Answer to Exercises

Microinstabilities in Magnetized Plasmas

Notes for Cours Ecole Doctorale

Advanced Theory of Plasmas

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1 Diamagnetic Drifts

This provides the solution to exercises 1 and 2 of section 1.1.2.

In a fluid representation, the momentum equation for a given species α (=electrons, ions) in a magnetized plasma reads:

$$m_\alpha N_\alpha \left[\frac{\partial \vec{u}_\alpha}{\partial t} + \vec{u}_\alpha \cdot (\nabla \vec{u}_\alpha) \right] = -\nabla p_\alpha + q_\alpha N_\alpha \left(\vec{E} + \vec{u}_\alpha \times \vec{B} \right), \tag{1}$$

where m_α is the mass and q_α the charge of the considered species. The fields $N_\alpha(\vec{r}, t)$, $p_\alpha(\vec{r}, t)$, and $\vec{u}_\alpha(\vec{r}, t)$ are respectively the density, pressure, and average velocity. $[\vec{E}(\vec{r}, t), \vec{B}(\vec{r}, t)]$ are possible electric and magnetic fields acting on the plasma.

Let us consider a slab-like magnetic equilibrium similar to the one depicted in Figures 1.3 and 1.4 of the notes. For this purpose one introduces an orthonormal coordinate system $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, such that $\vec{B} = B\vec{e}_z$, and such that all spatial gradients are along $\vec{e}_x$. Thus, in Eq. (1) one has $\partial \vec{u}_\alpha / \partial t = 0$, $\vec{u}_\alpha \cdot (\nabla \vec{u}_\alpha) = 0$, as well as $\vec{E} = 0$, and obtains:

$$0 = -\nabla p_\alpha + q_\alpha N_\alpha \vec{u}_\alpha \times \vec{B}. \tag{2}$$

Taking the vector product of Eq. (2) with $\vec{B}$, one derives the transverse velocity

$$\vec{u}_{\alpha,\perp} = \frac{-\nabla p_\alpha}{N_\alpha} \times \frac{\vec{B}}{q_\alpha B^2}, \tag{3}$$

which is indeed equivalent to the diamagnetic drift $\vec{v}_d$ obtained in Eq. (1.4) of the notes, in the frame of a kinetic description.

Note the scaling of diamagnetic drifts compared to the thermal velocity v_{th}:

$$v_d \sim \frac{|\nabla p|}{N} \frac{1}{qB} \sim \frac{1}{L} \frac{T}{qB} \sim v_{th} \frac{\lambda_L}{L},$$

where L is the characteristic scale length of pressure gradients, $v_{th}^2 = T/m$ the squared thermal velocity, $\lambda_L = v_{th}/\Omega$ the thermal Larmor radius, and $\Omega = qB/m$ the cyclotron frequency. Thus assuming $\epsilon = \lambda_L/L \ll 1$, one has $v_d/v_{th} \ll 1$.

The diamagnetic drifts can naturally carry a charge current:

$$\vec{j} = \sum_\alpha \vec{j}_\alpha = \sum_\alpha q_\alpha N_\alpha \vec{u}_\alpha = -\sum_\alpha \nabla p_\alpha \times \frac{\vec{B}}{B^2} = -\nabla P \times \frac{\vec{B}}{B^2}, \tag{4}$$

having made use of Eq. (3), and defined $P = \sum_\alpha p_\alpha$ the total plasma pressure.

Let us now address the self-consistency of the magnetostatic field $\vec{B}$ with respect to these currents. This is achieved by injecting (4) into Ampere's law:

$$\nabla \times \vec{B} = \mu_0 \vec{j} = -\mu_0 \nabla P \times \frac{\vec{B}}{B^2}.$$

Taking again the vector product of this last relation with $\vec{B}$ provides:

$$\frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B} = \frac{1}{\mu_0}\vec{B} \cdot (\nabla \vec{B}) - \frac{1}{\mu_0}(\nabla \vec{B}) \cdot \vec{B} = -\nabla \left(\frac{B^2}{2\mu_0}\right) = \nabla P,$$

where the curvature term $(1/\mu_0)\vec{B} \cdot (\nabla \vec{B})$ (the so-called magnetic tension force) is zero in the here considered slab geometry. From this last relation one thus obtains:

$$\nabla \left(P + \frac{B^2}{2\mu_0}\right) = 0, \quad \Longrightarrow \quad P + \frac{B^2}{2\mu_0} = \text{const.} \tag{5}$$

Relation (5) states that the thermal pressure P and magnetic pressure $B^2/2\mu_0$ compensate each other. Thus, if a plasma is submitted to an external magnetic field $\vec{B}_{\text{ext}}$ it generates, through its diamagnetic currents, an internal magnetic field $\vec{B}_{\text{int}}$ which opposes $\vec{B}_{\text{ext}}$ in such a way that (5) is verified:

$$P(x) + \frac{[B_{\text{ext}} + B_{\text{int}}(x)]^2}{2\mu_0} = \frac{B_{\text{ext}}^2}{2\mu_0}.$$

One says that the plasma generates a magnetic well. This is illustrated in figure 1. A plasma thus clearly has a **diamagnetic** behavior.

The diamagnetic nature of a plasma naturally results from the behavior of each individual particle. Each particle in a magnetic field $\vec{B}$, through its cyclotron motion, generates a small magnetic moment $\vec{\mu}$, which is always opposite the magnetic field as shown in Fig. 2. The amplitude μ of a magnetic moment is the product of the surface S of its current loop, times the current I: $\mu = SI$. In the case of a single charge q in a magnetic field $\vec{B}$, the surface is given in terms of the Larmor radius $\lambda_L = v_\perp/\Omega$ by $S = \pi\lambda_L^2$, and the current by $I = q\,\Omega/2\pi$. The magnetic moment thus becomes:

$$\mu = SI = \pi\lambda_L^2\, q\frac{\Omega}{2\pi} = \frac{mv_\perp^2}{2B}.$$

Averaging the squared perpendicular velocity $v_\perp^2$ over a distribution with temperature T, one obtains $(m/2) < v_\perp^2 >= T$, so that the average magnetic moment per particle becomes:

$$\langle\vec{\mu}\rangle = -T\frac{\vec{B}}{B^2}.$$

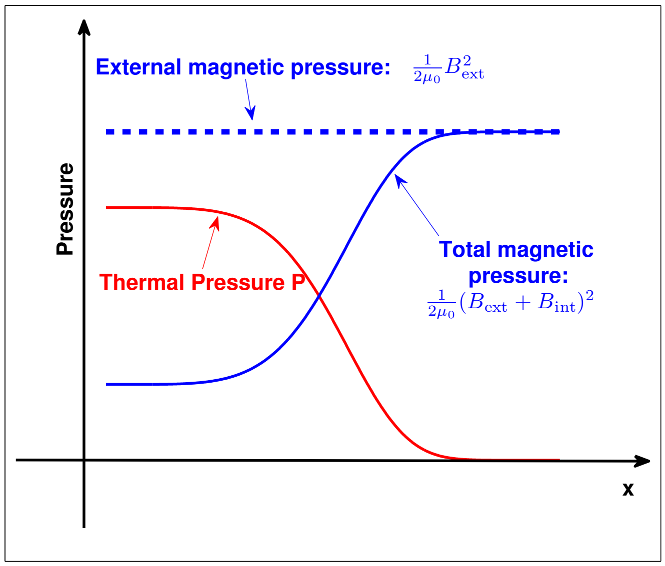

Figure 1: Magnetic well generated by a plasma as a result of its diamagnetic behavior

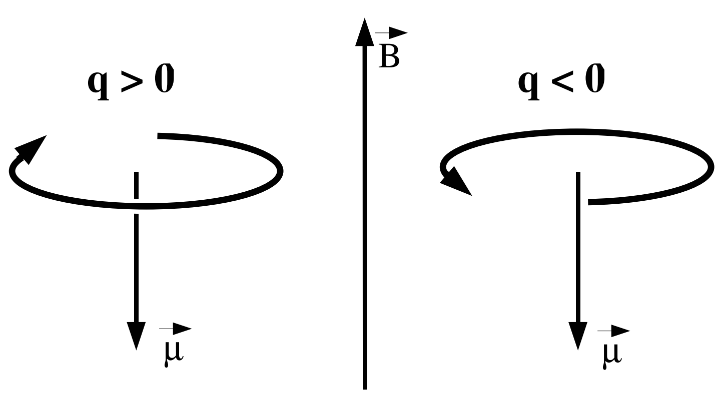

Figure 2: Magnetic moment $\vec{\mu}$ associated to the cyclotron motion of a particle in a magnetic field $\vec{B}$. The moment $\vec{\mu}$ is opposite the field $\vec{B}$ whether the charge q is positive or negative. This is the origin of the diamagnetic behavior of the plasma.

For a plasma species with density N, the magnetization field $\vec{M}$, which measures the density of magnetic moments, is thus given by

$$\vec{M} = N\langle\vec{\mu}\rangle = -p\frac{\vec{B}}{B^2},$$

where again p is the pressure of the considered species. In the presence of a pressure gradient, one can then compute the magnetization current:

$$\vec{j} = \nabla \times \vec{M} = -\nabla p \times \frac{\vec{B}}{B^2},$$

which clearly recovers Eq. (4).

2 Limits of Dielectric Function

This provides the solution to exercise 1 of section 1.2.3.

One considers the **dielectric function of an inhomogeneous, magnetized plasma** given by Eq. (1.20) in the notes:

$$\epsilon(\vec{k},\omega) = 1+\sum_{\text{species}} \frac{1}{(k\lambda_D)^2}\left\{1+(\omega-\omega_d')\sum_{n=-\infty}^{+\infty}\frac{1}{\omega-\omega_F-n\Omega}\left[W\left(\frac{\omega-\omega_F-n\Omega}{|k_z|v_{th}}\right)-1\right]\Lambda_n(\xi)\right\}. \tag{6}$$

In the limit of zero external forces ($\vec{F}=0 \implies \omega_F=0$), and no equilibrium gradients ($\omega_d'=0$), one obtains:

$$\epsilon(\vec{k},\omega) = 1+\sum_{\text{species}} \frac{1}{(k\lambda_D)^2}\left\{1+\sum_{n=-\infty}^{+\infty}\frac{\omega}{\omega-n\Omega}\left[W\left(\frac{\omega-n\Omega}{|k_z|v_{th}}\right)-1\right]\Lambda_n(\xi)\right\}. \tag{7}$$

This is the **dielectric function of an homogeneous magnetized plasma** [see for example Eq. (4.68) in S. Ichimaru, *Basic Principles of Plasma Physics. A Statistical Approach* (W. A. Benjamin, Inc., Reading, Massachusetts, 1973), or Eq. (3.54), K. Appert, *Théorie des Plasmas Chauds* (EPFL-Repro, EPFL, 2003)].

One now considers the limit of vanishing magnetic field: $B \to 0$. In this case $\Omega = qB/m \to 0$ and $\xi = (k_y v_{th}/\Omega)^2 \to +\infty$. One can thus make use of the asymptotic limit of the scaled modified Bessel functions:

$$\Lambda_n(\xi) \overset{\xi\to+\infty}{\to} \frac{1}{\sqrt{2\pi\xi}}\exp(-\frac{n^2}{2\xi}) = \frac{\Omega}{\sqrt{2\pi}k_y v_{th}}\exp\left[-\frac{1}{2}\left(\frac{n\Omega}{k_y v_{th}}\right)^2\right].$$

The dielectric function then becomes

$$\epsilon(\vec{k},\omega) = 1+\sum_{\text{species}}\frac{1}{(k\lambda_D)^2}\left[1+I\right], \tag{8}$$

having defined the term I:

$$I = \sum_{n=-\infty}^{+\infty}\frac{\Omega}{\sqrt{2\pi}k_y v_{th}}\frac{\omega}{\omega-n\Omega}\left[W\left(\frac{\omega-n\Omega}{|k_z|v_{th}}\right)-1\right]\exp\left[-\frac{1}{2}\left(\frac{n\Omega}{k_y v_{th}}\right)^2\right].$$

This term thus appears in the form of a Darboux sum, which in the limit of $\Omega \to 0$ converges to a continuous integral:

$$I = \sum_{n=-\infty}^{+\infty}\Omega\,\text{fct}(n\Omega) \overset{\Omega\to 0}{\to} \int_{-\infty}^{+\infty} dy\,\text{fct}(y),$$

having identified $y = n\Omega$ and $dy = \Omega$. In the limit of vanishing magnetic field, one can therefore write:

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dy}{k_y v_{th}} \frac{\omega}{\omega - y} \left[W\left(\frac{\omega - y}{|k_z| v_{th}}\right) - 1 \right] \exp\left[-\frac{1}{2} \left(\frac{y}{k_y v_{th}} \right)^2 \right].$$

Making use of the definition of the dispersion function $W(z)$:

$$W(z) = \frac{1}{\sqrt{2\pi}} \int dx \frac{x}{x - z} \exp(-x^2/2),$$

one furthermore obtains:

$$\begin{aligned}
I &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dy}{k_y v_{th}} \exp\left[-\frac{1}{2} \left(\frac{y}{k_y v_{th}} \right)^2 \right] \frac{\omega}{\omega - y} \int_{-\infty}^{+\infty} \frac{dv_z}{v_{th}} [\underbrace{\frac{v_z}{v_z - (\omega - y)/|k_z|} - 1}_{\frac{\omega - y}{k_z v_z + y - \omega}}] \exp\left[-\frac{1}{2} \left(\frac{v_z}{v_{th}} \right)^2 \right] \\
&\stackrel{v_y = y/k_y}{=} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dv_y}{v_{th}} \int_{-\infty}^{+\infty} \frac{dv_z}{v_{th}} \frac{\omega}{k_y v_y + k_z v_z - \omega} \exp\left[-\frac{1}{2} \left(\frac{v_y^2 + v_z^2}{v_{th}^2} \right) \right] \times \\
&\quad \underbrace{\frac{1}{\sqrt{2\pi}} \int \frac{dv_x}{v_{th}} \exp\left[-\frac{1}{2} \left(\frac{v_x}{v_{th}} \right)^2 \right]}_{=1} \\
&= \frac{1}{(2\pi)^{3/2}} \int \frac{dv_x dv_y dv_z}{v_{th}^3} \frac{\omega}{k_y v_y + k_z v_z - \omega} \exp\left[-\frac{1}{2} \left(\frac{v_x^2 + v_y^2 + v_z^2}{v_{th}^2} \right) \right].
\end{aligned}$$

As we have considered the wave vector $\vec{k} = k_x \vec{e}_x + k_y \vec{e}_y$, the resonant denominator in the above relation can be written $\vec{k} \cdot \vec{v} - \omega$, so that

$$\begin{aligned}
I &= \frac{1}{(2\pi v_{th}^2)^{3/2}} \int d\vec{v} \underbrace{\frac{\omega}{\vec{k} \cdot \vec{v} - \omega}}_{\frac{\vec{k}\cdot\vec{v}}{\vec{k}\cdot\vec{v} - \omega} - 1} \exp(-\frac{1}{2} \frac{v^2}{v_{th}^2}) \\
&= -1 + \frac{1}{\sqrt{2\pi}} \int \frac{dv_x}{v_{th}} \frac{v_x}{v_x - \omega/|k|} \exp(-\frac{1}{2} \frac{v_x^2}{v_{th}^2}) \\
&= W\left(\frac{\omega}{|k| v_{th}} \right) - 1. \qquad (9)
\end{aligned}$$

having re-aligned the coordinate system so that the direction Ox becomes parallel to the wave vector $\vec{k}$: $\vec{k} = k\vec{e}_x$.

By inserting (9) into (8), one thus finally obtains:

$$\epsilon(\vec{k}, \omega) \stackrel{B \to 0}{\longrightarrow} 1 + \sum_{\text{species}} \frac{1}{(k\lambda_D)^2} W\left(\frac{\omega}{|k| v_{th}} \right),$$

which is indeed the **dielectric function of an homogeneous, unmagnetized plasma**.

3 Electrostatic Waves in Homogeneous, Magnetized Plasma

This provides the solution to exercise 2 of section 1.2.3.

The dielectric function for electrostatic waves in an homogeneous magnetized plasma is given by Eq. (7):

$$\epsilon(\vec{k},\omega) = 1 + \sum_{\text{species}} \frac{1}{(k\lambda_D)^2} \left\{ 1 + \sum_{n=-\infty}^{+\infty} \frac{\omega}{\omega - n\Omega} \left[W\left(\frac{\omega - n\Omega}{|k_\parallel| v_{th}} \right) - 1 \right] \Lambda_n(\xi) \right\}, \tag{10}$$

with $\xi = (k_\perp v_{th}/\Omega)^2$, and using here the notations $k_\parallel$ and $k_\perp$ for the parallel and perpendicular components respectively to the magnetic field. Still considering an orthonormal coordinate system $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, such that $\vec{B} = B\vec{e}_z$, note that $\vec{k}_\perp$ can take any orientation in the Oxy plane as the homogeneous system is isotropic in Oxy (see Fig. 3).

Having fixed the wave vector $\vec{k}$, one solves the dispersion relation

$$\epsilon(\vec{k},\omega) = 0,$$

for the complex frequency $\omega = \omega_R + i\gamma$ by assuming $|\gamma/\omega_R| \ll 1$. In this case, the dispersion relation can be solved in the resonant approximation, so that:

$$\epsilon_R(\omega_R) = 0, \tag{11}$$

$$\gamma = -\frac{\epsilon_I(\omega_R)}{\partial\epsilon_R(\omega_R)/\partial\omega}, \tag{12}$$

where ϵ_R and ϵ_I are the real and imaginary parts respectively of the dielectric function $\epsilon = \epsilon_R + i\epsilon_I$. Equation (11) thus provides an equation for the real part ω_R of the frequency, while Eq. (12) provides a relation for the growth/damping rate γ.

In the following, one also makes use of the Taylor series expansion of the dispersion function $W(z)$ for $|z| \ll 1$:

$$W(z) = 1 - z^2 + \frac{z^4}{3} - \ldots + (-1)^n \frac{z^{2n}}{(2n-1)!!} + \ldots + i\sqrt{\frac{\pi}{2}} z \exp(-\frac{z^2}{2}), \tag{13}$$

as well as the asymptotic series for $|z| \gg 1$:

$$W(z) = -\frac{1}{z^2} - \frac{3}{z^4} - \ldots - \frac{(2n-1)!!}{z^{2n}} - \ldots + i\sqrt{\frac{\pi}{2}} z \exp(-\frac{z^2}{2}). \tag{14}$$

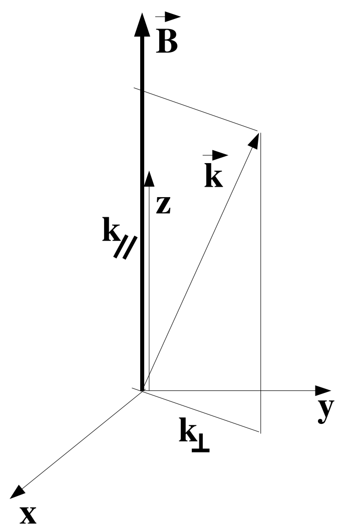

Figure 3: Wave vector in homogeneous, magnetized plasma

3.1 Electron Plasma Wave (EPW)

Here one wants to consider waves with frequency $\omega \sim \omega_{pe}$. For such high frequency modes, ions can be considered fixed, so that in the sum over species appearing in (10) only electrons provide a significant contribution. Furthermore, one assumes here that the magnetic field amplitude is sufficiently strong so that $|\omega_{pe}| \ll |\Omega_e|$. Note that this is not necessarily the case under magnetic fusion conditions (see table 1 and 2 in the notes). This scaling avoids however possible resonances with harmonics of the cyclotron frequency, so that in the sum $\sum_n$ appearing in (10), one needs to consider only $n = 0$.

The dispersion relation thus reduces here to

$$\epsilon(\vec{k}, \omega) = 1 + \frac{1}{(k\lambda_{De})^2} \left\{1 + [W(z_e) - 1]\, \Lambda_0(\xi_e)\right\} = 0,$$

with $z_e = \omega/|k_\parallel| v_{th\,e} \sim 1/(k\lambda_{De})$. Assuming $k\lambda_{De} \ll 1$, which turns out to be necessary for the resonant approximation to be valid (as will be shown on the final result), one thus has $|z_e| \gg 1$, and makes use of the asymptotic series for $W(z)$. See Fig. 4 for the relative position of the parallel phase velocity $\omega/k_\parallel$ with respect to the electron distribution in the case of the EPW. One then also has:

$$\xi_e = \left(\frac{k_\perp v_{th\,e}}{\Omega_e}\right)^2 \overset{|\omega_{pe}| \ll |\Omega_e|}{\ll} \left(\frac{k_\perp v_{th\,e}}{\omega_{pe}}\right)^2 < (k\lambda_{De})^2 \ll 1 \quad \Longrightarrow \quad \Lambda_0(\xi_e) \simeq 1.$$

In the resonant approximation, one thus obtains from (11):

$$\epsilon_R(\omega_R) \simeq 1 + \frac{1}{(k\lambda_{De})^2}\left\{\underbrace{1-\Lambda_0(\xi_e)}_{\simeq 0} - \left(\frac{k_\parallel v_{th\,e}}{\omega_R}\right)^2 \underbrace{\Lambda_0(\xi_e)}_{\simeq 1}\right\} = 0, \tag{15}$$

which gives for the real frequency:

$$\omega_R^2 = \frac{(k_\parallel v_{th\,e})^2}{(k\lambda_{De})^2} = \frac{k_\parallel^2}{k^2}\,\omega_{pe}^2 \qquad \Longrightarrow \qquad \omega_R = \pm\frac{k_\parallel}{k}\,\omega_{pe}. \tag{16}$$

This result is to be compared to the frequency $\omega_R \simeq \omega_{pe}$ in an homogeneous, *unmagnetized* plasma. The factor $k_\parallel/k$ appearing in (16) reflects the fact that the motion of particles is mainly constrained along the magnetic field. As a result, the plasma is very slow in responding to perturbations nearly perpendicular to $\vec{B}$ for which $|k_\parallel/k| \ll 1$.

One now computes the damping rate in the resonant approximation. For this one first derives the following relations, making again use of (14) as well as (15):

$$\epsilon_I(\omega_R) \simeq \sqrt{\frac{\pi}{2}}\frac{z_{e,R}}{(k\lambda_{De})^2}\exp(-\frac{z_{e,R}^2}{2}) = \sqrt{\frac{\pi}{2}}\frac{1}{(k\lambda_{De})^2}\frac{\omega_R}{|k_\parallel|v_{th\,e}}\exp\left[-\frac{1}{2}\left(\frac{\omega_R}{|k_\parallel|v_{th\,e}}\right)^2\right],$$

$$\frac{\partial\epsilon_R}{\partial\omega}(\omega_R) \simeq \frac{2}{(k\lambda_{De})^2}\frac{(|k_\parallel|v_{th\,e})^2}{\omega_R^3}.$$

So that from (12), one obtains:

$$\gamma = -\frac{\epsilon_I(\omega_R)}{\partial\epsilon_R(\omega_R)/\partial\omega} \simeq -\sqrt{\frac{\pi}{8}}\frac{\omega_R^4}{(|k_\parallel|v_{th\,e})^3}\exp\left[-\frac{1}{2}\left(\frac{\omega_R}{|k_\parallel|v_{th\,e}}\right)^2\right].$$

One clearly has $\gamma < 0$, which corresponds to damping. Making use of (16) one then finally obtains:

$$\left|\frac{\gamma}{\omega_R}\right| \simeq \sqrt{\frac{\pi}{8}}\frac{1}{(k\lambda_{De})^3}\exp\left[-\frac{1}{2}\frac{1}{(k\lambda_{De})^2}\right],$$

which is essentially the same relation for the relative Landau damping as in a non-magnetized plasma.

Note that for $k\lambda_{De} \ll 1$, this final result confirms the assumption $|\gamma/\omega_R| \ll 1$ required for solving the dispersion relation under the resonant approximation.

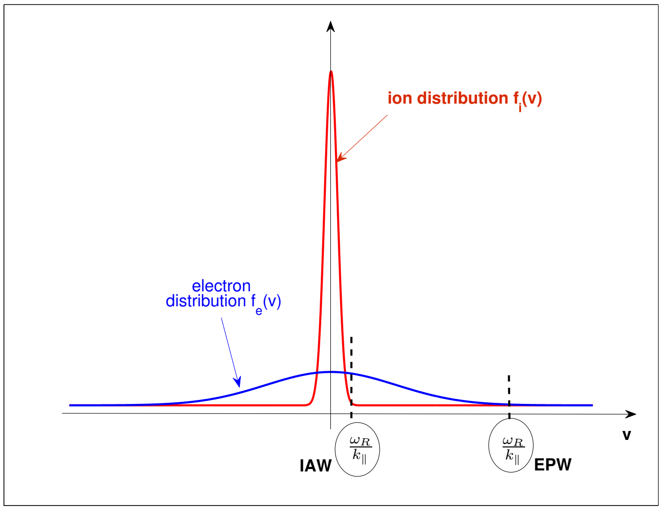

Figure 4: Phase velocity of EPW and IAW with respect to the electron and ion distributions

3.2 Ion Acoustic Wave (IAW)

One now considers low frequency modes such that

$$v_{th\,i} \ll |\frac{\omega}{k_\parallel}| \ll v_{th\,e}. \tag{17}$$

See figure 4, illustrating also the position of the parallel phase velocity of the IAW with respect to both the electron and ion distributions. The scaling defined by (17) will need to be checked on the final result. Furthermore, one shall again assume that the magnetic field is sufficiently strong, such that $|\omega| \ll |\Omega_i|, |\Omega_e|$, so that all harmonic resonances $|n| > 0$ of the cyclotron frequencies can be neglected.

The dispersion relation, with both electron and ion contributions, thus reduces here to

$$\epsilon(\vec{k},\omega) = 1 + \frac{1}{(k\lambda_{De})^2}\left\{1 + [W(z_e) - 1]\,\Lambda_0(\xi_e)\right\} + \frac{1}{(k\lambda_{Di})^2}\left\{1 + [W(z_i) - 1]\,\Lambda_0(\xi_i)\right\}$$
$$= 0,$$

with

$$z_e = \frac{\omega}{|k_\parallel| v_{th\,e}}, \qquad \text{and} \qquad z_i = \frac{\omega}{|k_\parallel| v_{th\,i}}.$$

The assumed scaling (17) implies:

$$
\begin{aligned}
|z_e| \ll 1 \quad &\Longrightarrow \quad W(z_e) \simeq 1 + i\sqrt{\frac{\pi}{2}} z_e, \\
|z_i| \gg 1 \quad &\Longrightarrow \quad W(z_i) \simeq -\frac{1}{z_i^2} + i\sqrt{\frac{\pi}{2}} z_i \exp(-\frac{z_i^2}{2}).
\end{aligned}
$$

having made use of Eqs. (13) and (14) respectively.

In the resonant approximation, one thus obtains from (11):

$$
\epsilon_R(\omega_R) \simeq 1 + \frac{1}{(k\lambda_{De})^2} + \frac{1}{(k\lambda_{Di})^2}\left\{1 - \Lambda_0(\xi_i) - \left(\frac{k_\parallel v_{th\,i}}{\omega_R}\right)^2 \Lambda_0(\xi_i)\right\} = 0. \quad (18)
$$

The electron contribution clearly reduces to its adiabatic response. Furthermore, assuming again sufficiently long wavelengths such that $k\lambda_{De} \ll 1$, one can neglect the first term, i.e. 1, in the above relation. This corresponds to neglecting the left hand side in Poisson's equation, thus imposing quasi-neutrality. In this way, one obtains the following real frequency:

$$
\omega_R^2 = (k_\parallel v_{th\,i})^2 \frac{\Lambda_0(\xi_i)}{(\lambda_{Di}/\lambda_{De})^2 + 1 - \Lambda_0(\xi_i)} = (k_\parallel c_s)^2 \frac{\Lambda_0(\xi_i)}{1 + (ZT_e/T_i)[1 - \Lambda_0(\xi_i)]},
$$

where $c_s^2 = ZT_e/m_i$ is the sound speed squared, Z the ionization degree, and having made use of $(\lambda_{Di}/\lambda_{De})^2 = T_i/ZT_e$.

In the limit of $\xi_i = (k_\perp \lambda_{Li})^2 \ll 1$, such that $\Lambda_0(\xi_i) \simeq 1 - \xi_i$, one then furthermore obtains:

$$
\omega_R^2 = \frac{(k_\parallel c_s)^2}{1 + (k_\perp \rho_s)^2}, \quad (19)
$$

where $\rho_s = c_s/\Omega_i$ is the ion Larmor radius estimated at the sound speed. The term $(k_\perp \rho_s)^2$ appearing in the denominator of Eq. (19) is the so-called polarization drift term.

One thus essentially obtains

$$
\omega_R = \pm k_\parallel c_s, \quad (20)
$$

which, similar to the EPW case, differs from the corresponding dispersion relation $\omega_R = \pm k c_s$ in an homogeneous, unmagnetized plasma by a factor $k_\parallel/k$.

To compute the damping rate in the resonant approximation, one derives the following relations:

$$\begin{aligned}
\epsilon_I(\omega_R) &\simeq \sqrt{\frac{\pi}{2}}\frac{\Lambda_0(\xi_e)}{(k\lambda_{De})^2}z_{e,R} + \sqrt{\frac{\pi}{2}}\frac{\Lambda_0(\xi_i)}{(k\lambda_{Di})^2}z_{i,R}\exp(-\frac{z_{i,R}^2}{2}) \\
&\simeq \sqrt{\frac{\pi}{2}}\frac{\Lambda_0(\xi_e)}{(k\lambda_{De})^2}\frac{\omega_R}{|k_\parallel|v_{th\,e}}, \\
\frac{\partial\epsilon_R}{\partial\omega}(\omega_R) &\simeq 2\frac{\Lambda_0(\xi_i)}{(k\lambda_{Di})^2}\frac{(|k_\parallel|v_{th\,i})^2}{\omega_R^3},
\end{aligned}$$

having again made use of (18), and having neglected the ion contribution to ϵ_I as it is exponentially small compared to the electron contribution.

From (12), one then obtains:

$$\gamma = -\frac{\epsilon_I(\omega_R)}{\partial\epsilon_R(\omega_R)/\partial\omega} \simeq -\sqrt{\frac{\pi}{8}}\frac{\Lambda_0(\xi_e)}{\Lambda_0(\xi_i)}\left(\frac{\lambda_{Di}}{\lambda_{De}}\right)^2\frac{\omega_R^4}{(|k_\parallel|v_{th\,i})^2(|k_\parallel|v_{th\,e})},$$

which is again negative, i.e. clearly corresponding to damping.

In the limit $\xi_{e,i} \ll 1$, so that $\Lambda_0(\xi_{e,i}) \simeq 1$, one then finally obtains:

$$\left|\frac{\gamma}{\omega_R}\right| = \sqrt{\frac{\pi}{8}}\frac{T_i}{ZT_e}\frac{c_s^3}{v_{th\,i}^2 v_{th\,e}} = \sqrt{\frac{\pi}{8}}\sqrt{\frac{Zm_e}{m_i}},$$

which is again essentially the same relation for the relative Landau damping of IAWs as in a non-magnetized plasma. As $m_e/m_i \ll 1$ one indeed has $|\gamma/\omega_R| \ll 1$, required for justifying the resonant approximation.

Finally, let us check the initial assumptions $|z_e| \ll 1$ and $|z_i| \gg 1$:

$$\begin{aligned}
|z_e| &\simeq \left|\frac{\omega_R}{k_\parallel v_{th\,e}}\right| \overset{(20)}{\simeq} \frac{c_s}{v_{th\,e}} = \sqrt{\frac{Zm_e}{m_i}} \ll 1 \quad \text{OK!} \\
|z_i| &\simeq \left|\frac{\omega_R}{k_\parallel v_{th\,i}}\right| \simeq \frac{c_s}{v_{th\,i}} = \sqrt{\frac{ZT_e}{T_i}} \overset{?}{\gg} 1 \quad \text{OK, if } T_e \gg T_i.
\end{aligned}$$

The requirement $T_e \gg T_i$ is naturally the same as for IAWs in an unmagnetized plasma.

4 Two-Fluid Model of the Slab-ITG Instability

This provides the solution to the exercise of section 1.5.3.

One derives here the linear dispersion relation for the slab-ITG instability starting from a two-fluid model. The "hot" ions are represented by the continuity equation, the momentum equation including a finite pressure term, and a heat equation representing convection in the flow $\vec{v}_E = (\vec{E} \times \vec{B})/B^2$:

$$\begin{aligned}
\frac{\partial N_i}{\partial t} + \nabla \cdot (N_i \vec{u}_i) &= 0, \\
m_i N_i \left[\frac{\partial \vec{u}_i}{\partial t} + \vec{u}_i \cdot (\nabla \vec{u}_i) \right] &= e N_i \left(\vec{E} + \vec{u}_i \times \vec{B} \right) - \nabla (N_i T_i), \\
\frac{\partial}{\partial t}(N_i T_i) + \nabla \cdot (N_i T_i \vec{v}_E) &= 0.
\end{aligned}$$

where N_i is the average density, $\vec{u}_i$ the average velocity, and T_i the temperature of the ions with mass m_i and charge e. The field $\vec{B}$ is the uniform magnetostatic field, and $\vec{E} = -\nabla\phi$ the electrostatic field associated to the perturbation, deriving from the potential ϕ. Electrons are assumed to respond adiabatically (valid if $|\omega/k_\parallel| \ll v_{th\,e}$), so that :

$$N_e = N_{e0} \exp(\frac{e\phi}{T_e}),$$

where N_e is the total density, N_{e0} the initial, homogeneous density, and T_e the temperature of the electrons. Assuming that the wavelengths are sufficiently large such that $k\lambda_{De} \ll 1$, one may invoke quasineutrality for closing the above system of equations:

$$N_e = N_i.$$

Let us now linearize this system of equations for small amplitude fluctuations $\delta N/N = e\phi/T_e \ll 1$. These perturbations are of the form $\phi \sim \exp i(\vec{k}\cdot\vec{r} - \omega t)$. The unperturbed fields are denoted $(N_i = N_0, N_e = N_0, \vec{u}_i = 0, T_i = T_{i0})$. The corresponding fluctuations are denoted $(\delta N_i, \delta N_e, \vec{u}_i, \delta T_i)$. The unperturbed fields are assumed all homogeneous, except for the ion temperature: $\nabla T_{i0} \neq 0$. The linearized equations then become:

$$\frac{\partial \delta N_i}{\partial t} + N_0 \nabla \cdot \vec{u}_i = 0, \tag{21}$$

$$m_i \frac{\partial \vec{u}_i}{\partial t} = e(\vec{E} + \vec{u}_i \times \vec{B}) - \nabla \delta T_i^\star, \tag{22}$$

$$\frac{\partial \delta T_i^\star}{\partial t} + \vec{v}_E \cdot \nabla T_{i0} = 0, \tag{23}$$

having defined $\delta T_i^\star = \delta T_i + (\delta N_i/N_0)T_{i0}$.

Equation (22) can be re-written:

$$\vec{u}_i = \underbrace{\frac{e\vec{E} - \nabla\delta T^\star}{-i\omega m_i}}_{\vec{a}} + \vec{u}_i \times \underbrace{\frac{e\vec{B}}{-i\omega m_i}}_{\vec{b}},$$

and is thus of the form $\vec{u} = \vec{a} + \vec{u} \times \vec{b}$, with solution $\vec{u} = (1+b^2)^{-1}(\vec{a} + \vec{a}\cdot\vec{b}\,\vec{b} + \vec{a}\times\vec{b}) = \vec{a}_\parallel + (1+b^2)^{-1}(\vec{a}_\perp + \vec{a}\times\vec{b})$, where $\vec{a}_\parallel$ and $\vec{a}_\perp$ are respectively the components of $\vec{a}$ parallel and perpendicular to $\vec{b}$. One thus obtains for the components of $\vec{u}$ parallel and perpendicular to $\vec{B}$:

$$u_{i\parallel} = \frac{eE_\parallel - \nabla_\parallel \delta T_i^\star}{-i\omega m_i}, \tag{24}$$

$$\begin{aligned} \vec{u}_{i\perp} &= \left[1 - \frac{\Omega_i^2}{\omega^2}\right]^{-1} \left[\frac{e\vec{E}_\perp - \nabla_\perp \delta T_i^\star}{-i\omega m_i} - \frac{e\vec{E} - \nabla\delta T_i^\star}{\omega^2 m_i^2} \times e\vec{B}\right] \\ &\overset{|\omega|\ll\Omega_i}{\simeq} \frac{e\vec{E} - \nabla\delta T^\star}{eB^2} \times \vec{B} = \vec{v}_E - \frac{\nabla\delta T^\star}{eB^2} \times \vec{B}, \end{aligned} \tag{25}$$

having neglected here the polarization drift term $(\omega/i\Omega_i^2 m_i))(e\vec{E}_\perp - \nabla_\perp\delta T_i^\star)$ as it is order $|\omega|/\Omega_i$ smaller than v_E.

Equation (24) clearly corresponds to the linearized, parallel momentum equation:

$$m_i \frac{\partial u_{i\parallel}}{\partial t} = eE_\parallel - \nabla_\parallel \delta T^\star,$$

while from equation (25) one concludes that $\nabla\cdot\vec{u}_{i\perp} = 0$, as

$$\nabla\cdot\vec{v}_E = \nabla\cdot\frac{-\nabla\phi\times\vec{B}}{B^2} = \nabla\phi\cdot\underbrace{(\nabla\times\frac{\vec{B}}{B^2})}_{=0} - \frac{\vec{B}}{B^2}\cdot\underbrace{(\nabla\times\nabla\phi)}_{=0} = 0,$$

and in the same way $\nabla\cdot[\nabla\delta T_i^\star \times \vec{B}/(eB^2)] = 0$. The linearized continuity equation (21) thus reduces to

$$\frac{\partial\delta N_i}{\partial t} + N_0 \nabla_\parallel u_{i\parallel} = 0.$$

The so-obtained set of linearized equations can thus be summarized as follows:

$$\frac{\partial \delta N_i}{\partial t} + N_0 \nabla_\parallel u_{i\parallel} = 0, \tag{26}$$

$$m_i \frac{\partial u_{i\parallel}}{\partial t} = -e\nabla_\parallel \phi - \nabla_\parallel \delta T_i^\star, \tag{27}$$

$$\frac{\partial \delta T_i^\star}{\partial t} + \vec{v}_E \cdot \nabla T_{i0} = 0, \tag{28}$$

$$\delta N_i = \delta N_e = N_0 \frac{e\phi}{T_e}, \tag{29}$$

with $\vec{v}_E = (-\nabla\phi \times \vec{B})/B^2$.

For perturbations of the form $\exp i(\vec{k}\cdot\vec{r} - \omega t)$, one obtains for Eqs. (26)-(28):

$$\delta N_i = N_0 \frac{k_\parallel u_{i\parallel}}{\omega},$$

$$u_{i\parallel} = \frac{k_\parallel}{\omega\, m_i}(e\phi + \delta T_i^\star),$$

$$\delta T_i^\star = -\frac{\omega_{T_i}}{\omega}\, e\phi,$$

where $\omega_{T_i} = \vec{v}_{T_i} \cdot \vec{k} = [-\nabla T_{i0} \times \vec{B}/(eB^2)] \cdot \vec{k}$ is the diamagnetic drift frequency related to the ion temperature gradient. From these last relations one then derives:

$$\frac{\delta N_i}{N_0} = \frac{(k_\parallel\, c_s)^2}{\omega^2}\left(1 - \frac{\omega_{T_i}}{\omega}\right)\frac{e\phi}{T_e},$$

which can then be inserted into Eq. (29), thus providing the dispersion relation:

$$1 - \frac{(k_\parallel\, c_s)^2}{\omega^2}\left(1 - \frac{\omega_{T_i}}{\omega}\right) = 0, \tag{30}$$

having defined $c_s^2 = T_e/m_i$ the squared sound speed. Equation (30) indeed agrees with Eq. (1.43) in the notes, obtained from the kinetic dispersion relation in the appropriate limit for the slab-ITG instability.