The Hamiltonian formulation of GR

- we would like to show that the GR Hamiltonian is a pure bnd. term on-shell (in part., $H=0$ for a manifold w/ closed spatial sections, e.g. dS

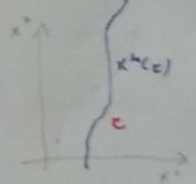
- this analysis will also show how to compute conserved charges in GR (energy, ang. momentum. etc) → not a trivial pb.

The above property of the GR Hamiltonian is related to the time reparam. invariance of GR. We will start w/ a simple toy model for such behaviour = the relativistic point particle

- The relativistic point particle

  - expect Lorentz-invar. dynamics; however, simplest descr. of the motion as $x^i(t)$ in some inertial frame, w/ action $S[x^i_{(t)}] = -m\int dt\,\sqrt{1-\dot{x}^i\dot{x}_i}$

  - to make Lorentz invar. manifest, intro some arbitrary param τ to label the position on the worldline, now given as $x^\mu(\tau)$. the action is



$$S[x^\mu(\tau)] = -m\int d\tau\,\sqrt{-\dot{x}^\mu\dot{x}_\mu} \qquad \text{w/ } \dot{} = \frac{d}{d\tau} \text{ now}$$

  - manifest Lorentz invar $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ (global symm.)
  - reparametrization invar $\tau \to \tau' = \tau'(\tau)$, $x^\mu(\tau) \to x'^\mu(\tau') = x^\mu(\tau)$ } local symm.

  - previous action obtained by gauge-fixing $x^0(\tau) = t$ (x^0 no longer dyn.) its time evol. fixed by the gauge cond.

The canonical momenta conjugate to x^μ are

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}$$

- they satisfy the constraint $\quad p_\mu p^\mu + m^2 = 0$

- note also one cannot uniquely det. the velocities $\dot{x}^\mu$ from the momenta (arbitrary rescaling $\dot{x}^\mu = \alpha p^\mu$)

- the Hamiltonian $H = p_\mu \dot{x}^\mu - \mathcal{L} = \alpha(p^2 + m^2) = 0$ on the constraint surface. this is a direct conseq. of the reparam. invar. of the action

- the dynamics of the system takes place entirely on the constraint surface $p^2 + m^2 = 0$. Moreover, configs that only differ by a gauge transf. are to be physically identified. In the quantum th., the constraint is taken to annih. physical states $(\hat{p}^2 + m^2)|\psi_{phys}\rangle = 0$

General Relativity

- the Einstein-Hilbert action is $\quad S_{EH} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^d x \, R\sqrt{g}$

- however, this action does not have a well-defined variational principle (namely $\delta S \neq 0$ on-shell for generic sols to the e.o.m)

Example : classical point particle

$$S = \int_{t=0,\, q_i}^{T,\, q_f} dt \, \frac{m\dot{q}^2}{2} \quad \text{has a well-def. var. principle}$$

$$\delta S = \int_{q_i,0}^{q_f,T} dt \, m\dot{q}\,\delta\dot{q} = \int_{q_i,0}^{q_f,T} dt \, \partial_t(m\dot{q}\,\delta q) - \int dt \, \underbrace{m\ddot{q}}_{e.o.m}\,\delta q = p\,\delta q\Big|_i^f - \int_{e.o.m}$$

$$\delta S_{on\text{-}shell} = p\,\delta q\Big|_i^f = 0 \quad \text{if } q_{f,i} \text{ fixed} \quad (\text{soln } \exists, \text{ as e.o.m } 2^{nd} \text{ order})$$

- on the other hand $S' = -\int dt \frac{m}{2} q\ddot{q}$ does not have a well-def var. princip.
(only differs from S by bnd. terms $\sim \frac{d}{dt}(q\dot{q})$)

$$\delta S' = -\frac{m}{2}\int dt\, \delta q\, \ddot{q} - \frac{m}{2}\int dt\, q\, \delta\ddot{q} = -\frac{m}{2}\int dt\, \partial_t(q\delta\dot{q}) + \frac{m}{2}\int dt\, \dot{q}\delta\dot{q} - \frac{m}{2}\int dt\, \ddot{q}\,\delta q$$

$$= \frac{m}{2}\int dt\, \partial_t(\dot{q}\delta q - q\delta\dot{q}) - m\int dt\, \underbrace{\ddot{q}}_{e.o.m}\delta q = \frac{m}{2}(p\delta q - q\delta p)\Big|_i^f - \int e.o.m.$$

- e.o.m identical; however $\delta S' \neq 0$ on-shell b/c we cannot fix both $q, \dot{q}$ @ each endpoint of the interval.

- The problem stems from the fact that, even though the e.o.m are only 2^{nd} order, the Lagrangian contains time deriv. of q higher than first. (Had the e.o.m been higher than 2^{nd} order, then there wouldn't have been a pb. w/ fixing derivatives higher than the first).

- fix: add a bnd. term $\frac{m}{2} q\dot{q}\Big|_i^f$ to S' (which does not affect the e.o.m) so that

$$\delta S' + \delta\left(\frac{m}{2} q\dot{q}\right) = m\dot{q}\,\delta q \quad \checkmark$$

well-def var. p. w/ Dirichlet bnd. cond. (q fixed)

- we could also have added $-\frac{m}{2} q\dot{q}$, case in which

$$\delta S' - \delta\left(\frac{m}{2} q\dot{q}\right) = -mq\,\delta\dot{q}$$

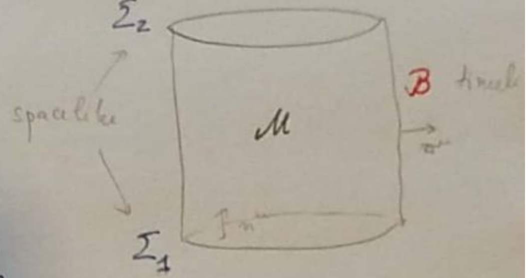
& the var. principle is well-def w/ Neumann bnd. cond ($\dot{q}$ fixed)

The Einstein-Hilbert action is precisely of the above kind, i.e. contain second derivatives of the metric. & the var. principle w/ fixing the metric on ∂M ($\mathcal{B}$) is not compatible

- this can be fixed by adding the so-called Gibbons-Hawking bnd. term, constructed from the extrinsic curvature of the boundary $\mathcal{B}$ inside M

def^n theory   initial / final states

- we will be simultaneously considering timelike and spacelike boundaries, characterized by the sign of the unit normal to the hypersurface
$n_\mu n^\mu = \varepsilon = \pm 1$



- we start by intro. the induced metric on $\partial\mathcal{M}$

$$\gamma_{\mu\nu} = g_{\mu\nu} - \varepsilon\, n_\mu n_\nu \qquad \text{w/ prop. } \gamma_{\mu\nu} n^\nu = 0$$

- the extrinsic curvature of the bnd. inside $\mathcal{M}$: how $\gamma_{\mu\nu}$ changes as we move along n^μ

$$K_{\mu\nu} \equiv \frac{1}{2} \mathcal{L}_{n^\mu} \gamma_{\mu\nu} = \frac{1}{2}\left(n^\lambda \nabla_\lambda \gamma_{\mu\nu} + \nabla_\mu n^\lambda \gamma_{\lambda\nu} + \nabla_\nu n^\lambda \gamma_{\lambda\mu} \right)$$

$$= \nabla_{(\mu} n_{\nu)} - \varepsilon\, n^\lambda n_{(\mu} \nabla_\lambda n_{\nu)} = \nabla_\mu n_\nu - \varepsilon\, n_\mu n^\lambda \nabla_\lambda n_\nu$$

symmetric
2nd term ok if n^μ || geod

where the last equality can be proven using the def^n of the normal: if the hypersurface is defined via $f(x^\mu)$, then $n_\mu \propto \partial_\mu f$.

- the trace of the extrinsic curvature is

$$K \equiv \gamma^{\mu\nu} K_{\mu\nu} = g^{\mu\nu} K_{\mu\nu} = \nabla_\mu n^\mu \qquad (\text{since } K_{\mu\nu} n^\nu = 0)$$

- it can be shown that the action

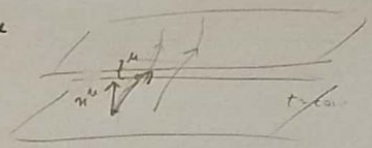
GHY bnd. term

$$S[g_{\mu\nu}] = \frac{1}{16\pi G} \int d^d x \sqrt{g}\, R + \frac{\varepsilon}{8\pi G} \int d^{d-1} x \sqrt{\gamma}\, K$$

does have a well-def var. principle for Dirichlet bnd. cond
there is one such Gibbons-Hawking-York bnd. term for each timelike ($\varepsilon = 1$) & spacelike ($\varepsilon = -1$) component of $\partial\mathcal{M}$.

- we would now like to pass to the Hamiltonian formalism. This necessarily introduces a split of the coord. x^μ into "time" & "space" → not natural from p.d.v. of general covariance.

- to start, we foliate the spacetime $\mathcal{M}$ by a family of spacelike hypersurf. Σ_t, param by a global time function t (so the normal to Σ_t $n_\mu \sim \partial_\mu t$) & some vector field t^μ that will be defining the "flow of time", norm. ∋ $t^\mu \partial_\mu t = 1$.



- the spt. metric may be parametrized as

$$ds^2 = -\underbrace{N^2}_{\text{lapse}} dt^2 + h_{ij}(dx^i + \underbrace{N^i}_{\text{shift}} dt)(dx^j + N^j dt)$$

here $N = -t^\mu n_\mu$ $\quad$ $n_\mu \propto \{1, 0 \ldots 0\}$ w/ $g^{00} n_0^2 = -1$ $\quad$ (so $g^{00} = \frac{-1}{N^2}$)

$\quad$ $t^\mu \propto \{1, 0 \ldots 0\}$ w/ $t^0 n_0 = 1$

$$N_\mu = h_{\mu\nu} t^\nu = \underbrace{(g_{\mu\nu} + n_\mu n_\nu)}_{\text{induced metric on } \Sigma} t^\nu \qquad n_\mu N^\mu = 0$$

- the functions N, N^i are arbitrary (arbitrary choice of slicing)

- decomposing the Lagrangian according to this "3+1" split, we find

$$R = \bar{R}[h_{ij}] \overset{+\epsilon}{-} (K^2 - K^{\mu\nu} K_{\mu\nu}) \overset{+\epsilon}{-} 2 \nabla_\mu (n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu)$$

so $$S_{EH} = \frac{1}{16\pi G} \left[\int d^d x \sqrt{h} N (\bar{R} - K^2 + K_{\mu\nu} K^{\mu\nu}) \overset{2\epsilon \times (+\epsilon)}{+2} \int d^{d-1} x \sqrt{h}\, n_\mu \underbrace{(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu)}_{K} \right.$$

intro $x^\mu = (t, y^i)$ $\quad$ $E^\mu_i = \left(\frac{\partial x^\mu}{\partial y^i}\right)_t$

$E^t_i = 0$, $E^j_i = \delta^j_i$

- for spacelike boundaries

$$S_{EH} + S_{GH} = \frac{1}{16\pi G} \int dt\, d^{d-1}x\, N \sqrt{h} \underbrace{\left(\bar{R}[h] - K^2 + K_{\mu\nu} K^{\mu\nu} \right)}_{\mathcal{L}_{ADM} = \bar{R} + G^{ijkl} K_{ij} K_{kl}}$$

$G^{ijkl} = \frac{1}{2}(h^{ik} h^{jl} + h^{il} h^{jk} - 2 h^{ij} h^{kl})$

w/ $$K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - \bar{\nabla}_i N_j - \bar{\nabla}_j N_i) = E^\mu_i E^\nu_j K_{\mu\nu}$$

- note this Lagr. does not contain $\dot{N}$ or $\dot{N}_i$ $\Rightarrow$ canonically conjugate momenta vanish identically $p_N = p_{N_i} = 0$

- given that N, N^i themselves corresp. to arbitrary choices of param., the canonical variables should be h_{ij} & π^{ij} (this is similar to E&M)

$$\pi^{ij} = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{h}_{ij}} = \frac{\partial}{\partial \dot{h}_{ij}} \left[\sqrt{h}\, N \left(K^{ij} K_{ij} - K^2 + \bar{R} \right) \right] = \sqrt{h} \left(K^{ij} - K h^{ij} \right)$$

(ignoring the factor of $16\pi G$).

- the Hamiltonian density is

($\perp$ difs to the surface)

$$\mathcal{H}_{ADM} = \pi^{ij} \dot{h}_{ij} - \mathcal{L}_{ADM} = N\sqrt{h} \overbrace{\left(\frac{\pi^{ij}\pi_{ij}}{h} - \frac{\pi^2}{(d-2)h} - \bar{R} \right)}^{\mathcal{H} \,=\, \text{Hamiltonian constraint}} + 2\pi^{ij} \bar{\nabla}_i N_j$$

$$= N\sqrt{h}\, \mathcal{H} - 2 N_j \underbrace{\bar{\nabla}_i \pi^{ij}}_{\mathcal{H}^j \,=\, \text{momentum constraint}} + 2 \bar{\nabla}_i \left(\pi^{ij} N_j \right)$$

(implements spatial diffeos)

- the terminology "constraints" stems from the fact that $\mathcal{H}, \mathcal{H}^j$ appear in the Hamilt. multiplied by N, N^i, which act as Lagrange multipliers

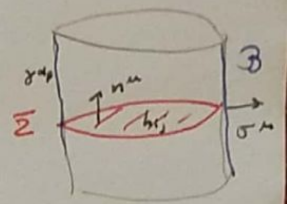
- thus (ignoring the bnd. terms for now) $\mathcal{H}_{ADM} = N\mathcal{H} + N^i \mathcal{H}_i = 0$ when the constraints are satisfied

- in an initial value formulation of GR (specify $h_{ij}, \dot{h}_{ij}$ on initial surf) they are constraints on the initial data (preserved by time evol.).

- note $\mathcal{H}$ is highly non-linear in h & π $\rightarrow$ very hard to solve

- Let us now turn to computing the energy of a given spacetime. Clearly, the answer can only be nonzero if $\partial\Sigma \neq 0$, or $\mathcal{M}$ has a non-triv. timelike bnd. B.

- for this, we need to evaluate the boundary terms in the Hamiltonian. We denote by σ^μ the unit normal to B, & assume that $n^\mu \sigma_\mu = 0$



- two sources of bnd. terms: 1) $S^{(B)}_{GHY} = \frac{1}{8\pi G}\int_B d^{d-1}x\sqrt{\gamma}\, K_B$ $\quad K_B = \nabla_\mu\sigma^\mu - \sigma^\mu\sigma^\nu\nabla_\mu\sigma_\nu$

  $\uparrow$ constr. from $\gamma_{\alpha\beta} = g_{\alpha\beta} - \sigma_\alpha\sigma_\beta$

  2) when writing the EH Lagrangian in Gauss-Codacci form, $\exists$ a bnd term $-2\nabla_\mu(n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu)$. Its contr. on $\Sigma_{i,f}$ canceled that of GHY. However, @ the timelike bnd. B it contributes as

$$\Delta S_{ADM} = \frac{1}{16\pi G}(-2)\int d^{d-1}x\sqrt{\gamma}\,\sigma_\mu\left(n^\nu\nabla_\nu n^\mu - \overset{\text{orthogonality}}{\cancel{n^\mu\nabla_\nu n^\nu}}\right)$$

$$= \frac{1}{8\pi G}\int d^{d-1}x\sqrt{\gamma}\, n^\mu n^\nu\nabla_\nu\sigma_\mu$$

- their sum is thus $\Delta S = \frac{1}{8\pi G}\int d^{d-1}x\sqrt{\gamma}\,\overbrace{\sigma^{\mu\nu}\nabla_\mu\sigma_\nu}^{k_{\partial\Sigma}}$

$\sigma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu - \sigma_\mu\sigma_\nu$ is the induced metric on $\partial\Sigma$

$\sigma_{\mu\nu}n^\mu = \sigma_{\mu\nu}\sigma^\mu = 0 \qquad \sqrt{\gamma} = N\sqrt{\sigma}$

- the contr. of this term to the ADM Hamiltonian is $\oplus$ bnd. term when obt. mom. constr.

$$H_{ADM} \to H_{ADM} - \frac{1}{8\pi G}\int_{\partial\Sigma} d^{d-2}x\sqrt{\sigma}\,N\,k_{\partial\Sigma} + \frac{1}{8\pi G}\int_{\partial\Sigma} d^{d-2}x\, N_i\pi^{ij}\sigma_j$$

- the bnd terms repr. the value of the Hamilt. evaluated on an on-shell config that satisfies the constraints

- in the asymptotically flat case, for asympt. static sols $N=1$ & $N_i=0$ then, the energy is

$$E = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^{d-2}x\,\sqrt{\sigma}\,k_{\partial\Sigma}$$

- taking $\partial\Sigma$ to be an asympt. large sphere of radius r_0, the expr. for the candidate ADM energy is $\lim_{r_0\to\infty} -\frac{1}{8\pi G}\oint_{S^2_{r_0}} d^{d-2}x\,\sqrt{\sigma}\,k_{r_0}$

- this is however divergent even for flat sp. $d\bar{s}^2 = dr^2 + r^2 d\Omega_2^2$ $\underline{3d}$

$k_{ab} = \frac{1}{2}\cdot 2_r\, r^2 \bar{\sigma}_{ab}$

$k = \frac{2}{r}$ $\quad \sqrt{\sigma} = r^2$

- energy differences should nonetheless make sense. We thus finally arrived @ the def^n of the ADM energy

$$\boxed{E_{ADM} = -\frac{1}{8\pi G}\lim_{r_0\to\infty}\oint_{S^{d-2}_{r_0}} d^{d-2}x\,\sqrt{\sigma}\left(k_{S^{d-2}} - k^{(\eta)}_{S^{d-2}}\right)}$$

- this expression was obtained via a very direct evaluation of the GR Hamiltonian

- we will, however, find a somewhat different expr. useful when dealing w/ $\neq$ asymptotics