Answer to Exercises
Non-Linear Effects in Plasmas

Notes for Cours Ecole Doctorale

Advanced Theory of Plasmas

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1 Energy Conservation for Vlasov-Poisson System

We present here the proof of the energy conservation for the Vlasov-Poisson system:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\,\frac{\partial f}{\partial v} = 0, \tag{1}$$

$$\frac{\partial E}{\partial x} = \frac{1}{\epsilon_0}\left(-e\int dv f + q_i N_{i,0}\right), \tag{2}$$

assuming the system is periodic over the length $x \in [0, \lambda]$.

The total kinetic energy Kin and potential energy Pot are given by

$$\mathrm{Kin} = \frac{m}{2}\int_0^{\lambda_0} dx \int dv\, v^2 f, \tag{3}$$

$$\mathrm{Pot} = \frac{\epsilon_0}{2}\int_0^{\lambda_0} dx\, E^2. \tag{4}$$

Let us show that the total energy $E_{\mathrm{tot}} = \mathrm{Kin} + \mathrm{Pot}$ is a constant.

One starts by deriving the temporal variation of kinetic energy:

$$\begin{aligned}
\frac{d}{dt}\mathrm{Kin} &= \frac{m}{2}\int_0^{\lambda_0} dx \int dv\, v^2 \frac{\partial f}{\partial t} \\
&= -\frac{m}{2}\int dv\, v^3 \underbrace{\int dx \frac{\partial f}{\partial x}}_{=0} + \frac{e}{2}\int dx\, E \int dv\, v^2 \frac{\partial f}{\partial v} \\
&= \int dx\, E\,(-e)\int dv\, v\, f = \int dx\, j\, E,
\end{aligned} \tag{5}$$

having made use of the Vlasov equation (1), the periodic boundaries in x, integrated by parts in v, and having defined $j = (-e)\int dv\, v\, f$ the charge current. By taking the time derivative of the Poisson Eq. (2), one obtains:

$$\begin{aligned}
\frac{\partial^2 E}{\partial x \partial t} &= \frac{-e}{\epsilon_0}\int dv \frac{\partial f}{\partial t} \\
&= \frac{e}{\epsilon_0}\int dv\, v \frac{\partial f}{\partial x} - \frac{e^2}{m\,\epsilon_0}E \underbrace{\int dv \frac{\partial f}{\partial v}}_{=0} \\
&= \frac{\partial}{\partial x}\left(\frac{e}{\epsilon_0}\int dv\, v\, f\right) = -\frac{\partial}{\partial x}\frac{j}{\epsilon_0} \\
\Longrightarrow &\qquad \epsilon_0 \frac{\partial E}{\partial t} + j = 0.
\end{aligned} \tag{6}$$

This last relation is simply Ampere's law written for zero magnetic field, and is useful for deriving the temporal variation of the potential energy:

$$\frac{d}{dt}\mathrm{Pot} = \epsilon_0 \int dx\, E\, \frac{\partial E}{\partial t} = -\int dx\, j\, E. \tag{7}$$

From Eqs. (5) and (7) one then finally obtains:

$$\frac{d}{dt}E_{\mathrm{tot}} = \frac{d}{dt}\mathrm{Kin} + \frac{d}{dt}\mathrm{Pot} = 0.$$

2 Linear Landau Damping of Langmuir Waves

This provides the solution to exercise 1.2.2.1. The goal here is to derive the dispersion relation, i.e the equation providing the relation between the wave number k and the frequency ω, of Langmuir waves.

The starting equations are given by the linearized Vlasov-Poisson system for electrons:

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} = \frac{e}{m}E\,\frac{\partial f_0}{\partial v}, \tag{8}$$

$$\frac{\partial E}{\partial x} = -\frac{e}{\epsilon_0}\int dv f_1. \tag{9}$$

Where $f_0 = f_0(v)$ is the background velocity distribution and $f_1(x, v, t)$ the fluctuating part of the distribution. As the unperturbed system is homogeneous in x and t, one may consider fluctuations of the form:

$$f_1 \;\sim\; E \;\sim\; e^{i(kx-\omega t)}.$$

In this Fourier representation, Eqs (8) and (9) become:

$$i(kv-\omega)f_1 = \frac{e}{m}E\,\frac{\partial f_0}{\partial v}, \tag{10}$$

$$ik\,E = -\frac{e}{\epsilon_0}\int dv f_1. \tag{11}$$

From Eq. (10) one obtains

$$f_1 = \frac{e}{m}\frac{E}{i(kv-\omega)}\frac{\partial f_0}{\partial v},$$

which can then be inserted into Eq. (11), providing:

$$ik\left(1 - \frac{\omega_p^2}{k^2}\int dv\frac{\frac{\partial f_0/N}{\partial v}}{v-\omega/k}\right)E = ik\,\epsilon(k,\omega)\,E = 0,$$

having defined the dielectric function $\epsilon(k, \omega)$:

$$\epsilon(k,\omega) = 1 - \frac{\omega_p^2}{k^2}\int dv\frac{\frac{\partial f_0/N}{\partial v}}{v-\omega/k},$$

where $\omega_p^2 = N\,e^2/m\,\epsilon_0$ is the squared plasma frequency.

The dispersion relation is thus given by

$$\epsilon(k,\omega) = 0. \tag{12}$$

Let us now solve the dispersion relation (12) in the **resonant approximation**. For a given wave number k, one solves for the frequency ω, assuming that

$$\omega = \omega_R + i\gamma, \qquad \text{with} \qquad |\frac{\gamma}{\omega_R}| \ll 1,$$

where ω_R and γ are respectively the real and imaginary part of the complex frequency. One then expands the complex dispersion function ϵ in Eq. (12) as follows:

$$\begin{aligned} \epsilon(\omega_R + i\gamma) &= \underbrace{\epsilon_R(\omega_R + i\gamma)}_{\text{real part}} + i\,\underbrace{\epsilon_I(\omega_R + i\gamma)}_{\text{imag. part}} \\ &= \epsilon_R(\omega_R) + i\gamma\frac{\partial\epsilon_R}{\partial\omega}(\omega_R) + i\,\epsilon_I(\omega_R) - \gamma\,\frac{\partial\epsilon_I}{\partial\omega}(\omega_R) = 0 \end{aligned} \tag{13}$$

The term $\gamma\,\partial\epsilon_I/\partial\omega$ is dropped as $|\epsilon_I/\epsilon_R| \sim |\gamma/\omega_R| \ll 1$.

From the real and imaginary parts of Eq. (13) one then obtains:

$$\epsilon_R(\omega_R) = 0, \tag{14}$$

$$\gamma = -\frac{\epsilon_I(\omega_R)}{\partial\epsilon_R(\omega_R)/\partial\omega}. \tag{15}$$

From equation (14) one computes the real frequency ω_R, which then enables to directly estimate the growth/damping rate γ using (15).

To apply the resonant approximation relations (14) and (15) requires to evaluate $\epsilon_R(\omega_R)$ and $\epsilon_I(\omega_R)$ with ω_R the real frequency. One has

$$\epsilon(\omega_R) = \epsilon_R(\omega_R) + i\,\epsilon_I(\omega_R) = 1 - \frac{\omega_p^2}{k^2}\int_\Gamma dv \frac{\frac{\partial f_0/N}{\partial v}}{v - \omega_R/k}. \tag{16}$$

The integral path Γ over v is essentially along the real axis, except at the resonance $v = \omega_R/k$. Correctly avoiding this pole is determined by imposing causality: At time $t = -\infty$, the field $E(x,t) = E\,\exp i(kx - \omega t)$ does not yet exist, one thus considers $\omega = \omega + i\nu$, with ν a small *positive* imaginary part. The pole ω_R/k is thus displaced above the real axis, indicating that in general the path Γ must avoid the pole ω/k from below. Noting that

$$\begin{aligned} \frac{1}{v - \frac{\omega_R + i\nu}{k}} &= \frac{v - \omega_R/k}{(v - \omega_R/k)^2 + (\nu/k)^2} + i\frac{\nu/k}{(v - \omega_R/k)^2 + (\nu/k)^2} \\ &\overset{\nu\to 0}{=} \frac{\mathcal{P}}{v - \omega_R/k} + i\,\pi\,\delta(v - \omega_R/k), \end{aligned} \tag{17}$$

where $\mathcal{P}$ stands for the principal part and $\delta(x)$ is the Dirac function. Inserting (17) into (16) leads to

$$\epsilon_R(\omega_R) = 1 - \frac{\omega_p^2}{k^2} \mathcal{P}\!\!\int_{-\infty}^{+\infty} dv \frac{\frac{\partial f_0/N}{\partial v}}{v - v_\phi}, \tag{18}$$

$$\epsilon_I(\omega_R) = -\pi \frac{\omega_p^2}{k^2} \left.\frac{\partial f_0/N}{\partial v}\right|_{v=v_\phi}, \tag{19}$$

where $v_\phi = \omega_R/k$ is the phase velocity of the wave.

In the case of Langmuir waves, one typically has $|v_\phi| \gg v_{\rm th}$, where $v_{\rm th} = (T/m)^{1/2}$ is the thermal velocity. In this limit, the resonant denominator of (18) can be expanded as follows

$$\frac{1}{v - v_\phi} = -\frac{1}{v_\phi}\frac{1}{1 - v/v_\phi} = -\frac{1}{v_\phi}\left[1 + \frac{v}{v_\phi} + (\frac{v}{v_\phi})^2 + (\frac{v}{v_\phi})^3 \dots\right],$$

so that after integrating by parts

$$\begin{aligned}\epsilon_R(\omega_R) &= 1 - \frac{\omega_p^2}{\omega_R k}\int_{-\infty}^{+\infty} dv\, \frac{f_0}{N}\left[\frac{k}{\omega_R} + 2(\frac{k}{\omega_R})^2 v + 3(\frac{k}{\omega_R})^3 v^2 \dots\right] \\ &= 1 - \frac{\omega_p^2}{\omega_R^2}(1 + 3\frac{k^2 v_{\rm th}^2}{\omega_R^2} + \dots) \simeq 1 - \frac{\omega_p^2}{\omega_R^2 - 3\,(k v_{\rm th})^2},\end{aligned} \tag{20}$$

having assumed that $f_0(v)$ is even (e.g. Maxwellian) and used the definitions

$$N = \int dv f_0, \tag{21}$$

$$N\, v_{\rm th}^2 = \int dv f_0\, v^2. \tag{22}$$

Making use of the approximate relation (20), one can now solve Eq. (14), which provides the well-known thermal dispersion of Langmuir waves (Bohm-Gross relation):

$$\omega^2 = \omega_p^2 + 3\, k^2 v_{\rm th}^2.$$

From (20) one also estimates

$$\frac{\partial \epsilon_R}{\partial \omega}(\omega_R) \simeq 2\,\frac{\omega_p^2}{\omega_R^3} \simeq \frac{2}{\omega_p},$$

having neglected here the higher order thermal corrections, so that $\omega_R \simeq \omega_p$. Together with (19) the damping rate (15) can thus finally be evaluated:

$$\gamma = \frac{\pi}{2}\frac{\omega_p^3}{k^2}\left.\frac{\partial f_0/N}{\partial v}\right|_{v=v_\phi} = \frac{\pi}{2}\frac{\omega_p^2}{k}\, v \left.\frac{\partial f_0/N}{\partial v}\right|_{v=v_\phi}, \tag{23}$$

which, accounting for the different sign convention on γ, indeed agrees with Eq. (1.25) of the notes.

Note also that using $\omega_R \simeq \omega_p$, the condition $|v_\phi| \gg v_{\text{th}}$ is equivalent to $|k\lambda_D| \ll 1$. This assumption had been made for deriving Eq. (20).

Considering f_0 to be Maxwellian, one obtains from (23):

$$\left|\frac{\gamma}{\omega_p}\right| = \sqrt{\frac{\pi}{8}}\frac{1}{(k\lambda_D)^3}\exp -\frac{1}{2}\frac{1}{(k\lambda_D)^2},$$

so that the condition $|k\lambda_D| \ll 1$ also enables to justify the original assumption $|\gamma/\omega_R| \ll 1$ made for the resonant approximation derivation.

3 Wave Energy in a Dispersive Media

This provides the solution to exercise 1.2.2.2. The derivation given here is similar to the one in section 61, Vol. 8, “Electrodynamics of Continuous Media”, of the Landau & Lifshitz series.

The starting point are Maxwell's equations in a continuous media:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{24}$$

$$\nabla \times \vec{H} = \mu_0 \, \vec{j}^{\text{ext}} + \frac{1}{c^2}\frac{\partial \vec{D}}{\partial t}, \tag{25}$$

$$\nabla \cdot \vec{B} = 0, \tag{26}$$

$$\nabla \cdot \vec{D} = \frac{\rho^{\text{ext}}}{\epsilon_0}, \tag{27}$$

where one has defined the electric displacement field $\vec{D} = \overleftrightarrow{\epsilon}\vec{E}$ and the magnetic field $\vec{B} = \overleftrightarrow{\mu}\vec{H}$, $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$ standing respectively for the dielectric tensor and the magnetic permeability tensor.

From Eqs. (24)-(27) one can derive a continuity equation for local energy conservation. Starting as follows:

$$\begin{aligned}
\epsilon_0 \, \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad &\overset{(25)}{=} \quad \epsilon_0 \, \vec{E} \cdot c^2 (\nabla \times \vec{H} - \mu_0 \, \vec{j}^{\text{ext}}) \\
&\overset{c^2 = 1/\epsilon_0 \mu_0}{=} \quad \frac{1}{\mu_0} \underbrace{\vec{E} \cdot (\nabla \times \vec{H})}_{-\nabla\cdot(\vec{E}\times\vec{H}) + \vec{H}\cdot(\nabla\times\vec{E})} - j^{\text{ext}} \cdot \vec{E} \\
&\overset{(24)}{=} \quad -\frac{\vec{H}}{\mu_0} \cdot \frac{\partial \vec{B}}{\partial t} - \nabla \cdot \left(\frac{\vec{E} \times \vec{H}}{\mu_0} \right) - j^{\text{ext}} \cdot \vec{E},
\end{aligned}$$

one finally obtains:

$$\epsilon_0 \, \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \frac{1}{\mu_0} \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \quad \nabla \cdot \left(\frac{\vec{E} \times \vec{H}}{\mu_0} \right) = -j^{\text{ext}} \cdot \vec{E}. \tag{28}$$

In a **media without dispersion**, when $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$ are real constants, one has $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$, so that the rate of change of energy per unit volume becomes

$$\epsilon_0 \, \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \frac{1}{\mu_0} \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t} \left[\frac{1}{2} \epsilon_0 \epsilon \, E^2 + \frac{B^2}{2 \, \mu_0 \mu} \right] = \frac{\partial U}{\partial t},$$

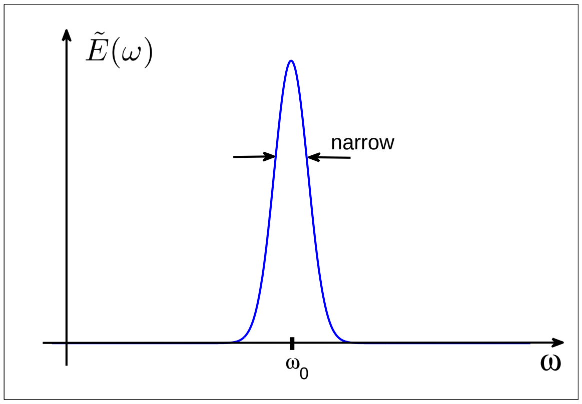

Figure 1: Electric field spectrum

where $U = \epsilon_0 \epsilon\, E^2/2 + B^2/2\, \mu_0 \mu$ can be identified as the wave energy density.

In a **dispersive media**, one has in general

$$\begin{aligned} \tilde{\vec{D}}(\vec{k},\omega) &= \overleftrightarrow{\epsilon}(\vec{k},\omega)\tilde{\vec{E}}(\vec{k},\omega) \\ \tilde{\vec{B}}(\vec{k},\omega) &= \overleftrightarrow{\mu}(\vec{k},\omega)\tilde{\vec{H}}(\vec{k},\omega) \end{aligned}$$

these relations having been written in Fourier space.

The goal here is again to be able to identify an appropriate relation for the wave energy U.

Let us consider a wave with wave vector $\vec{k}_0$ and dominant frequency ω_0. To study the energy build-up of the wave as it grows to finite amplitude requires taking account of a certain frequency spread around ω_0. For example, the spectrum $\tilde{E}(\omega)$ of the electric field takes on a form as shown in Fig. 1, and one obtains [to lighten notation, $\exp(i\vec{k}_0 \cdot \vec{x})$ is understood to be factored into $\tilde{E}(\omega)$]:

$$\begin{aligned} \vec{E}(\vec{x},t) &= \frac{1}{2}\left[\int \tilde{\vec{E}}(\omega)\, e^{i\omega t}\, d\omega + \underbrace{\text{c.c}}_{\text{complex conj.}}\right] \\ &= \frac{1}{2}\left[e^{i\omega_0 t}\int \tilde{\vec{E}}(\omega_0+\Delta\omega)\, e^{i\Delta\omega t}\, d\Delta\omega \,+\, \text{c.c}\right] \\ &= \frac{1}{2}\left[\vec{\mathcal{E}}(t)\, e^{i\omega_0 t} + \text{c.c}\right], \end{aligned} \tag{29}$$

having defined the time varying complex amplitude $\vec{\mathcal{E}}(t)$:

$$\vec{\mathcal{E}}(t) = \int \vec{\tilde{E}}(\omega_0 + \Delta\omega)\, e^{i\Delta\omega t}\, d\Delta\omega. \tag{30}$$

The fact that the frequency spread in Fig. 1 is narrow compared to ω_0 implies the time scale separation $|(1/\mathcal{E})d\mathcal{E}/dt| \ll |\omega_0|$.

In real space, the electric displacement field in turn becomes:

$$\vec{D}(\vec{x}, t) = \frac{1}{2}\left[\int \overleftrightarrow{\epsilon}(\omega)\vec{\tilde{E}}(\omega)\, e^{i\omega t}\, d\omega \;+\; \text{c.c}\right],$$

so that its time derivative can be written as

$$\begin{aligned}
\frac{\partial \vec{D}}{\partial t}(\vec{x}, t) &= \frac{1}{2}\left[\int i\omega\overleftrightarrow{\epsilon}(\omega)\vec{\tilde{E}}(\omega)\, e^{i\omega t}\, d\omega \;+\; \text{c.c}\right] \\
&= \frac{1}{2}\left[e^{i\omega_0 t}\int i(\omega_0 + \Delta\omega)\overleftrightarrow{\epsilon}(\omega_0 + \Delta\omega)\vec{\tilde{E}}(\omega_0 + \Delta\omega)\, e^{i\Delta\omega t}\, d\Delta\omega \;+\; \text{c.c}\right].
\end{aligned}$$

In this last relation one expands

$$(\omega_0 + \Delta\omega)\overleftrightarrow{\epsilon}(\omega_0 + \Delta\omega) = \omega_0\overleftrightarrow{\epsilon}(\omega_0) + \Delta\omega \underbrace{\left[\overleftrightarrow{\epsilon}(\omega_0) + \omega_0\frac{\partial\overleftrightarrow{\epsilon}(\omega_0)}{\partial\omega}\right]}_{\frac{\partial}{\partial\omega}(\omega\overleftrightarrow{\epsilon})|_{\omega_0}} + \mathcal{O}(\Delta\omega^2),$$

so that

$$\begin{aligned}
\frac{\partial \vec{D}}{\partial t}(\vec{x}, t) &= \frac{1}{2}\left[i\omega_0\overleftrightarrow{\epsilon}(\omega_0)\, e^{i\omega_0 t}\int \vec{\tilde{E}}(\omega_0 + \Delta\omega)\, e^{i\Delta\omega t}\, d\Delta\omega \;+\; \text{c.c}\right] \\
&\quad + \frac{1}{2}\left[\frac{\partial}{\partial\omega}(\omega\overleftrightarrow{\epsilon})\Big|_{\omega_0}\, e^{i\omega_0 t}\int i\Delta\omega\vec{\tilde{E}}(\omega_0 + \Delta\omega)\, e^{i\Delta\omega t}\, d\Delta\omega \;+\; \text{c.c}\right] \\
&= \frac{1}{2}\left[i\omega_0\overleftrightarrow{\epsilon}(\omega_0)\, \vec{\mathcal{E}}(t)\, e^{i\omega_0 t} \;+\; \text{c.c}\right] \\
&\quad + \frac{1}{2}\left[\frac{\partial}{\partial\omega}(\omega\overleftrightarrow{\epsilon})\Big|_{\omega_0}\, \frac{\partial\vec{\mathcal{E}}(t)}{\partial t}\, e^{i\omega_0 t} \;+\; \text{c.c}\right],
\end{aligned} \tag{31}$$

having made use of Eq. (30).

One now estimates the time-average of the first term $\epsilon_0\,\vec{E}\cdot\partial\vec{D}/\partial t$ on the left hand side of Eq. (28) over the fast time scale ω_0. For this one makes use of the following: Let A and B be two fields with rapid time variation of frequency ω_0:

$$A = \frac{1}{2}\left(\mathcal{A}e^{i\omega_0 t} + \text{c.c}\right),$$
$$B = \frac{1}{2}\left(\mathcal{B}e^{i\omega_0 t} + \text{c.c}\right),$$

where $\mathcal{A}$ and $\mathcal{B}$ are the complex amplitudes of the fields A and B respectively.

The time average of the product of these two fields over the fast time variation is then given by

$$\begin{aligned}\langle A\,B\rangle_t &= \frac{1}{4}(\mathcal{A}\,\mathcal{B}\underbrace{\langle e^{2i\omega_0 t}\rangle_t}_{=0} + \mathcal{A}\,\mathcal{B}^\star + \mathcal{A}^\star\,\mathcal{B} + \mathcal{A}^\star\,\mathcal{B}^\star\underbrace{\langle e^{-2i\omega_0 t}\rangle_t}_{=0}) \\ &= \frac{1}{4}\left(\mathcal{A}\,\mathcal{B}^\star + \mathcal{A}^\star\,\mathcal{B}\right) = \frac{1}{2}\mathrm{Re}(\mathcal{A}\,\mathcal{B}^\star).\end{aligned}$$

Applying this relation to $\epsilon_0\,\vec{E}\cdot\partial\vec{D}/\partial t$ using Eqs. (29) and (31) then provides

$$\begin{aligned}\langle\epsilon_0\,\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\rangle_t &= \frac{1}{4}\Big[-i\omega_0\,\epsilon_0\,\vec{\mathcal{E}}\cdot\overset{\leftrightarrow}{\epsilon}^\star(\omega_0)\vec{\mathcal{E}}^\star + i\omega_0\,\epsilon_0\,\vec{\mathcal{E}}^\star\cdot\overset{\leftrightarrow}{\epsilon}(\omega_0)\vec{\mathcal{E}} \\ &\quad + \epsilon_0\,\vec{\mathcal{E}}\cdot\frac{\partial}{\partial\omega}(\omega\overset{\leftrightarrow}{\epsilon}^\star)\Big|_{\omega_0}\frac{\partial\vec{\mathcal{E}}^\star(t)}{\partial t} + \epsilon_0\,\vec{\mathcal{E}}^\star\cdot\frac{\partial}{\partial\omega}(\omega\overset{\leftrightarrow}{\epsilon})\Big|_{\omega_0}\frac{\partial\vec{\mathcal{E}}(t)}{\partial t}\Big] \quad (32)\end{aligned}$$

Assuming that dissipation is small, the dielectric tensor $\overset{\leftrightarrow}{\epsilon}$ is essentially a Hermitian matrix. As a result, the first two terms in Eq. (32) cancel, and the last two combine to give:

$$\langle\epsilon_0\,\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\rangle_t = \frac{\partial}{\partial t}\left[\frac{\epsilon_0}{4}\vec{\mathcal{E}}^\star\cdot\frac{\partial}{\partial\omega}(\omega\overset{\leftrightarrow}{\epsilon})\Big|_{\omega_0}\vec{\mathcal{E}}(t)\right].$$

A similar relation can be derived for $\frac{1}{\mu_0}\vec{H}\cdot\partial\vec{B}/\partial t$, so that the time average of the rate of change of energy on the left hand side of Eq. (28) can finally be written

$$\langle\epsilon_0\,\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \frac{1}{\mu_0}\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\rangle_t = \frac{\partial}{\partial t}\left[\frac{\epsilon_0}{4}\vec{\mathcal{E}}^\star\cdot\frac{\partial}{\partial\omega}(\omega\overset{\leftrightarrow}{\epsilon})\Big|_{\omega_0}\vec{\mathcal{E}} + \frac{1}{4\mu_0}\vec{\mathcal{H}}^\star\cdot\frac{\partial}{\partial\omega}(\omega\overset{\leftrightarrow}{\mu})\Big|_{\omega_0}\vec{\mathcal{H}}\right],$$

where $\vec{\mathcal{H}}$ is the complex amplitude of field $\vec{H}$:

$$\vec{H} = \frac{1}{2}(\vec{\mathcal{H}}\,e^{i\omega_0 t} + \text{c.c}).$$

In the case of low dissipation, the time-averaged wave energy density can thus be identified as

$$\bar{U} = \frac{\epsilon_0}{4}\vec{\mathcal{E}}^\star\cdot\frac{\partial}{\partial\omega}(\omega\overset{\leftrightarrow}{\epsilon})\Big|_{\omega_0}\vec{\mathcal{E}} + \frac{1}{4\mu_0}\vec{\mathcal{H}}^\star\cdot\frac{\partial}{\partial\omega}(\omega\overset{\leftrightarrow}{\mu})\Big|_{\omega_0}\vec{\mathcal{H}}, \quad (33)$$

where $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ are the complex amplitudes of fields $\vec{E}$ and $\vec{H}$ respectively.

In the case of an electrostatic wave in a media with dielectric function ϵ, one in particular has

$$\bar{U} = \frac{\epsilon_0}{4}\frac{\partial}{\partial\omega}(\omega\epsilon_R)\bigg|_{\omega_0}|\vec{\mathcal{E}}|^2 = \frac{\partial}{\partial\omega}(\omega\epsilon_R)\bigg|_{\omega_0}\text{Pot}, \tag{34}$$

where $\text{Pot} = \epsilon_0|\vec{\mathcal{E}}|^2/4$ is the potential energy of the wave. In a non-magnetized plasma, the dielectric function is given by

$$\epsilon(k,\omega) = 1 + \sum_{\text{species}}\chi = 1 - \sum_{\text{species}}\frac{\omega_p^2}{k^2}\int dv\frac{\frac{\partial f_0/N}{\partial v}}{v-\omega/k}, \tag{35}$$

where χ is the contribution to the electric susceptibility from each species.

3.1 Case of Electron Plasma Waves (EPWs) = Langmuir waves

In this case, one has from Eq. (20):

$$\epsilon_R(\omega_R) = 1 - \frac{\omega_p^2}{\omega_R^2 - 3(kv_{\text{th,e}})^2}.$$

At the frequency ω_0 of the EPW, verifying the dispersion relation $\epsilon_R(\omega_0) = 0$, one thus has:

$$\frac{\partial}{\partial\omega}(\omega\epsilon_R)\bigg|_{\omega_0} = \underbrace{\epsilon_R(\omega_0)}_{=0} + \omega_0\frac{\partial\epsilon_R}{\partial\omega}(\omega_0) = 2\,\frac{\omega_p^2\,\omega_0^2}{[\omega_0^2 - 3(k_0v_{\text{th,e}})^2]^2} = 2\,\frac{\omega_0^2}{\omega_p^2},$$

so that from Eq. (34), the wave energy density for EPWs reads

$$\bar{U} = 2\,\frac{\omega_0^2}{\omega_p^2}\,\text{Pot}.$$

As $\bar{U} = \text{Kin}+\text{Pot}$, where Kin is the kinetic energy of oscillation in the presence of the wave, one has

$$\text{Kin} = [2\frac{\omega_0^2}{\omega_p^2} - 1]\text{Pot} \simeq \text{Pot},$$

having invoked $\omega_0 \simeq \omega_p$.

3.2 Case of Ion Acoustic Waves (IAWs)

Here one needs to consider the dielectric function (35) with both electron and ion contributions in the appropriate limit.

With respect to electrons, the phase velocity of IAWs is such that $|\omega/k| \ll v_{\text{th,e}}$, which implies an essentially adiabatic response from this species, and their contribution to the electric susceptibility derives from

$$\chi^{\text{el.}} = -\frac{\omega_{pe}^2}{k^2}\int dv \frac{\frac{\partial f_{0e}/N_e}{\partial v}}{v - \omega/k} \overset{|v|\gg|\omega/k|}{\simeq} -\frac{\omega_{pe}^2}{k^2}\int dv \frac{\frac{\partial f_{0e}/N_e}{\partial v}}{v} = \frac{1}{(k\lambda_{De})^2},$$

having used $\partial f_{0e}/\partial v = -(v/v_{\text{th,e}}^2) f_{0e}$, assuming f_{0e} Maxwellian.

With respect to ions, $|\omega/k| \gg v_{\text{th,i}}$, i.e. similar to the relation between the phase velocity of Langmuir waves and $v_{\text{th,e}}$, so that the contribution of ions to the dielectric function in the case of IAWs is the same as the one of electrons to (20).

The dielectric function for IAWs thus finally reduces to

$$\epsilon_R(\omega_R) \simeq 1 + \frac{1}{(k\lambda_{De})^2} - \frac{\omega_{pi}^2}{\omega_R^2 - 3\,(kv_{\text{th,i}})^2} \simeq \frac{1}{(k\lambda_{De})^2} - \frac{\omega_{pi}^2}{\omega_R^2 - 3\,(kv_{\text{th,i}})^2},$$

having furthermore invoked the assumption $k\lambda_{De} \ll 1$.

The dispersion relation for IAWs thus reads:

$$\epsilon_R(\omega_0) = \frac{1}{(k\lambda_{De})^2} - \frac{\omega_{pi}^2}{\omega_0^2 - 3\,(kv_{\text{th,i}})^2} = 0,$$

which leads to

$$\omega_0^2 = (k\lambda_{De})^2\omega_{pi}^2 + 3(kv_{\text{th,i}})^2 = c_s^2\left(1 + 3\frac{T_i}{Z\,T_e}\right)k^2,$$

where $c_s^2 = ZT_e/m_i$ is the squared sound speed, Z the ionization degree of ions, and having used the condition of global neutrality $N_e = ZN_i$.

Neglecting the ion thermal corrections, so that $\omega_0 \simeq kc_s$, one obtains:

$$\frac{\partial}{\partial\omega}(\omega\epsilon_R)\bigg|_{\omega_0} = \underbrace{\epsilon_R(\omega_0)}_{=0} + \omega_0\frac{\partial\epsilon_R}{\partial\omega}(\omega_0) \simeq 2\frac{\omega_{pi}^2}{\omega_0^2} \simeq \frac{2}{(k\lambda_{De})^2}$$

The wave energy density for IAWs thus finally reads

$$\bar{U} = \text{Kin} + \text{Pot} \simeq \frac{2}{(k\lambda_{De})^2}\text{Pot}.$$

For $|k\lambda_{De}| \ll 1$, one therefore has Kin $\gg$ Pot in this case.

3.3 Transverse Electromagnetic Waves in a Non-Magnetized Plasma

In this case, the wave has both an electric and magnetic component. The dispersion is however entirely characterized by the dielectric tensor $\overleftrightarrow{\epsilon}$, so that $\overleftrightarrow{\mu} = \mathbf{1}$, and the wave energy becomes:

$$\bar{U} = \frac{\epsilon_0}{4}\frac{\partial}{\partial\omega}(\omega\epsilon_R)\bigg|_{\omega_0}|\vec{\mathcal{E}}|^2 + \frac{1}{4\mu_0}|\vec{\mathcal{B}}|^2, \tag{36}$$

where $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are respectively the amplitudes of the electric and magnetic components of the wave.

Let us derive the dielectric function and dispersion relation for transverse EM waves in a non-magnetized plasma. One starts from Faraday and Ampere's law:

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad \nabla\times\vec{B} = \mu_0\,\vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t},$$

so that

$$\nabla\times(\nabla\times\vec{E}) = \underbrace{\nabla(\nabla\cdot\vec{E})}_{=0,\text{trans. waves}} - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\nabla\times\vec{B}) = -\mu_0\frac{\partial\vec{j}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}. \tag{37}$$

As the transverse waves are high frequency, $\omega > \omega_{pe}$, ions can be considered essentially immobile, so that the current is dominantly carried by electrons:

$$\vec{j} = \sum_{\text{species}} q\,N\,\vec{v} \simeq -e\,N_e\,\vec{v}_e.$$

The equation of motion of electrons reads

$$m_e\frac{\partial\vec{v}_e}{\partial t} = -e\,\vec{E},$$

so that one obtains

$$\frac{\partial\vec{j}}{\partial t} = -e\,N_e\frac{\partial\vec{v}_e}{\partial t} = \epsilon_0\,\omega_{pe}^2\,\vec{E}.$$

Inserting this last relation into Eq. (37) gives the wave equation for $\vec{E}$:

$$\nabla^2\vec{E} = \frac{1}{c^2}\left(\omega_{pe}^2\,\vec{E} + \frac{\partial^2\vec{E}}{\partial t^2}\right) = -\frac{\omega^2}{c^2}\underbrace{\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)}_{\epsilon_R(\omega)}\vec{E}.$$

For a wave with wave vector $\vec{k}$ and frequency ω_0, the dispersion relation thus becomes:

$$\epsilon_R(\omega_0) = \left(\frac{kc}{\omega}\right)^2, \quad \text{with} \quad \epsilon_R(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2},$$

which provides:

$$\omega_0^2 = \omega_{pe}^2 + (kc)^2. \tag{38}$$

To compute the wave energy, one again needs to estimate:

$$\frac{\partial}{\partial\omega}(\omega\epsilon_R)\bigg|_{\omega_0} = \underbrace{\epsilon_R(\omega_0)}_{=(kc/\omega_0)^2} + \omega_0\frac{\partial\epsilon_R}{\partial\omega}(\omega_0) = \left(\frac{kc}{\omega_0}\right)^2 + 2\frac{\omega_{pe}^2}{\omega_0^2} = \frac{(kc)^2 + 2\omega_{pe}^2}{\omega_0^2}. \tag{39}$$

Using the relation $\mathcal{B} = (k/\omega_0)\mathcal{E}$ from Faraday's law, one then finally obtains by inserting (39) into (36):

$$\bar{U} = \frac{\epsilon_0}{4}\left[\frac{2(kc)^2 + 2\omega_{pe}^2}{\omega_0^2}\right]\mathcal{E}^2 = \frac{1}{2}\epsilon_0\mathcal{E}^2,$$

having made use of the dispersion relation (38). In this case, the wave energy U is thus the combination of three contributions: (1) The energy Pot in the electric field $\vec{E}$, (2) the energy U_{mag} in the magnetic field $\vec{B}$, and (3) the kinetic energy of oscillation Kin. These contributions are given by

$$\mathrm{Pot} = \frac{\epsilon_0}{4}\mathcal{E}^2,$$

$$U_{\mathrm{mag}} = \frac{\epsilon_0}{4}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)\mathcal{E}^2,$$

$$\mathrm{Kin} = \frac{\epsilon_0}{4}\left(\frac{\omega_{pe}}{\omega}\right)^2\mathcal{E}^2.$$

4 Trapping in Langmuir Waves

This provides the solution to exercise 1.2.3.1.

The wave-frame energy of particles in a sinusoidal wave is given by

$$W = \frac{1}{2} m\, v^2 - e\phi_0 \cos(k_0 x). \tag{40}$$

For **deeply trapped** electrons, located at the potential minimas $x_{\rm min} = n\, 2\pi/k_0$, one can make a Taylor expansion:

$$W = \frac{1}{2} m\, v^2 + \frac{1}{2} e\phi_0\, k_0^2 (x - x_{\rm min})^2 - e\phi_0.$$

This corresponds to the energy of a harmonic oscillator with frequency ω_b, such that

$$\omega_b^2 = \frac{e\,\phi_0\, k_0^2}{m}. \tag{41}$$

The frequency ω_b is identified as the **bounce frequency** of deeply trapped electrons at the bottom of each potential well of the Langmuir wave.

The potential field $\phi(x) = \phi_0 \cos(k_0 x)$ verifies Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\epsilon_0} (-e\, \delta N),$$

where δN is the electron density perturbation relative to the Langmuir wave. One thus obtains:

$$k_0^2 \phi_0 = \frac{e\, \delta N_0}{\epsilon_0},$$

δN_0 being the amplitude of δN, so that Eq. (41) becomes:

$$\omega_b^2 = \frac{\delta N_0\, e^2}{m\, \epsilon_0},$$

and finally

$$\frac{\omega_b}{\omega_p} = \sqrt{\frac{\delta N_0}{N}}. \tag{42}$$

By definition, the **trapping width** $\Delta v_{\rm trap}$ is the maximum velocity of **marginally trapped** electrons. The energy level in Eq. (40) of marginally trapped particles is $W = e\phi_0$. The velocity is maximum when the potential energy is minimum, i.e. $\phi = -e\phi_0$:

$$W = \frac{1}{2} m\, \Delta v_{\rm trap}^2 - e\phi_0 = e\phi_0 \qquad \Longrightarrow \qquad \Delta v_{\rm trap} = 2 \left(\frac{e\phi_0}{m} \right)^{1/2}.$$

Finally, using Eqs. (41) and (42) as well as the fact that the lab frequency ω_0 of the Langmuir wave is such that $\omega_0 \simeq \omega_p$, one obtains:

$$\Delta v_{\mathrm{trap}} = 2\,\frac{\omega_b}{k_0} = 2\,\frac{\omega_b}{\omega_0}\,\underbrace{\frac{\omega_0}{k_0}}_{v_\phi} \overset{\omega_0 \simeq \omega_p}{=} 2\,\sqrt{\frac{\delta N_0}{N}}v_\phi.$$

5 Particle in Magnetic Well of a Tokamak

This provides the solution to exercise 1.2.2.3.

A particle in a spatially varying magnetic field has at least two constants of motion, the kinetic energy K and the magnetic moment μ (adiabatic invariant):

$$K = \frac{m}{2}v^2 = \frac{m}{2}(v_\parallel^2 + v_\perp^2),$$
$$\mu = \frac{m\,v_\perp^2}{2\,B},$$

where B is the amplitude of the magnetic field, and $(v_\parallel, v_\perp)$ are respectively the velocities parallel and perpendicular to the magnetic field (see Fig. 2). One thus can write

$$K = \frac{m}{2}v_\parallel^2 + \mu\,B = \frac{m}{2}(v_\varphi^2 + v_\theta^2) + \mu\,B,$$

with (v_φ, v_θ) the toroidal and poloidal components respectively of the parallel velocity $\vec{v}_\parallel = v_\parallel \vec{B}/B$.

In a large aspect ratio tokamak with circular cross section (see Fig. 3 for notations), one has

$$\frac{v_\varphi}{v_\theta} = \frac{R\dot{\varphi}}{\rho\,\dot{\theta}} = \frac{R\,q_s}{\rho},$$

where $q_s = \Delta\varphi/\Delta\theta$ is the so-called safety factor, which determines the pitch of the magnetic field as it winds around the magnetic surface. One clearly has $v_\varphi \gg v_\theta$ as the inverse aspect ratio $\epsilon = \rho/R$ is such that $\epsilon \ll 1$ and the safety factor is typically such that $q_s \stackrel{\sim}{>} 1$.

In a large aspect ratio tokamak, one also has

$$B \simeq B_\varphi \simeq B_0 \frac{R}{r} = \frac{B_0\,R}{R + \rho\cos\theta} = \frac{B_0}{1 + \epsilon\cos\theta} \simeq B_0\,(1 - \epsilon\cos\theta). \tag{43}$$

The kinetic energy can thus be approximated by:

$$K \simeq \frac{m}{2}v_\varphi^2 + \mu\,B = \frac{m}{2}(R\,q_s)^2\dot{\theta}^2 \; + \; \mu\,B_0\,(1 - \epsilon\cos\theta),$$

By rescaling the kinetic energy to K' one obtains:

$$K' = \frac{K - \mu\,B_0}{(R\,q_s)^2} = \frac{m}{2}\dot{\theta}^2 \; - \; \frac{\epsilon\,\mu B_0}{(R\,q_s)^2}\cos\theta = \text{const.}, \tag{44}$$

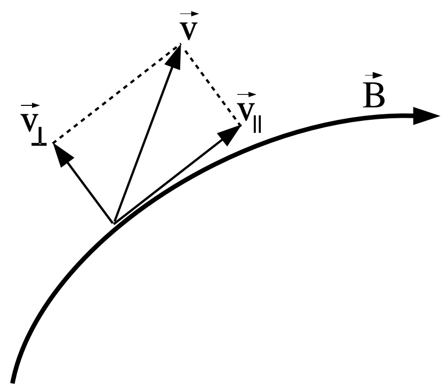

Figure 2: Parallel and perpendicular components of velocity in a magnetic field $\vec{B}$

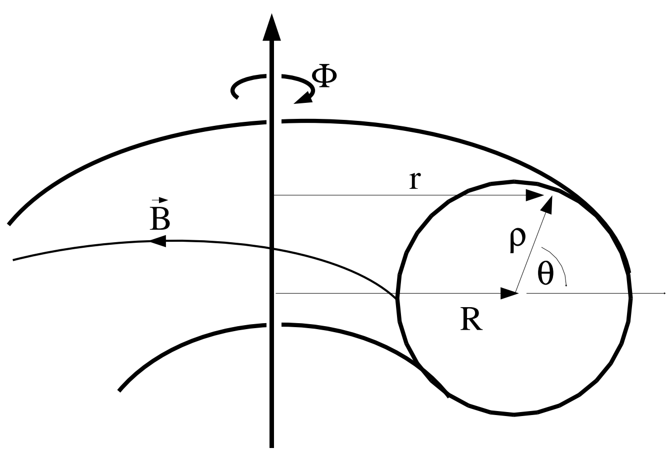

Figure 3: Large aspect ratio tokamak with circular cross section

which can now be compared to the energy of an electron in a sinusoidal wave, as given by Eq. (40):

$$\begin{aligned} & W = \frac{m}{2}\,\dot{x}^2 - e\phi_0 \cos(k_0 x) = \text{const.}, \\ \Longrightarrow \quad & W' = k_0^2\, W = \frac{m}{2}\,(k_0\,\dot{x})^2 - e\,k_0^2\,\phi_0 \cos(k_0 x) = \text{const.} \end{aligned} \tag{45}$$

Comparing Eqs. (44) and (45), one can now identify:

Particle in electrostatic wave field		Particle in a magnetic well
$k_0 x$	$\longleftrightarrow$	θ
$k_0^2\, W$	$\longleftrightarrow$	$(K - \mu\, B_0)/(R\, q_s)^2$
$e\, k_0^2\, \phi_0$	$\longleftrightarrow$	$\epsilon\, \mu B_0/(R\, q_s)^2$

Using this identification table, one can make use of the relations in appendix A of the notes to obtain:

	Particle in electrostatic wave field	Particle in a magnetic well
Frequency of deeply trapped particles	$\omega_b = (e\, k_0^2\, \phi_0/m)^{1/2}$	$\omega_b = [\epsilon\, \mu B_0/(R\, q_s)^2\, m]^{1/2}$ $= \sqrt{\epsilon}\, v_{\rm th}/R\, q_s$
Energy-like variable κ	$\kappa^2 = 2\, e\phi_0/(W + e\,\phi_0)$	$\kappa^2 = 2\, \epsilon\mu B_0/(K - \mu B_{\rm min})$
Energy bounds for passing particles	$0 < \kappa < 1 \Longrightarrow W > e\phi_0$	$0 < \kappa < 1 \Longrightarrow K > \mu\, B_{\rm max}$
Transit time for passing particles	$\tau_t = (2\kappa/\omega_b) F(\kappa^2)$	$\tau_t = (2\kappa/\omega_b) F(\kappa^2)$
Energy bounds for trapped particles	$1 < \kappa < +\infty \Longrightarrow -e\phi_0 < W < e\phi_0$	$1 < \kappa < +\infty \Longrightarrow$ $\mu B_{\rm min} < K < \mu B_{\rm max}$
Bounce period for trapped particles	$\tau_b = (4/\omega_b) F(1/\kappa^2)$	$\tau_b = (4/\omega_b) F(1/\kappa^2)$

In the above table, $B_{\rm min} = B_0(1-\epsilon)$ and $B_{\rm max} = B_0(1+\epsilon)$ are respectively the minimum and maximum values of B on the magnetic surface $\rho =$ constant, as given by Eq. (43). To derive ω_b one has also made use of $< \mu > B_0/m =< v_\perp^2 > /2 \sim v_{\rm th}^2$.

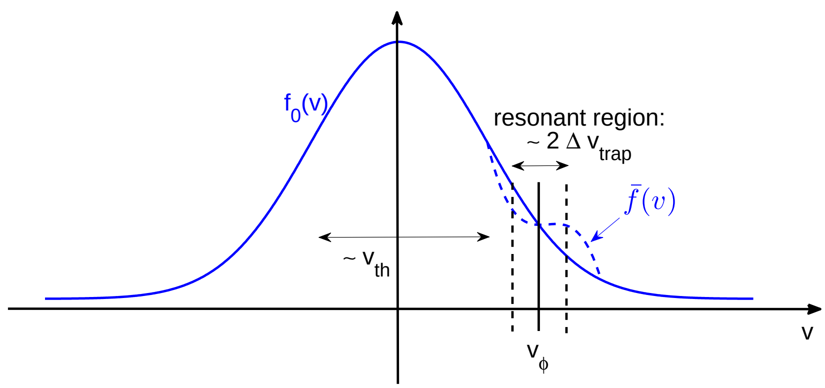

Figure 4: Flattened distribution in resonant region of width $\sim 2\Delta v_{\rm trap}$ around the phase velocity v_ϕ.

6 Energy Cost for Distribution Flattening

This provides the solution to exercise 1.2.3.4.

Let us estimate the variation in kinetic energy which would result from flattening the initial distribution f_0 in the resonant region, i.e. in an interval of width $\sim 2\,\Delta v_{\rm trap}$ around the phase velocity v_ϕ:

$$\Delta\mathrm{Kin}^{\rm flat} = \frac{m}{2}\int_{\rm res}\left[\bar f(v) - f_0(v)\right]\,v^2\,dv,$$

where $\bar f(v)$ is the flattened distribution (see Fig. 4). Assuming $\Delta v_{\rm trap} \ll v_{\rm th}$, one can consider a Taylor expansion for evaluating $f_0(v)$ in the vicinity of v_ϕ:

$$f_0(v) \simeq f_0(v_\phi) + \frac{\partial f_0(v_\phi)}{\partial v}(v - v_\phi) + \mathcal{O}(\frac{\Delta v_{\rm trap}}{v_{\rm th}})^2.$$

Using this expansion, and based on particle conservation, the flattened distribution $\bar f$ in the vicinity of v_ϕ can be estimated as:

$$\begin{aligned}\bar f(v) &= \frac{1}{2\Delta v_{\rm trap}}\int_{v_\phi-\Delta v_{\rm trap}}^{v_\phi+\Delta v_{\rm trap}} dv\, f_0(v)\\ &\simeq \frac{1}{2\Delta v_{\rm trap}}\int_{v_\phi-\Delta v_{\rm trap}}^{v_\phi+\Delta v_{\rm trap}} dv\,\left[f_0(v_\phi) + \frac{\partial f_0(v_\phi)}{\partial v}(v - v_\phi)\right] = f_0(v_\phi),\end{aligned}$$

One thus has

$$\bar f(v) - f_0(v) \simeq -\frac{\partial f_0(v_\phi)}{\partial v}(v - v_\phi) \qquad \text{for} \qquad |v - v_\phi| < \Delta v_{\rm trap}.$$

The variation in kinetic energy can thus be estimated by

$$\begin{aligned}
\Delta\mathrm{Kin}^{\mathrm{flat}} &\simeq -\frac{m}{2}\frac{\partial f_0(v_\phi)}{\partial v}\int_{v_\phi-\Delta v_{\mathrm{trap}}}^{v_\phi+\Delta v_{\mathrm{trap}}} dv\, v^2\,(v-v_\phi) \\
&\overset{v=v_\phi+v'}{=} -\frac{m}{2}\frac{\partial f_0(v_\phi)}{\partial v}\int_{-\Delta v_{\mathrm{trap}}}^{+\Delta v_{\mathrm{trap}}} dv'\,(v_\phi+v')^2\,v' \\
&= -m\,v\left.\frac{\partial f_0}{\partial v}\right|_{v_\phi}\int_{-\Delta v_{\mathrm{trap}}}^{+\Delta v_{\mathrm{trap}}} dv'\,v'^2 = -\frac{2\,m}{3}\,v\left.\frac{\partial f_0}{\partial v}\right|_{v_\phi}\Delta v_{\mathrm{trap}}^3. \qquad (46)
\end{aligned}$$

Note, that apart from the multiplicative constant, this last relation provides essentially the same result as Eq. (1.51) in the notes.

In the case of a Langmuir wave, the wave energy is given by

$$E_{\mathrm{wave}} = \frac{\epsilon_0}{4}\left.\frac{\partial}{\partial\omega}(\omega\epsilon)\right|_{\omega_0} E_0^2 \simeq \frac{\epsilon_0}{2}\,E_0^2, \qquad (47)$$

having used $\epsilon \simeq 1-\omega_p^2/\omega^2$ and $\omega \simeq \omega_p$.

Inserting Eqs. (46) and (47) into the condition $\Delta\mathrm{Kin}^{\mathrm{flat}} > E_{\mathrm{wave}}$ provides:

$$-\frac{2\,m}{3}\,v\left.\frac{\partial f_0}{\partial v}\right|_{v_\phi}\Delta v_{\mathrm{trap}}^3 > \frac{\epsilon_0}{2}\,E_0^2.$$

Making use of Eq. (23) for γ_L, the above relation then leads to

$$\gamma_L > \frac{3\pi}{8}\frac{\epsilon_0}{m\,N}\frac{\omega_p^2}{k_0}\frac{E_0^2}{\Delta v_{\mathrm{trap}}^3} = \frac{3\pi}{64}\omega_b,$$

having made use of $\Delta v_{\mathrm{trap}} = 2(eE_0/mk_0)^{1/2}$ and $\omega_b = k_0\Delta v_{\mathrm{trap}}/2$ (see exercise 1.2.3.1) This last relation thus confirms the equivalence of the orderings

$$\Delta\mathrm{Kin}^{\mathrm{flat}} > E_{\mathrm{wave}} \qquad \Longleftrightarrow \qquad \gamma_L > \omega_b,$$

which is the necessary condition for linear Landau damping to explain the full evolution of the collisionless damping of a wave.

7 Effective Non-Linear Landau Damping

This provides the solution to exercise 1.2.4.1.

The goal here is to show that one can recover relation (1.52) in the notes for the time integrated damping rate $\int_0^\infty dt\,\gamma(t)$ from relation (1.39) for $\gamma(t)$ obtained by O'Neil [T. O'Neil, Physics of Fluids **8**, 2255 (1965)].

Equation (1.39) provided

$$\gamma(t) = \gamma_L \sum_{n=1}^{\infty} \frac{64}{\pi} \int_0^1 d\kappa \,\{\, \underbrace{\frac{2n\pi^2 \sin\left[\frac{n\pi\,\omega_b t}{\kappa F}\right]}{\kappa^5 F^2 (1+q^{2n})(1+q^{-2n})}}_{\text{passing}} + \underbrace{\frac{(2n-1)\pi^2 \kappa \sin\left[\frac{(2n-1)\pi\,\omega_b t}{2F}\right]}{F^2(1+q^{2n-1})(1+q^{-2n+1})}}_{\text{trapped}} \,\}\,.$$

Notice that the harmonic $n=0$ does not contribute to the first term related to passing particles, and that one has made the change of variable $n+1 \rightarrow n$ for the second term related to trapped particles.

One then carries out the time integral:

$$\begin{aligned}
\int_0^t dt'\gamma(t') &= \gamma_L \sum_{n=1}^{\infty} \frac{64}{\pi} \int_0^1 d\kappa \left\{ \frac{\frac{2\pi\,\kappa F}{\omega_b}\left(1-\cos\left[\frac{n\pi\,\omega_b t}{\kappa F}\right]\right)}{\kappa^5 F^2 (1+q^{2n})(1+q^{-2n})} \right. \\
&\qquad \left. + \frac{\frac{2\pi\,\kappa F}{\omega_b}\left(1-\cos\left[\frac{(2n-1)\pi\,\omega_b t}{2F}\right]\right)}{F^2(1+q^{2n-1})(1+q^{-2n+1})} \right\} \\
&= \frac{\gamma_L}{\omega_b} \sum_{n=1}^{\infty} \frac{64}{\pi} \int_0^1 d\kappa \left\{ \frac{2\pi}{\kappa^4 F}\, \frac{1-\cos\left[\frac{n\pi\,\omega_b t}{\kappa F}\right]}{(1+q^{2n})(1+q^{-2n})} \right. \\
&\qquad \left. + \frac{2\pi\,\kappa}{F}\, \frac{1-\cos\left[\frac{(2n-1)\pi\,\omega_b t}{2F}\right]}{(1+q^{2n-1})(1+q^{-2n+1})} \right\}.
\end{aligned}$$

For large values of $\omega_b t$, the integral over the energy variable κ of the phase terms $\cos[n\pi\,\omega_b t/\kappa F(\kappa)]$ and $\cos[(2n-1)\pi\,\omega_b t/2F(\kappa)]$ phase-mix to zero. These phase terms are in fact harmonics of the transit period $\tau_t = (2\kappa/\omega_b)F(\kappa^2)$ and bounce period $\tau_b = (4/\omega_b)F(\kappa^2)$: $\sin[n(2\pi/\tau_t)t]$ and $\sin[(2n-1)(2\pi/\tau_b)t]$. The phase-mixing is thus clearly the result of the difference in transit/bounce periods between orbits with neighboring energy levels. One thus obtains:

$$\int_0^t dt'\gamma(t') = \frac{\gamma_L}{\omega_b} \sum_{n=1}^{\infty} \frac{64}{\pi} \int_0^1 d\kappa \left\{ \frac{1}{\kappa^4}\frac{2\pi}{F}\, \frac{1}{(1+q^{2n})(1+q^{-2n})} + \frac{\kappa^3}{\pi}\frac{2\pi^2}{\kappa^2 F}\, \frac{1}{(1+q^{2n-1})(1+q^{-2n+1})} \right\}.$$

Now making use of the relations

$$\begin{aligned}
\frac{2\pi}{F(\kappa^2)} \sum_{n=1}^{\infty} \frac{1}{(1+q^{2n})(1+q^{-2n})} &= \frac{E(\kappa^2)}{\pi} - \frac{\pi}{4\,F(\kappa^2)}, \\
\frac{2\pi^2}{\kappa^2\,F(\kappa^2)} \sum_{n=1}^{\infty} \frac{1}{(1+q^{2n-1})(1+q^{-2n+1})} &= \left(1 - \frac{1}{\kappa^2}\right) F(\kappa^2) + \frac{1}{\kappa^2} E(\kappa^2),
\end{aligned}$$

one finally obtains

$$\int_0^\infty dt\,\gamma(t) = \frac{\gamma_L}{\omega_b}\,\frac{64}{\pi}\int_0^1 d\kappa \left\{ \underbrace{\frac{1}{\kappa^4}\left[\frac{E(\kappa^2)}{\pi} - \frac{\pi}{4\,F(\kappa^2)}\right]}_{\text{passing}} + \underbrace{\frac{\kappa}{\pi}\left[E(\kappa^2) + (\kappa^2-1)F(\kappa^2)\right]}_{\text{trapped}} \right\},$$

which indeed agrees with Eq. (1.52) from the notes.

8 Analyzing Simulations of Non-Linear Landau Damping

This provides the solution to exercise 1.2.4.2.

When interpreting the plots of Fig. 1.1, one must be aware that they correspond to the results for a standing wave. The study of non-linear Landau damping presented in the notes is however for a single propagating Fourier mode.

In a linear system, a standing wave can naturally be interpreted as the superposition of a forward and backward propagating wave. For example for a standing electrostatic field:

$$E(x,t) = E_0^{\mathrm{std}} \cos(k_0 x)\cos(\omega_0 t) \tag{48}$$

$$= E_0^{\mathrm{fwd}} \cos(k_0 x - \omega_0 t) + E_0^{\mathrm{bwd}} \cos(k_0 x + \omega_0 t), \tag{49}$$

where the amplitudes E_0^{fwd} and E_0^{bwd} of the forward and backward propagating waves respectively are equal, and such that

$$E_0^{\mathrm{fwd}} = E_0^{\mathrm{bwd}} = E_0^{\mathrm{std}}/2, \tag{50}$$

where E_0^{std} is the amplitude of the standing wave.

One must be careful here as one is addressing a non-linear problem, so that a superposition principle in general does not apply. However, the non-linear effects are localized in the resonant regions relative to each wave, i.e. out in the positive velocity tail of the electron distribution for the forward propagating Langmuir wave, and out in the negative velocity tail for the backward propagating wave. These two resonant regions are in general clearly separated as the characteristic width of the resonance Δv_{trap} is typically such that $\Delta v_{\mathrm{trap}} << v_{\mathrm{th}}$ (this assumption is in fact made in the analytical analysis of non-linear Landau damping). As a result, there is no significant interference between the non-linear dynamics of the forward and backward propagating waves. Applying a superposition principle in this case is thus valid.

From the space-averaged potential energy $\langle \mathrm{Pot} \rangle_x$ in Fig. 1.1, one can estimate E_0^{fwd}. Indeed, the values of $\langle \mathrm{Pot} \rangle_x$ in this figure have been computed as follows:

$$\langle \mathrm{Pot} \rangle_x = \langle \frac{1}{2}\epsilon_0 E(x,t)^2 \rangle_x,$$

where $\langle \rangle_x$ stands for the spatial average. From Eq. (48) one thus obtains

$$\langle \mathrm{Pot} \rangle_x = \frac{1}{4}\epsilon_0 (E_0^{\mathrm{std}})^2 \cos^2(\omega_0 t) = \mathcal{P}\ \cos^2(\omega_0 t),$$

with $\mathcal{P} = \epsilon_0 (E_0^{\text{std}})^2/4$ the time envelope of the potential energy trace $\langle \text{Pot} \rangle_x(t)$. Together with (50), one thus obtains:

$$\mathcal{P} = \epsilon_0 (E_0^{\text{fwd}})^2.$$

The amplitude of the normalized electrostatic potential $e\phi_0^{\text{fwd}}/T$ for the forward propagating wave can then be estimated from $\mathcal{P}/NT$ as

$$\frac{e\phi_0^{\text{fwd}}}{T} = \frac{1}{k_0 \lambda_D} \frac{e\, E_0^{\text{fwd}}\, \lambda_D}{T} = \frac{1}{k_0 \lambda_D} \left(\frac{\mathcal{P}}{NT} \right)^{1/2}. \tag{51}$$

having used $k_0 \phi_0^{\text{fwd}} = E_0^{\text{fwd}}$.

8.1 Bounce Period and Time Modulation of Amplitude

In exercise 1.2.3.1 one derived the bounce frequency of deeply trapped particles [Eq.(41)]:

$$\omega_b = \left(\frac{e\, \phi_0^{\text{fwd}}\, k_0^2}{m} \right)^{1/2},$$

from which one obtains the normalized relation

$$\frac{\omega_b}{\omega_p} = k_0 \lambda_D \left(\frac{e\phi_0^{\text{fwd}}}{T} \right)^{1/2} = (k_0 \lambda_D)^{1/2} \left(\frac{\mathcal{P}}{NT} \right)^{1/4},$$

having used Eq. (51). This bounce frequency is computed based on a *time-averaged* estimate from Fig. 1.1 of $\mathcal{P}/NT$ for the two scenarios:

Case $\delta N_0/N = 0.01$: $\quad \dfrac{\mathcal{P}}{NT} \simeq 5 \cdot 10^{-5} \quad \Longrightarrow \dfrac{\omega_b}{\omega_p} \simeq \sqrt{0.3}\,(5 \cdot 10^{-5})^{1/4} = 4.6\,10^{-2}.$

Case $\delta N_0/N = 0.1$: $\quad \dfrac{\mathcal{P}}{NT} \simeq 1 \cdot 10^{-2} \quad \Longrightarrow \dfrac{\omega_b}{\omega_p} \simeq \sqrt{0.3}\,(1 \cdot 10^{-2})^{1/4} = 0.17.$

From Fig. 1.4, which plots the instantaneous damping rate $\gamma(t)$ of the non-linear Langmuir wave as given by relation (1.39), one can estimate the first extremas of the wave amplitude [$\Longleftrightarrow \gamma(t) = 0$] as predicted by the (approximate!) theory:

$$\begin{aligned} \text{First minimas:} \quad & t\,\omega_b \simeq 2\pi\,[0.65,\ 1.88,\ 3.00] = [4.06,\ 11.79,\ 18.90]. \\ \text{First maximas:} \quad & t\,\omega_b \simeq 2\pi\,[1.29,\ 2.44,\ 3.45] = [8.08,\ 15.31,\ 21.70]. \end{aligned}$$

These analytical results need to be compared against the simulation results in Fig. 1.1. For the case $\delta N_0/N = 0.01$ one obtains:

First minimas: $\quad t\,\omega_p \simeq [75,\ 250] \quad \Longrightarrow \quad t\,\omega_b = t\,\omega_p \dfrac{\omega_b}{\omega_p} \simeq t\,\omega_p \cdot 4.6\,10^{-2} = [3.5,\ 11.5].$

First maximas: $\quad t\,\omega_p \simeq 155 \quad \Longrightarrow \quad t\,\omega_b \simeq 7.1.$

And for the case $\delta N_0/N = 0.1$ one obtains:

First minimas: $t\,\omega_p \simeq [25,\ 72,\ 130] \quad \Longrightarrow \quad t\,\omega_b = t\,\omega_p\,\frac{\omega_b}{\omega_p} \simeq t\,\omega_p \cdot 0.17 = [4.3,\ 12.2,\ 22.1].$

First maximas: $t\,\omega_p \simeq [45,\ 100,\ 150] \quad \Longrightarrow \quad t\,\omega_b \simeq [7.6,\ 17.0,\ 25.5].$

The simulation results for the modulation of the amplitude thus agree with the theoretical ones within $\sim 15\%$.

8.2 Effective Asymptotic Damping

The theoretical model predicts for the time integrated damping rate: [Eq. (1.52) in the notes]:

$$\int_0^\infty dt\,\gamma(t) \simeq 1.96\,\frac{\gamma_L}{\omega_b}.$$

In the case of the considered simulation, involving perturbations of a Maxwellian plasma, the linear damping rate γ_L is estimated from Eq. (1.26) (derived assuming $|\gamma_L| \ll |\omega_0|$):

$$\begin{aligned}
\frac{\gamma_L}{\omega_p} &\simeq \sqrt{\frac{\pi}{8}}\frac{1}{(k_0\,\lambda_D)^3}\frac{\omega_0^2}{\omega_p^2}\exp-\frac{1}{2}\left(\frac{\omega_0}{\omega_p}\frac{1}{k_0 v_{\rm th}}\right)^2 \\
&= \sqrt{\frac{\pi}{8}}\,\frac{1+3\,(k_0\,\lambda_D)^2}{(k_0\,\lambda_D)^3}\exp-\frac{1}{2}\left[\frac{1}{(k_0\,\lambda_D)^2}+3\right] \\
&\overset{k_0\,\lambda_D=0.3}{\simeq} 2.5\,10^{-2},
\end{aligned}$$

having used the Bohm-Gross relation $\omega_0^2 = \omega_p^2 + 3(k_0\,\lambda_D)^2$. This value for γ_L is relatively near to the numerical value $\gamma_L/\omega_p = 1.26\cdot 10^{-2}$ obtained by solving the linear dispersion relation (12) for $k_0\,\lambda_D = 0.3$ with no further approximation. We shall use this more accurate numerical value here.

For the case $\delta N_0/N = 0.1$ one thus obtains:

$$\int_0^\infty dt\,\gamma(t) \simeq 1.96\,\frac{\gamma_L}{\omega_p}\frac{\omega_p}{\omega_b} \simeq 1.96\frac{1.26\cdot 10^{-2}}{0.17} = 0.15.$$

As $E_0(t) \sim \exp - \int_0^t \gamma(t')dt'$, the theoretical model would predict:

$$\frac{\mathcal{P}(t=\infty)}{\mathcal{P}(t=0)} = \frac{E_0^2(t=\infty)}{E_0^2(t=0)} = \exp -2\int_0^\infty \gamma(t')dt' \simeq \exp -(2\cdot 0.15) = 0.74.$$

From Fig. 1.1 d., the potential energy seems to settle at:

$$\frac{\mathcal{P}(t=\infty)}{\mathcal{P}(t=0)} \simeq \frac{1\cdot 10^{-2}}{2.8\cdot 10{-}2} = 0.35.$$

Thus there is a factor 2 difference between the approximate theoretical result given by Eq. (1.52) and the effective damping observed in the simulation results of Fig. 1.1.d. The initial evolution of the simulation contains however certain strongly damped transients, which lead to the very sharp decrease of the wave amplitude within the first period $2\pi/\omega_p$, as clearly seen in Fig. 1.1.d. These transients are not taken account of in the theoretical analysis. A more appropriate estimate is thus $\mathcal{P}(t=0)/NT \simeq 1.5 \cdot 10^{-2}$, leading to

$$\frac{\mathcal{P}(t=\infty)}{\mathcal{P}(t=0)} \simeq \frac{1 \cdot 10^{-2}}{1.5 \cdot 10{-}2} = 0.67,$$

clearly in better agreement with theory.

8.3 Trapping width

In exercise 1.2.3.1 one also derived the relation for the trapping width:

$$\Delta v_{\text{trap}} = 2\left(\frac{e\phi_0}{m}\right)^{1/2},$$

from which one obtains the normalized relation

$$\frac{\Delta v_{\text{trap}}}{v_{\text{th}}} = 2\left(\frac{e\phi_0}{T}\right)^{1/2} = \frac{2}{(k_0\lambda_D)^{1/2}}\left(\frac{\mathcal{P}}{NT}\right)^{1/4}.$$

At the end of the runs, i.e. at time $t\omega_p = 250$, one has

$$\text{Case } \delta N_0/N = 0.01: \quad \frac{\mathcal{P}}{NT} \simeq 3.5 \cdot 10^{-5} \quad \Longrightarrow 2\frac{\Delta v_{\text{trap}}}{v_{\text{th}}} \simeq 4\frac{(3.5 \cdot 10^{-5})^{1/4}}{\sqrt{0.3}} = 0.56,$$
$$\text{Case } \delta N_0/N = 0.1: \quad \frac{\mathcal{P}}{NT} \simeq 1.1 \cdot 10^{-2} \quad \Longrightarrow 2\frac{\Delta v_{\text{trap}}}{v_{\text{th}}} \simeq 4\frac{(1.1 \cdot 10^{-2})^{1/4}}{\sqrt{0.3}} = 2.37,$$

These estimates agree well with the trapping widths $2\Delta v_{\text{trap}}$ pointed out in the phase space plots of Figs. 1.1. c & d.

9 Frequency Shift for Wave Generated Adiabatically

This provides the solution to exercise 1.2.6.1.

In the notes, one has in fact studied the situation of a plasma which is "suddenly" submitted to an initial sinusoidal perturbation of finite amplitude. Here one considers the situation of a wave turned on adiabatically, typically generated through an external drive applied to the system over a finite amount of time. The wave is then let to propagate freely, at which time one is interested in the possible frequency shift with respect to the linear dispersion relation. This "adiabatic" case was first considered by Dewar [Dewar, Phys. Fluids **15**, 712 (1972)].

One studies again the system in the wave frame. Adiabatic growth of the wave means that the amplitude $E_0(t)$ of the wave increases slowly with respect to the bounce period: $|(1/E_0)(dE_0/dt)| \ll \omega_b$. In this case, the action integral $\int dx\, v$ is an adiabatic invariant for each particle. If v_0 is the initial velocity of a particle before the presence of the wave, and $W = (m/2)v^2 - e\phi(x)$ its energy in the presence of the wave so that $v(x, W) = \sigma[(2/m)(W + e\phi(x))]$ is its velocity as a function of position x, one thus has

$$\begin{aligned}
\text{const.} &= \int_{-\lambda_0/2}^{\lambda_0/2} dx\, v \\
&= \lambda_0\, v_0 \qquad \text{(initially)} \\
&= \sigma \int_{-\lambda_0/2}^{\lambda_0/2} dx\, H(W + e\phi) \left[\frac{2}{m}(W + e\phi)\right]^{1/2} \qquad \text{(finally)},
\end{aligned}$$

H being the Heaviside step function. As a result

$$v_0 = \frac{\sigma}{\lambda_0} \int_{-\lambda_0/2}^{\lambda_0/2} dx\, H(W + e\phi) \left[\frac{2}{m}(W + e\phi)\right]^{1/2} = \sigma\, \bar{v},$$

where $\bar{v}$ is the spatial average of the velocity for the particle with energy W, given by relations (A.5) and (A.7) in Appendix A for untrapped and trapped particles respectively.

Invoking the invariance of the full distribution along the non-linear trajectories, one thus obtains for the final distribution seen from the wave frame:

$$\sum_{\sigma=\pm 1} f_\infty^{\text{ad}}(W, \sigma) = \sum_{\sigma=\pm 1} f_0(\sigma \bar{v}),$$

which replaces Eq. (1.62) in the notes for the "sudden" case.

To obtain the non-linear frequency shift for the here considered adiabatic scenario, one follows basically the same procedure as in Sec. 1.2.6.

Working in the wave frame, the second order Taylor expansion of f_0 around $v = 0$ simply leads to

$$\sum_{\sigma=\pm 1} f_\infty^{\mathrm{ad}}(W,\sigma) \simeq 2f_0(0) + \frac{d^2 f_0(0)}{dv^2}\,\bar{v}^2,$$

as well as

$$\sum_{\sigma=\pm 1} \Delta f_{\mathrm{NL}}^{\mathrm{ad}} = \sum_\sigma (f_\infty^{\mathrm{ad}} - f_{\mathrm{L}}) \simeq -\frac{2}{m}\frac{d^2 f_0(0)}{dv^2}\left(W - \frac{m}{2}\bar{v}^2\right), \tag{52}$$

which are the equivalent of Eqs. (1.64) and (1.66) in the notes for the "sudden" case. Inserting (52) into (1.67) leads to

$$\epsilon_L(k_0,\omega) = -\frac{4}{m\epsilon_0 E_0^2}\frac{d^2 f_0(0)}{dv^2}\int_{\min(q\phi)}^{+\infty} dW\,(W - \frac{m}{2}\bar{v}^2)(W\bar{v}' - \frac{\bar{v}}{2}). \tag{53}$$

Finally, inserting (A.5) and (A.7) for $\bar{v}$ provides in the "adiabatic" case

$$\begin{aligned}
\epsilon_L(k_0,\omega) \;&=\; -\frac{\omega_p^2}{k_0^2}\Delta v_{\mathrm{trap}}\left.\frac{d^2(f_0/N)}{dv^2}\right|_{v_\phi}\frac{16}{\pi}\int_0^1 d\kappa\;\times \\
&\left\{\underbrace{\frac{1}{\kappa^4}\left[\left(\frac{1}{\kappa^2}-\frac{1}{2}\right)-\frac{4}{\pi^2\kappa^2}E^2\right]\left[2(F-E)-\kappa^2 F\right]}_{\text{passing}}\right. \\
&\left.+\underbrace{\kappa\left[(\kappa^2-\frac{1}{2})-\frac{4}{\pi^2}\left((\kappa^2-1)F+E\right)^2\right](F-2E)}_{\text{trapped}}\right\},
\end{aligned} \tag{54}$$

which is the equivalent of Eq. (1.69) for the "sudden case".

The integral $\alpha^{\mathrm{ad}} = (16/\pi)\int d\kappa\ldots$ in (54) is a constant, and is again integrated numerically, providing the value $\alpha^{\mathrm{ad}} = 0.544$, which is composed of the contribution $\alpha^{\mathrm{u}} = 0.0519$ from untrapped particles and $\alpha^{\mathrm{t}} = 0.493$ from trapped particles. This value is to be compared to the value $\alpha^{\mathrm{sudden}} = \alpha^{\mathrm{u}} + \alpha^{\mathrm{t}} = 0.117 + 0.705 = 0.823$ which was obtained for the "sudden" case.

The difference in the parameter α between the "sudden" and "adiabatic" scenario thus directly affects the frequency shift of the Langmuir wave as given by

$$\delta\omega = -\frac{\alpha}{2}\frac{\omega_p^3}{k_0^2}\Delta v_{\text{trap}} \left.\frac{d^2(f_0/N)}{dv^2}\right|_{v_\phi} .$$

In absolute value, the non-linear frequency shift is thus slightly weaker in the "adiabatic" case compared to the "sudden" one. The two scenarios clearly illustrate how for a given wave amplitude, the non-linear dispersion is not unique, and depends on the wave's history.

10 System of Three Coupled Damped Oscillators

This provides the solution to exercise 1.3.1.1.

The equations of motion for three *damped*, coupled harmonic oscillators are given by

$$\ddot{x}_1 + \omega_1^2 x_1 + \gamma_1 \dot{x}_1 = -V x_2 x_3, \tag{55}$$
$$\ddot{x}_2 + \omega_2^2 x_2 + \gamma_2 \dot{x}_2 = -V x_1 x_3, \tag{56}$$
$$\ddot{x}_3 + \omega_3^2 x_3 + \gamma_3 \dot{x}_3 = -V x_1 x_2, \tag{57}$$

where γ_j is the damping rate of the j^{th} oscillator. These equations generalize Eqs. (1.71)-(1.73) in the notes for the system of undamped, coupled oscillators.

To obtain equations for the slow time scale varying, complex amplitudes $A_j(t)$, one again considers

$$\begin{aligned}
x_j(t) &= \frac{1}{2}\left[A_j(t)e^{i\omega_j t} + \text{c.c.}\right], \\
\dot{x}_j(t) &= \frac{1}{2}\left[(\dot{A}_j + i\omega_j A_j)e^{i\omega_j t} + \text{c.c.}\right], \\
\ddot{x}_j(t) &= \frac{1}{2}\left[(\ddot{A}_j + 2i\omega_j \dot{A}_j - \omega_j^2 A_j)e^{i\omega_j t} + \text{c.c.}\right] \\
&= \frac{1}{2}\left[(\ddot{A}_j + 2i\omega_j \dot{A}_j)e^{i\omega_j t} + \text{c.c.}\right] - \omega_j^2 x_j,
\end{aligned}$$

which are inserted into Eqs. (55)-(57).

For example, for oscillator #1, one obtains:

$$\begin{aligned}
&\frac{1}{2}\left[(\ddot{A}_1 + 2i\omega_1 \dot{A}_1)e^{i\omega_1 t} + \text{c.c.}\right] + \frac{\gamma_1}{2}\left[(\dot{A}_1 + i\omega_1 A_1)e^{i\omega_1 t} + \text{c.c.}\right] \\
&\qquad = -\frac{V}{4}\left[A_2 A_3\, e^{i(\omega_2+\omega_3)t} + A_2 A_3^\star e^{i(\omega_2-\omega_3)t} + \text{c.c.}\right].
\end{aligned}$$

In this last relation, one can neglect $\ddot{A}_1$ with respect to $\omega_1 \dot{A}_1$, as well as $\dot{A}_1$ with respect to $\omega_1 A_1$, under the assumed scaling $|\dot{A}_j/A_j| \ll |\omega_j|$. Furthermore, after multiplying Eq. (10) by $\exp(-i\omega_1 t)$ one obtains:

$$\begin{aligned}
&2i\omega_1 \dot{A}_1 - 2i\omega_1 \dot{A}_1^\star e^{-2i\omega_1 t} + i\omega_1\gamma_1 A_1 - i\omega_1\gamma_1 A_1^\star e^{-2i\omega_1 t} \\
&\qquad = -\frac{V}{2}\left[A_2 A_3 e^{-i(\omega_1-\omega_2-\omega_3)t} + A_2 A_3^\star e^{-i(\omega_1-\omega_2+\omega_3)t}\right. \\
&\qquad\qquad \left. + A_2^\star A_3 e^{-i(\omega_1+\omega_2-\omega_3)t} + A_2^\star A_3^\star e^{-i(\omega_1+\omega_2+\omega_3)t}\right].
\end{aligned}$$

By averaging over the fast time scale of the eigenfrequencies ω_j, and assuming that the condition of frequency matching

$$\omega_1 = \omega_2 + \omega_3 + \delta\omega, \qquad \text{with} \qquad |\delta\omega| \ll |\omega_j|,$$

is met, one finally obtains:

$$2i\omega_1 \dot{A}_1 + i\omega_1\gamma_1 A_1 = -\frac{V}{2} A_2 A_3 e^{-i\,\delta\omega\, t}, \tag{58}$$

$$2i\omega_2 \dot{A}_2 + i\omega_2\gamma_2 A_2 = -\frac{V}{2} A_1 A_3^\star e^{+i\,\delta\omega\, t}, \tag{59}$$

$$2i\omega_3 \dot{A}_3 + i\omega_3\gamma_3 A_3 = -\frac{V}{2} A_1 A_2^\star e^{+i\,\delta\omega\, t}, \tag{60}$$

having at the same time written the equations for the amplitudes A_2 and A_3, which have been derived in a similar way.

One now carries out a stability analysis for the state of the system where one of the oscillators, e.g. oscillator # 1, has been initialized with a much larger amplitude than the two other oscillators: $|A_1| \gg |A_2|, |A_3|$. For this purpose, one considers Eqs. (58)-(60) retaining only terms linear in A_2 and A_3.

For Eq. (58) one then obtains

$$2i\omega_1 \dot{A}_1 + i\omega_1\gamma_1 A_1 = 0.$$

For oscillator #1 we shall neglect damping, so that $\dot{A}_1 = 0$, which implies that $A_1 = A_{1,0} = \text{const}$. Equations (59) and (60) thus become linear in A_2 and A_3:

$$\begin{aligned} 2i\omega_2 \dot{A}_2 + i\omega_2\gamma_2 A_2 &= -\frac{V}{2} A_{1,0} A_3^\star e^{+i\,\delta\omega\, t}, \\ 2i\omega_3 \dot{A}_3 + i\omega_3\gamma_3 A_3 &= -\frac{V}{2} A_{1,0} A_2^\star e^{+i\,\delta\omega\, t}, \end{aligned}$$

Considering the Ansatz

$$\begin{aligned} A_2(t) &= a_2 \exp(\gamma + i\,\delta\omega/2)t, \\ A_3(t) &= a_3 \exp(\gamma^\star + i\,\delta\omega/2)t, \end{aligned}$$

where in general γ is a complex value, leads to the following linear system for $a_{2,3}$:

$$\begin{aligned} i\omega_2 \left[2(\gamma + i\frac{\delta\omega}{2}) + \gamma_2\right] a_2 + \frac{V}{2} A_{1,0}\, a_3^\star &= 0, \\ i\omega_3 \left[2(\gamma^\star + i\frac{\delta\omega}{2}) + \gamma_3\right] a_3 + \frac{V}{2} A_{1,0}\, a_2^\star &= 0, \end{aligned}$$

which can finally be written

$$\underbrace{\begin{pmatrix} i\omega_2\left[2(\gamma+i\frac{\delta\omega}{2})+\gamma_2\right] & \frac{V}{2}A_{1,0} \\ \frac{V}{2}A_{1,0}^{\star} & -i\omega_3\left[2(\gamma-i\frac{\delta\omega}{2})+\gamma_3\right] \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} a_2 \\ a_3^{\star} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To obtain a non-trivial solution $\{a_2, a_3\}$ to this system, thus requires

$$\begin{aligned}
& \det(\mathbf{M}) = 0 \\
\Longleftrightarrow \quad & \omega_2\omega_3\left[2(\gamma+i\frac{\delta\omega}{2})+\gamma_2\right]\left[2(\gamma-i\frac{\delta\omega}{2})+\gamma_3\right] - \left(\frac{V}{2}\right)^2 |A_{1,0}|^2 = 0 \\
\Longleftrightarrow \quad & \gamma^2 + \gamma\,\frac{\gamma_2+\gamma_3}{2} + \left(\frac{\gamma_2}{2}+i\frac{\delta\omega}{2}\right)\left(\frac{\gamma_3}{2}-i\frac{\delta\omega}{2}\right) - \left(\frac{V}{4}\right)^2 \frac{|A_{1,0}|^2}{\omega_2\omega_3} = 0 \\
\Longleftrightarrow \quad & \gamma = -\frac{\gamma_2+\gamma_3}{4} \pm \left[\left(\frac{\gamma_2+\gamma_3}{4}\right)^2 - \left(\frac{\gamma_2}{2}+i\frac{\delta\omega}{2}\right)\left(\frac{\gamma_3}{2}-i\frac{\delta\omega}{2}\right) + \left(\frac{V}{4}\right)^2 \frac{|A_{1,0}|^2}{\omega_2\omega_3}\right]^{1/2} \\
\Longleftrightarrow \quad & \gamma = -\frac{\gamma_2+\gamma_3}{4} \pm \left[\left(\frac{\gamma_2-\gamma_3}{4}+i\frac{\delta\omega}{2}\right)^2 + \left(\frac{V}{4}\right)^2 \frac{|A_{1,0}|^2}{\omega_2\omega_3}\right]^{1/2} .
\end{aligned}$$

Considering already the case of perfect frequency matching, $\delta\omega = 0$, one clearly sees that there is in addition of the condition $\omega_2\omega_3 > 0$ an amplitude threshold for instability:

$$\begin{aligned}
& \left[\left(\frac{\gamma_2-\gamma_3}{4}\right)^2 + \left(\frac{V}{4}\right)^2 \frac{|A_{1,0}|^2}{\omega_2\omega_3}\right]^{1/2} > \frac{\gamma_2+\gamma_3}{4} \\
\Longleftrightarrow \quad & \left(\frac{V}{4}\right)^2 \frac{|A_{1,0}|^2}{\omega_2\omega_3} > \left(\frac{\gamma_2+\gamma_3}{4}\right)^2 - \left(\frac{\gamma_2-\gamma_3}{4}\right)^2 = \frac{\gamma_2\gamma_3}{4} \\
& |A_{1,0}| > 2\,\frac{(\gamma_2\gamma_3\omega_2\omega_3)^{1/2}}{|V|}.
\end{aligned}$$

The stability diagram in the $(|A_{1,0}, \delta\omega|)$ plane is then finally given in Fig. 5.

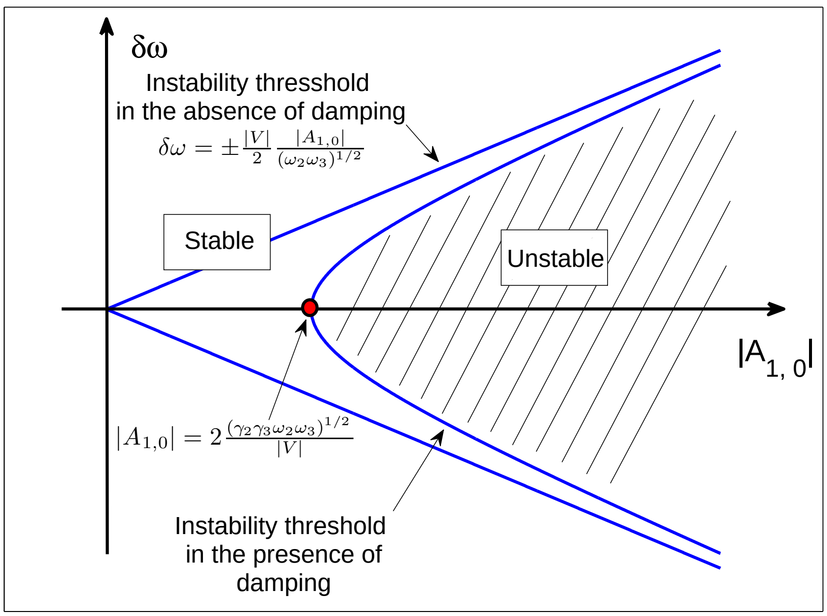

Figure 5: Stability diagram for three damped, coupled oscillators. $|A_{1,0}|$ is the initial amplitude of oscillator #1, assumed such that $|A_{1,0}| \gg |A_{2,0}|, |A_{3,0}|$, and δw is the frequency mismatch.

11 Equation for Slow Variation of EPW Amplitude

This provides the solution to exercise 1.3.2.1

One starts from equation (1.98):

$$\frac{\partial^2 E_x}{\partial t^2} - 3v_{\rm th}^2 \frac{\partial^2 E_x}{\partial x^2} + \omega_p^2 E_x = -\omega_p^2 \frac{1}{2}\frac{\partial}{\partial x}\frac{eA_z^2}{m} = -\frac{e}{m}\omega_p^2 A_z \frac{\partial A_z}{\partial x},$$

and inserts the Ansatz for the vector potential A_z and the electrostatic field E_x:

$$\begin{aligned} A_z(x,t) &= \underbrace{\frac{1}{2}\left[\mathcal{A}_0(x,t)\, e^{i(k_0 x - \omega_0 t)} + \text{c.c.}\right]}_{\text{Incident EM}} + \underbrace{\frac{1}{2}\left[\mathcal{A}_s(x,t)\, e^{i(k_s x - \omega_s t)} + \text{c.c.}\right]}_{\text{Scattered EM}}, \\ E_x(x,t) &= \underbrace{\frac{1}{2}\left[\mathcal{E}(x,t)\, e^{i(k_e x - \omega_e t)} + \text{c.c.}\right]}_{\text{EPW}}. \end{aligned}$$

So that by using relations

$$\begin{aligned} \partial_{xx} E_x &= \frac{1}{2}\left[(\partial_{xx}\mathcal{E} + 2ik_e\, \partial_x \mathcal{E} - k_e^2 \mathcal{E})\, e^{i(k_e x - \omega_e t)} + \text{c.c.}\right], \\ \partial_{tt} E_x &= \frac{1}{2}\left[(\partial_{tt}\mathcal{E} - 2i\omega_e\, \partial_t \mathcal{E} - \omega_e^2 \mathcal{E})\, e^{i(k_e x - \omega_e t)} + \text{c.c.}\right], \\ \partial_x A_z &= \sum_{0,s} \frac{1}{2}\left[(\partial_x \mathcal{A} + ik\mathcal{A})\, e^{i(kx - \omega t)} + \text{c.c.}\right], \end{aligned}$$

one obtains:

$$\begin{gathered} \frac{1}{2}\left[(\partial_{tt}\mathcal{E} - 2i\omega_e\, \partial_t\mathcal{E} - \underbrace{\omega_e^2\mathcal{E}}_{1})\, e^{i(k_e x - \omega_e t)} + \text{c.c.}\right] \\ -3v_{\rm th}^2 \frac{1}{2}\left[(\partial_{xx}\mathcal{E} + 2ik_e\, \partial_x\mathcal{E} - \underbrace{k_e^2\mathcal{E}}_{2})\, e^{i(k_e x - \omega_e t)} + \text{c.c.}\right] \\ +\omega_p^2 \frac{1}{2}\left[\underbrace{\mathcal{E}}_{3}\, e^{i(k_e x - \omega_e t)} + \text{c.c.}\right] \\ = -\frac{e}{m}\omega_p^2 \frac{1}{2}\left[\mathcal{A}_0\, e^{i(k_0 x - \omega_0 t)} + \text{c.c.} + \mathcal{A}_s\, e^{i(k_s x - \omega_s t)} + \text{c.c.}\right] \times \\ \frac{1}{2}\left[(\partial_x\mathcal{A}_0 + ik_0\mathcal{A}_0)\, e^{i(k_0 x - \omega_0 t)} + \text{c.c.} + (\partial_x\mathcal{A}_s + ik_s\mathcal{A}_s)\, e^{i(k_s x - \omega_s t)} + \text{c.c.}\right]. \end{gathered} \tag{61}$$

Note how the terms 1, 2, and 3 in the above equation cancel out as (k_e, ω_e) are assumed to verify the Bohm-Gross dispersion relation for EPWs:

$$\omega_e^2 = \omega_p^2 + 3(k_e v_{\rm th})^2.$$

Furthermore, the term $\partial_{tt}\mathcal{E}$ can be neglected with respect to $\omega_e\,\partial_t\mathcal{E}$, as well as $\partial_{xx}\mathcal{E}$ with respect to $k_e\,\partial_x\mathcal{E}$, as well as $\partial_x\mathcal{A}$ with respect to $k\mathcal{A}$, under the assumption of slow space and time variation of the envelopes:

$$|\frac{1}{\mathcal{E}}\frac{\partial\mathcal{E}}{\partial x}| \ll |k_e|, \qquad \text{and} \qquad |\frac{1}{\mathcal{E}}\frac{\partial\mathcal{E}}{\partial t}| \ll |\omega_e|,$$

with similar scalings for $\mathcal{A}_0$ and $\mathcal{A}_s$.

To obtain a slow scale variation equation for $\mathcal{E}$, one multiplies Eq. (61) by $\exp -i(k_e x - \omega_e t)$ and averages over the fast space ($\sim 1/k$) and time ($\sim 1/\omega$) scales. So that at least some terms on the right hand side of Eq. (61) do not cancel out as a result of this averaging, the wave numbers and frequencies must verify certain matching conditions. One assumes:

$$\begin{aligned} k_0 &= k_s + k_e, \\ \omega_0 &= \omega_s + \omega_e + \delta\omega, \end{aligned}$$

having allowed for a small mismatch of order $|\delta\omega| \ll |\omega_{0,s,e}|$ on the frequencies. For these matching conditions, one can easily convince oneself that the only terms on the right hand side of Eq. (61) that survive the averaging process are the ones $\sim \mathcal{A}_0\mathcal{A}_s^\star$. One thus obtains:

$$\begin{aligned} -2i\omega_e\,\partial_t\mathcal{E} - 3v_{\rm th}^2\,2ik_e\,\partial_x\mathcal{E} &= -\frac{e}{2m}\omega_p^2\,i\,\underbrace{(k_0 - k_s)}_{k_e}\,\mathcal{A}_0\mathcal{A}_s^\star\,e^{-i\delta\omega t} \\ &= -\frac{e}{2m}\omega_p^2\,ik_e\,\mathcal{A}_0\mathcal{A}_s^\star\,e^{-i\delta\omega t}, \end{aligned}$$

having again used the matching condition on the wave numbers. Dividing this last relation by $-2i\omega_e$ and identifying the group velocity $v_{\rm g,e} = d\omega_e/dk_e = 3v_{\rm th}^2 k_e/\omega_e$ for the EPW, one finally gets the slow scale variation equation for the amplitude $\mathcal{E}$ of the electrostatic field:

$$\partial_t\mathcal{E} + v_{\rm g,e}\,\partial_x\mathcal{E} = \frac{e}{4m}\frac{k_e}{\omega_e}\,\omega_p^2\,\mathcal{A}_0\,\mathcal{A}_s^\star\,e^{-i\delta\omega t}.$$

12 Matching Conditions for SBS and LDI

This provides the solution to exercises 1.3.3.1 and 1.3.3.2.

Let us start by considering the process of Stimulated Brillouin Scattering (SBS), which as mentioned in the introduction to Sec. 1.3, involves the interaction of the following three types of modes:

1. An incident EM wave, with wave number k_0 and frequency ω_0
2. A scattered EM wave, with wave number k_s and frequency ω_s.
3. An Ion Acoustic Wave (IAW), with wave number k_i and frequency ω_i.

One shall again limit the study to a one-dimensional system, where all waves propagate either forward or backward along a given direction x. As in SRS, the first two modes are again transverse electromagnetic waves, so that their (wave number, frequency) pairs verify the same dispersion relation:

$$\omega_0^2 = \omega_p^2 + (k_0 c)^2, \tag{62}$$
$$\omega_s^2 = \omega_p^2 + (k_s c)^2, \tag{63}$$

c being the speed of light. The IAW however verifies the essentially linear dispersion relation

$$\omega_i = k_i c_s, \tag{64}$$

where $c_s^2 = ZT_e/m_i$ is the squared sound speed. Note that for an electron temperature of $T_e \simeq 1$keV, one has $c_s \simeq 10^{-3}c$, i.e. typically $c_s \ll c$.

The 3 dispersion relations (62)-(64), together with the 2 matching conditions for wavenumbers and frequencies:

$$k_0 = k_s + k_i, \tag{65}$$
$$\omega_0 = \omega_s + \omega_i, \tag{66}$$

define a system of 5 equations, which in general determine (k_0, ω_0), (k_s, ω_s), and (k_i, ω_i), once e.g. the frequency ω_0 of the incident laser light is fixed.

One solves this system by inserting (65) and (66) into (63):

$$(\omega_0 - \omega_i)^2 = \omega_p^2 + (k_0 - k_i)^2 c^2.$$

Making use of Eq. (62), one obtains:

$$-2\omega_0\omega_i + \omega_i^2 = -2k_0 k_i c^2 + k_i^2 c^2,$$

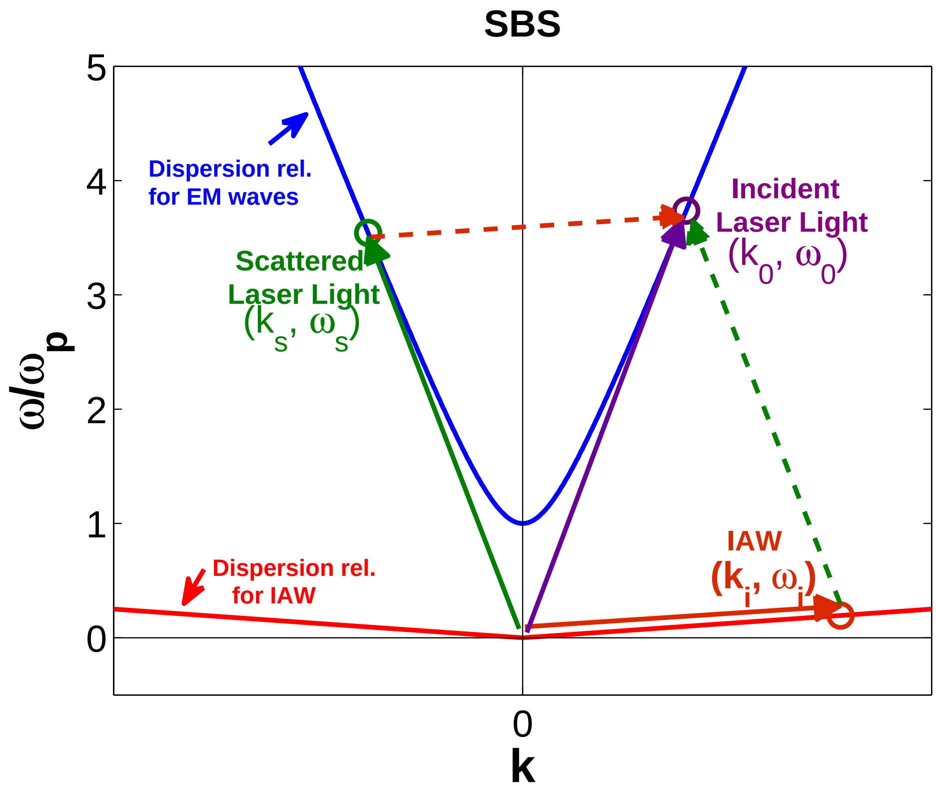

Figure 6: Graphical solution to matching conditions for the SBS process. Note that for this figure the relatively large ratio $c_s/c \simeq 3 \cdot 10^{-2}$ was considered.

and finally inserting (64) provides:

$$
\begin{aligned}
k_i &= 2\frac{k_0 c^2 - \omega_0 c_s}{c^2 - c_s^2} \overset{c_s \ll c}{\simeq} 2k_0 - \frac{2w_0}{c}\frac{c_s}{c}, \\
k_s &\simeq -k_0 + \frac{2w_0}{c}\frac{c_s}{c}.
\end{aligned}
$$

This unique solution, clearly involving backward scattering, is shown graphically in Fig. 6.

Assuming with no loss of generality that $\omega_0 > 0$, the above solution is not valid for $0 < k_0 < c_s\omega_0/c^2$ as it gives $k_i < 0$, which according to (64) provides a negative frequency ω_i. Together with the frequency matching condition (66) this would imply $|\omega_0| < |\omega_s|$, which prevents any decay of the incident light. From Fig. 6 one can clearly understand this (small) limit $k_{0,\text{lim}}$ on k_0 as the point in the dispersion relation of transverse EM waves where $d\omega/dk(k_{0,\text{lim}}) = c_s$. For $k_0 < k_{0,\text{lim}}$ their is then clearly no solution to the matching conditions. From (62) one indeed obtains $d\omega/dk = k_0c^2/\omega_0$, so that $d\omega/dk = c_s \Longrightarrow k_{0,\text{lim}} = c_s\omega_0/c^2$.

The problem of addressing the matching conditions for the Langmuir Decay Instability (LDI) is in fact analogous to the problem of matching conditions for SBS, as it also involves two waves with quadratic dispersion, the two EPWs, and an IAW. Indeed, recall that the dispersion relation for EPWs reads:

$$\omega^2 = \omega_p^2 + 3(kv_{\text{th}})^2,$$

and that $c_s/v_{\text{th}} = (Zm_e/m_i)^{1/2} \sim 10^{-2}$. Identifying c^2 appearing in SBS with $3v_{\text{th}}^2$ in LDI, the solution to the dispersion relations and matching conditions for these two processes becomes identical.

13 Manley Rowe Relations for Three Coupled Oscillators

This provides the solution to exercise 1.3.4.1

One starts by deriving the appropriate relation for the action amplitude a for each oscillator. The action amplitude a_j and the energy E_j of the j^{th} oscillator are by definition related by

$$E_j = n_j\,\omega_j = |a_j|^2\,\omega_j, \tag{67}$$

where ω_j is the eigenfrequency and $n_j = |a_j|^2$ the action of the oscillator.

For a free harmonic oscillator, the total energy is the sum of its kinetic and potential energy:

$$E_j = \frac{1}{2}\dot{x}_j^2 + \frac{1}{2}\omega_j^2 x_j^2. \tag{68}$$

When studying the system of coupled harmonic oscillators, the position x_j of oscillator # j was written in terms of its complex amplitude A_j as

$$x_j = \frac{1}{2}\left[A_j e^{i\omega_j t} + \text{c.c.}\right]. \tag{69}$$

Inserting Eq. (69) into Eq. (68) leads to:

$$\begin{aligned} E_j &= \frac{1}{2}\frac{1}{4}\left[i\omega_j A_j e^{i\omega_j t} - i\omega_j A_j^\star e^{-i\omega_j t}\right]^2 + \frac{\omega_j^2}{2}\frac{1}{4}\left[A_j e^{i\omega_j t} + A_j^\star e^{-i\omega_j t}\right]^2 \\ &= \frac{1}{2}\frac{1}{4}\left[-\omega_j^2 A_j^2 e^{2i\omega_j t} + 2\omega_j^2|A_j|^2 - \omega_j^2 A_j^{\star 2} e^{-2i\omega_j t}\right] + \frac{\omega_j^2}{2}\frac{1}{4}\left[A_j^2 e^{2i\omega_j t} + 2|A_j|^2 + A_j^{\star 2} e^{-2i\omega_j t}\right] \\ &= \frac{1}{2}\omega_j^2|A_j|^2. \end{aligned} \tag{70}$$

Comparing Eqs. (67) and (70) thus leads to the following relation for the action amplitude a_j of oscillator # j in terms of the complex amplitude A_j of its position x_j:

$$a_j = -i\left(\frac{\omega_j}{2}\right)^{1/2} A_j. \tag{71}$$

Notice that the relation between a_j and A_j is determined through (67) only up to a phase factor $\exp(i\theta)$. Here one chose the phase factor $\exp(-i\pi/2) = -i$, as it will provide coupled equations for the action amplitudes which are most analogous to the system (1.125)-(1.127) derived in the notes for the action amplitudes of the three coupled waves involved in the SRS process.

Inserting (71) into Eqs. (1.79)-(1.81) from the notes indeed provides the following equations for the action amplitudes:

$$\begin{aligned}
\dot{a_1} &= -\Gamma a_2 a_3 e^{-i\,\delta\omega\,t}, & (72)\\
\dot{a_2} &= +\Gamma a_1 a_3^\star e^{+i\,\delta\omega\,t}, & (73)\\
\dot{a_3} &= +\Gamma a_1 a_2^\star e^{+i\,\delta\omega\,t}, & (74)
\end{aligned}$$

having defined the normalized non-linear coupling parameter

$$\Gamma = \frac{V}{2\sqrt{2}} \frac{1}{(\omega_1\omega_2\omega_3)^{1/2}}.$$

Comparing Eqs. (72)-(74) with Eqs. (1.125)-(1.127), note however the change in sign of the phases related to the frequency mismatch. This is the result of the different choice of signs for the frequencies in the complex representation of the oscillator positions, as defined by Eq. (69), compared to the choice of sign of the frequencies in the complex representations of the modes involved in the SRS process, as given by Eqs. (1.99) and (1.100).

Let us now derive the Manley-Rowe relations for the action densities $n_j = |a_j|^2$:

$$\begin{aligned}
\frac{d}{dt}(n_1 + n_2) &= 0, & (75)\\
\frac{d}{dt}(n_1 + n_3) &= 0, & (76)
\end{aligned}$$

which respectively state the conservation of action transfer from oscillator 1 to 2, as well as from oscillator 1 to 3. These relations result directly from Eqs. (72)-(74). For instance Eq. (75) is obtained from Eqs. (72) and (73) as follows:

$$\begin{aligned}
\frac{d}{dt}(n_1 + n_2) &= \frac{d}{dt}(a_1 a_1^\star + a_2 a_2^\star) = \dot{a_1}\, a_1^\star + a_1\, \dot{a_1^\star} + \dot{a_2}\, a_2^\star + a_2\, \dot{a_2^\star}\\
&= -\Gamma a_1^\star a_2 a_3 e^{-i\,\delta\omega\,t} - \Gamma a_1 a_2^\star a_3^\star e^{+i\,\delta\omega\,t}\\
&\quad +\Gamma a_1 a_2^\star a_3^\star e^{+i\,\delta\omega\,t} + \Gamma a_1^\star a_2 a_3 e^{-i\,\delta\omega\,t}\\
&= 0.
\end{aligned}$$

From Eqs. (72) and (74) one naturally derives (76) in a similar way.

The energy conservation relation is then obtained by multiplying Eq. (75) by ω_2 and Eq. (76) by ω_3, adding these relations together, and invoking the matching condition $\omega_1 = \omega_2 + \omega_3$:

$$\frac{d}{dt}[\underbrace{(\omega_2 + \omega_3)}_{\omega_1} n_1 + \omega_2 n_2 + \omega_3 n_3] = 0 \quad \overset{E_j = n_j\omega_j}{\Longrightarrow} \quad \frac{d}{dt}\sum_{j=1}^{3} E_j = 0.$$

Equations (72)-(74) are clearly only valid under the assumption of the frequency matching condition $\omega_1 = \omega_2 + \omega_3 + \delta\omega$, as they result from Eqs. (1.79)-(1.81) which have been derived under this same assumption. In fact when deriving (72)-(74), one made the additional assumption of $\omega_j > 0$, as the action amplitudes were derived from the positive energies through the relations $E_j = |a_j|^2\omega_j$ [see Eq. (67)]. The considered matching condition together with the constraint of positive frequencies results in $|\omega_1| \geq |\omega_2|, |\omega_3|$, so that the only possible process is the decay from oscillator 1 into oscillators 2 and 3 (or the opposite, i.e. recombination of 2 and 3 into 1). This is reflected by the Manley-Rowe relations (75) and (76) describing action transfer from oscillator 1 into 2 and 3 (or its inverse).

To be more general, and allow for any sign of the frequencies ω_j, one should be more careful in the definition of the action amplitudes and write:

$$E_j = n_j\,|\omega_j| = |a_j|^2\,|\omega_j|.$$

In this way the action amplitude relations are given in terms of the absolute values of the frequencies:

$$a_j = -i\left(\frac{|\omega_j|}{2}\right)^{1/2} A_j.$$

Equations (72)-(74) can then be generalized to

$$\begin{aligned}
\dot{a_1} &= -\sigma_1\Gamma a_2 a_3 e^{-i\,\delta\omega\,t}, \qquad (77)\\
\dot{a_2} &= +\sigma_2\Gamma a_1 a_3^\star e^{+i\,\delta\omega\,t}, \qquad (78)\\
\dot{a_3} &= +\sigma_3\Gamma a_1 a_2^\star e^{+i\,\delta\omega\,t}, \qquad (79)
\end{aligned}$$

where $\sigma_j = \mathrm{sgn}(\omega_j)$ and

$$\Gamma = \frac{V}{2\sqrt{2}}\frac{1}{(|\omega_1||\omega_2||\omega_3|)^{1/2}}.$$

One can then for example consider $\omega_1 < 0$, $\omega_2 < 0$ and $\omega_3 > 0$, so that from the matching condition $\omega_1 = \omega_2 + \omega_3$ one obtains:

$$\underbrace{-\omega_2}_{>0} = \underbrace{-\omega_1}_{>0} + \underbrace{\omega_3}_{>0} \quad \Longrightarrow \quad |\omega_2| > |\omega_1|, |\omega_3|.$$

The only possible decay in this case is from oscillator 2 into 1 and 3 (or its inverse). This can be verified from the system (77)-(79), using the appropriate signs $\sigma_1 = -1$, $\sigma_2 = -1$, and $\sigma_3 = +1$, either by performing the corresponding stability analysis (linearize with respect to a_1 and a_3 assuming $|a_2| \gg |a_{1,3}|$), or by deriving the Manley-Rowe relations which now read:

$$\frac{d}{dt}(n_2 + n_1) = 0,$$
$$\frac{d}{dt}(n_2 + n_3) = 0.$$

The energy conservation naturally reads in all cases:

$$\frac{d}{dt}(|\omega_1|n_1 + |\omega_2|n_2 + |\omega_3|n_3) = 0.$$