}{2\pi}$.

This temperature diverges as the obs. approaches the horizon $(a \to \infty)$ & $\to 0$ if obs is @ ∞.



The Hawking effect (analogy)

Rindler	Schwarzschild
acc. obs.	obs following orbits of ∂_t (schw. time) @ fixed r.
Minkowski spt	Kruskal diagram
Minkowski vacuum	infalling obs vacuum (eternal b.h)
acc. a	surface gravity $\kappa = \frac{1}{4GM}$ $T_H = \frac{\hbar}{8\pi GM}$
Rindler vac.	"Boulware" vac (no part. det. by schw obs → singular on future (past hor).
	*"Unruh vac → b.h. from collapse. backreaction (evaporates) → BH info paradox

- greybody factors: prop. from hor. to ∞

Implications

- the fact that $\frac{\kappa\hbar}{2\pi}$ is the physical temperature of the b.h $\Rightarrow$ $S_{BH} = \frac{A_H}{4G\hbar}$

  should be its entropy, as implied by the 1^{st} law

- if so, then expect S_{BH} to have a "microscopic interpretation" as counting the number of microst. of the b.h., even if these are invisible in gravity

Obs : since $S \propto \frac{A_H}{\ell_p^2}$, it looks as if the d.o.f. of the b.h "live" on the surface, w/ ~ 1 d.o.f / Planck-size region ($S \sim A$ very few d.o.f. as compared to the standard $S \propto V$ in local th.)

- on the other hand, A_H/ℓ_p^2 is huge. Exercise $S_{BH}(\odot)$

- since $S \propto A$ one would imagine that standard matter w/ $S \propto V$ could easily be more entropic than b.h.s (in a fixed spt. region). However, this is not the case, due to gravitational collapse (generic $\leftarrow$ Penrose)

To see this, imagine a finite-temp (T) gas (described by standard QFT CFT) in a region R of spt. (size R). At high T, we have

$$E_{gas} \sim T^4 V \sim T^4 R^3 \qquad S_{gas} \sim T^3 V \sim (TR)^3$$

- we would like that $S_{gas} > S_{BH}(R) = \frac{A_H}{4G} \sim \frac{R^2}{G}$ entropy of b.h. that fills region R

- on the other hand, the eng. of the gas in R is limited by

$$GE < R \qquad \text{(lest it undergo grav. collapse)}$$

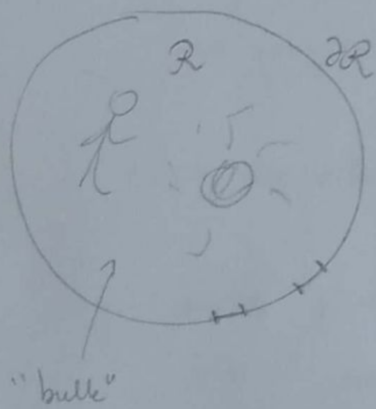
$$\Rightarrow \quad T < \left(\frac{1}{R^2 G}\right)^{1/4} \quad \Rightarrow \quad S_{gas} < \left(\frac{R^2}{G}\right)^{3/4} \sim \left(\frac{A_H}{G}\right)^{3/4} \ll \frac{A_H}{G} \quad \text{for } R \gg \ell_p.$$

⇒ the gas cannot have larger entropy than a b.h. filling the region

The far-reaching implication of this apparently simple obs

$$S_{\text{th in } R} < S_{BH}(R) = \frac{A_R}{4G\hbar}$$

is that we can encode all that happens $\subset R$ by d.o.f. that live on the surface of region R ⇒ HOLOGRAPHY ('t Hooft '93)



- the QG d.o.f live in fewer dims than the cls. limit indicates
- can encode th. w/ gravity in R by th. w/o grav. in ∂R
- bulk locality : approximate @ best, spt → emergent
- this arg. is very general, but also v. vague