Lecture 7

Fast ion effects on pressure driven long wavelength instabilities

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Inclusion of fast ion effects in MHD equations

We may refer to the derivation of the perpendicular MHD equation from week 1. These hold for weakly collisional populations as well as the usual collisional ones:

$$
\begin{aligned}
& -i\rho\omega\boldsymbol{\delta u}_\perp = \boldsymbol{J}\times\boldsymbol{\delta B} + \boldsymbol{\delta J}\times\boldsymbol{B} + (\delta P_\perp - \delta P_\parallel)\boldsymbol{\kappa} - \boldsymbol{\nabla}_\perp \delta P_\perp \\
& \boldsymbol{\delta u}_\perp = -i\omega\boldsymbol{\xi}_\perp = \frac{\boldsymbol{\delta E}\times\boldsymbol{B}}{B^2} \\
& \boldsymbol{\delta}\dot{\boldsymbol{B}} = -\boldsymbol{\nabla}\times\boldsymbol{\delta E} \\
& \boldsymbol{\delta J} = \boldsymbol{\nabla}\times\boldsymbol{\delta B} \\
& \boldsymbol{\nabla}\cdot\boldsymbol{\delta B} = 0, \\
& \frac{d\rho_j}{dt} + \rho_j\boldsymbol{\nabla}\cdot\boldsymbol{\delta u_j} = 0.
\end{aligned}
$$

The perpendicular momentum equation, derived and shown in Eq. (1.8) has exploited quasi-neutrality:

$$\sum_j q_j \delta n_j = 0.$$

We have the following definitions

$$\rho = \sum_j m_j n_j, \quad \delta P_\perp = \sum_j \delta P_{j\perp}, \quad \delta P_\parallel = \sum_j \delta P_{j\parallel}, \quad \boldsymbol{J} = \sum_j q_j n_j u_j.$$

Notice that we have not yet applied Ohm's law. Ideal Ohm's law sets $\boldsymbol{b}\cdot\boldsymbol{\delta E} = 0$. We have seen that parallel electric fields arise from dissipation (resistive MHD). But it can also come from weakly collisional kinetic corrections, e.g. ions and electrons having different kinetic behaviour (different orbit widths, different drift frequencies) will require parallel electric fields to maintain quasi-neutrality.

Notes

Diamagnetic corrections

There is a generalisation to the momentum equation written on the previous slide. It has been rigourously shown that there are diamagnetic corrections in both collisional and collisionless descriptions. See Lanthanler, Graves, Pfefferlé, Cooper Plasma Phys. Control. Fusion **61** (2019) 074006:

$$\sum_j \rho_j(-i\omega + \boldsymbol{u}_{*,j} \cdot \boldsymbol{\nabla})\boldsymbol{\delta u}_\perp = \boldsymbol{J} \times \boldsymbol{\delta B} + \boldsymbol{\delta J} \times \boldsymbol{B} + (\delta P_\perp - \delta P_\parallel)\boldsymbol{\kappa} - \boldsymbol{\nabla}_\perp \delta P_\perp$$

where the diamagnetic velocity for species j is

$$\boldsymbol{u}_{*,j} = \frac{\boldsymbol{B} \times \boldsymbol{\nabla} P_{\perp,j}}{q_j n_j B^2},$$

and $P_{\perp,j}$ is the perpendicular pressure associated with the equilibrium distribution of species j:

$$P_{\perp,j} = m_j \int d^3v\, v_\perp^2 F_j.$$

Usually fast ion physics is neglected from the LHS of the momentum equation because $\rho_h \ll \rho_i$. Electrons are usually neglected from the LHS since $m_e \ll m_i$ so that $\rho_e \ll \rho_i$, But, with diamagnetic corrections it might be important to retain fast ion physics on the LHS if $P'_{\perp h} \sim P'_i$ since

$$\rho_j \boldsymbol{u}_{*,j} \sim m_j \frac{\boldsymbol{B} \times \boldsymbol{\nabla} P_{\perp,j}}{q_j B^2}.$$

Clearly in a fusion plasma one can have $P'_{\perp h} \sim P'_i$ even if $n_h \ll n_i$, because of the energy of the fast ions, in particular since $T_{\perp h} \sim m_h v_\perp^2/2 \gg T_i$.

Inclusion of fast ion effects in MHD equations

Consider the electric field

$$\boldsymbol{\delta E} = -\boldsymbol{\nabla}\delta\Phi - \dot{\boldsymbol{\delta A}}. \tag{7.1}$$

We are free to choose a gauge $\boldsymbol{b} \cdot \boldsymbol{\delta A} = 0$, so that $\boldsymbol{\delta A}$ is perpendicular to $\boldsymbol{B}$, i.e.

$$\boldsymbol{\delta A} = \boldsymbol{\delta \xi} \times \boldsymbol{B}. \tag{7.2}$$

Thus, any parallel electric field is connected to non-zero $\delta\Phi$:

$$\delta E_{\|} \equiv \boldsymbol{\delta E} \cdot \boldsymbol{b} = -\boldsymbol{b} \cdot \boldsymbol{\nabla}\delta\Phi.$$

Consider now quasi-neutrality. We assume in this course that ions and electrons are highly collisional, while fast ions are not. But, if the fast ion density fulfills $n_h \ll (n_i, n_e)$ then the fast ions may be neglected in the quasi-neutrality condition. Hence, only ions and electrons enter quasi-neutrality, and these are independent of $\delta\Phi$ (see next slide). Hence $\delta\Phi$ does not enter the quasi-neutrality problem, and thus we may take $\delta\Phi = 0$, as in ideal MHD. Hence,

$$\boldsymbol{\delta E} = -\dot{\boldsymbol{\delta A}} = \dot{\boldsymbol{\delta \xi}} \times \boldsymbol{B}$$

and noting that we already have

$$-i\omega\boldsymbol{\xi}_{\perp} = \frac{\boldsymbol{\delta E} \times \boldsymbol{B}}{B^2}$$

then it follows that $\boldsymbol{\delta \xi} = \boldsymbol{\xi}$ and thus:

$$\begin{aligned}
\boldsymbol{\delta E} - i\omega\boldsymbol{\xi} \times \boldsymbol{B} &= 0, \\
\boldsymbol{\delta B} &= \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}), \\
\boldsymbol{\delta J} &= \boldsymbol{\nabla} \times \boldsymbol{\delta B} = \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B})].
\end{aligned}$$

Inclusion of fast ion effects in MHD equations

We require equations for the moments appearing in the equation of motion, in particular the perturbed pressure tensor. Consider first the electrons and ions. For collisional electrons and ions the adiabatic equation of state applies:

$$\frac{d}{dt}\left(P_j \rho_j^{-\gamma}\right) = 0, \quad i.e. \quad \frac{1}{\rho_j}\frac{d\rho_j}{dt} = -\gamma \frac{1}{P_j}\frac{dP_j}{dt}$$

where j indicates ion ($j = i$) or electron ($j = e$). From continuity equation we have,

$$\frac{1}{\rho_j}\frac{d\rho_j}{dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{\delta u_j}.$$

Equating these relations, linearising and using convective derivative definition yields:

$$\frac{\partial \delta P_j}{\partial t} + \boldsymbol{\delta u_j} \cdot \boldsymbol{\nabla} P_j = -\gamma P_j \boldsymbol{\nabla} \cdot \boldsymbol{\delta u}_j.$$

At this point we recall that $\boldsymbol{\delta u}_j = -i\omega \boldsymbol{\xi}_\perp$ is a common perpendicular velocity. Assume now that the parallel velocity is also common, as in the ideal MHD model, i.e. $\delta u_{j\parallel} = -i\omega\xi_\parallel$, so that,

$$\delta P_j = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla} P_j - \gamma P_j \boldsymbol{\nabla} \cdot \boldsymbol{\xi}.$$

The total thermal pressure $\delta P_t = \delta P_i + \delta P_e$ (thermal ion and electron) is therefore,

$$\delta P_t = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla} P_t - \gamma P_t \boldsymbol{\nabla} \cdot \boldsymbol{\xi}$$

where $P_t = P_i + P_e$ is the total thermal equilibrium pressure.

Momentum equation

The perturbed fast ion pressure tensor depends on the three components of the electric field, i.e. on $\boldsymbol{\xi}_\perp$ and $\delta\Phi$. But since we have argued that $\delta\Phi = 0$ we have that $\delta P_{\perp h} = \delta P_{\perp h}(\boldsymbol{\xi}_\perp)$ and $\delta P_{\| h} = \delta P_{\| h}(\boldsymbol{\xi}_\perp)$. The perpendicular momentum equation is then

$$-\rho_i\omega^2\boldsymbol{\xi}_\perp = \boldsymbol{J}\times\boldsymbol{\delta B}+\boldsymbol{\delta J}\times\boldsymbol{B}+[\delta P_{\perp h}(\boldsymbol{\xi}_\perp)-\delta P_{\| h}(\boldsymbol{\xi}_\perp)]\boldsymbol{\kappa}-\boldsymbol{\nabla}_\perp\left[\delta P_{\perp h}(\boldsymbol{\xi}_\perp) - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}P_t - \gamma P_t\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right].$$

The problem is still not closed because we need an extra equation in order to resolve the parallel displacement (note parallel displacement appears in δP_t, so it appears in perpendicular momentum equation in form $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$).

This problem can be modelled by adopting the parallel component of the ideal MHD momentum equation (without fast ions):

$$-\rho_i\omega^2\boldsymbol{\xi}_\| = -\boldsymbol{b}\cdot\boldsymbol{\nabla}\delta P_t = \boldsymbol{b}\cdot\boldsymbol{\nabla}\left[\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P_t + \gamma P_t\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right].$$

The parallel momentum equation and plasma compressibility has to do with corrections to the inertia $1\rightarrow 1+2q^2$. As discussed, minority ions do not have significant inertia, to the parallel momentum equation model appears reasonable. Hence, the full momentum equation is,

$$-\rho_i\omega^2\boldsymbol{\xi} = \boldsymbol{J}\times\boldsymbol{\delta B}+\boldsymbol{\delta J}\times\boldsymbol{B}+[\delta P_{\perp h}-\delta P_{\| h}]\boldsymbol{\kappa}-\boldsymbol{\nabla}_\perp\delta P_{\perp h}+\boldsymbol{\nabla}\left[\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P_t + \gamma P_t\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right]. \quad (7.3)$$

We now produce quadratic forms, as in lecture 3. We note that the force associated with fast particles isn't always self-adjoint (depends on the problem of interest), but alternative analysis has shown that the variation of the associated energy nevertheless recovers a valid dispersion relation, at least for the internal kink mode and interchange modes. The energy principle associated with the sign of δW (indicating stability or instability) is not always correct if the force isn't self-adjoint. We will consider the forms of $\delta P_{\perp h}$ and $\delta P_{\| h}$ later. Operating Eq. (7.3) with $-(1/2)\int d^3x\,\boldsymbol{\xi}^*\cdot$ we obtain,

$$\delta K+\delta W = 0,\quad \delta K = -\frac{\omega^2}{2}\int d^3x\,\rho\,|\xi|^2,\quad \delta W = \overline{\delta W}+\delta W_t+\delta W_h,\quad \overline{\delta W}(\boldsymbol{\xi},\boldsymbol{\xi}^*) = \frac{1}{2}\int_P d^3x\,\gamma P_t(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2$$

We have that

$$\delta W_h = -(1/2)\int d^3x\,\left\{\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla}\delta P_{\perp h} - \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa}[\delta P_{\perp h} - \delta P_{\parallel h}]\right\}$$

Consider the first term, use that $\boldsymbol{\nabla}\cdot(\phi\boldsymbol{A}) = \phi\boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\nabla}\phi$. Hence

$$\int d^3x\,\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla}\delta P_{\perp h} = \int d^3x\,\boldsymbol{\nabla}\cdot(\boldsymbol{\xi}_\perp^*\,\delta P_{\perp h}) - \int d^3x\,\delta P_{\perp h}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*.$$

Moreover the first term on the right hand side vanishes upon applying the divergence theorem, and assuming that the displacement vanishes at the plasma edge, i.e. the surface $\boldsymbol{S}$ bounding the volume $\int dx^3$:

$$\int_V dx^3\,\boldsymbol{\nabla}\cdot(\boldsymbol{\xi}_\perp^*\,\delta P_{\perp h}) = \int_S \delta P_{\perp h}\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{dS} = 0,$$

then

$$\delta W_h = -\frac{1}{2}\int d^3x\,\left[\delta P_{\perp h}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*) - (\delta P_{\parallel h} - \delta P_{\perp h})\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa}\right]. \tag{7.4}$$

Approximate form

We will return to Eq. (7.4) later. For now we can write an approximate form by adopting the MHD minimisation of Eq. (3.27), and at the next order (Eq. (3.30)), i.e. we use

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa} = 0,$$

so that we have approximately

$$\delta W_h = \frac{1}{2}\int d^3x\,(\delta P_{\parallel h} + \delta P_{\perp h})\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa}. \tag{7.5}$$

We will verify now the conditions under which this approximation holds.

Weakly anisotropic equilibrium

In order to rely on the properties of ideal MHD equilibria we should ensure that the equilibrium pressure depends only weakly on θ. In particular

$$\frac{P_\perp + P_\parallel - \overline{P_\perp + P_\parallel}}{\overline{P_\perp + P_\parallel}} \sim \epsilon$$

or less, where $\overline{X}$ represents a flux average. Since the thermal plasma is isotropic, this requires that

$$\frac{P_{\perp h} + P_{\parallel h} - \overline{P_{\perp h} + P_{\parallel h}}}{2P_t + \overline{P_{\perp h} + P_{\parallel h}}} \sim \epsilon$$

This ensures that the essential expansion of the Grad-Shafranov equation obtained in Chapter 2 holds (for a strong anisotropic equilibrium expansion see e.g. [Graves, PRL 84, 1204 (2000)] and references therein).

In practice weak poloidal dependence in the total pressure can be achieved in two ways:

1. The fast ions are distributed isotropically (e.g. alphas) or with an arbitrary excess of passing ions (e.g. with tangential NBI). Even if all the hot particles are *deeply* passing we have $P_{\parallel h} = \overline{P_{\parallel h}}(1 + O(\epsilon \cos \Theta))$. Under those conditions the fast ions can have associated pressure comparable to the thermal pressure,

$$P_{\parallel h} \sim P_t$$

2. The fast ions are distributed with a strong excess of trapped particles so that $P_{\perp h} \gg P_{\parallel h}$, but the perpendicular hot ion pressure is weaker than the thermal pressure

$$P_{\perp h} \sim \epsilon P_t, \quad \text{while} \quad P_{\perp h} \gg P_{\parallel h}$$

Under those conditions, for all equilibrium expansions, including the definition of the curvature, the Shafranov shift etc, we may define a total scalar pressure P which independent of θ at leading order in ϵ:

$$P = P_t + (P_{\perp h} + P_{\parallel h})/2,$$

Leading order in ϵ is sufficient for the equilibrium expansion. Anisotropic corrections are important for the perturbed quantities.

Hot adiabatic response

Assuming weak equilibrium pressure anisotropy the convenient form for δW holds for the thermal plasma (t), which for internal modes (dropping boundary contributions) is then

$$\delta W_t = \frac{1}{2}\int d^3x \left[|\delta B_\perp|^2 + B^2 \left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right|^2 - 2(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}P_t)(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*) - J_\parallel(\boldsymbol{\xi}_\perp^*\times\boldsymbol{b})\cdot\boldsymbol{\delta B}_\perp\right] \tag{7.6}$$

where have from week 6,

$$\boldsymbol{\kappa} = \left(\frac{1}{B^2}\right)\left[\boldsymbol{\nabla} - \boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla})\right]\left(\frac{B^2}{2} + P\right).$$

where total $P = P_t + (P_{\perp h} + P_{\parallel h})/2$ is taken to be independent of Θ at the required order, and indeed B will be the isotropic equilibrium solution of the Grad-Shafranov equilibrium.

We will show that the fast ion perturbed distribution function can be written in the form,

$$\delta F_h = \delta F_{hf} + \delta F_{hk}$$

where δF_{hf} is the adiabatic, or fluid like solution, and δF_{hk} is the kinetic solution. In the limit of strong collisions $\delta F_{hk} \to 0$, which gives us a reason for investigating problems where we include only the adiabatic or fluid-like fast ion response. We will see that,

$$\delta F_{hf} = -\xi^r \frac{\partial F_h}{\partial r}, \quad F_h = F_h(r, \mathcal{E}, \mu) \tag{7.7}$$

where F_h is written in terms of the constants of motion $\mathcal{E} = m_h v^2/2$, $\mu = m_h v_\perp^2/(2B)$, and r which is a constant of motion in the thin banana limit (toroidal canonical momentum is an exact invariant, but corresponds to constant r to leading order in Larmor radius (particles have vanishing radial orbit width to leading order in a Larmor radius expansion)).

Hot Adiabatic Response

We note the definitions of the equilibrium and perturbed pressure components:

$$P_{\perp h} = m_h \int d^3v\, \frac{v_\perp^2}{2} F_h\,, \quad P_{\parallel h} = m_h \int d^3v\, v_\parallel^2 F_h\,, \quad \delta P_{\perp h} = m_h \int d^3v\, \frac{v_\perp^2}{2} \delta F_h\,, \quad \delta P_{\parallel h} = m_h \int d^3v\, v_\parallel^2 \delta F_h\,.$$

So that, the adiabatic perturbed pressures and associated adiabatic δW of Eq. (7.5) are,

$$\delta P_{\perp h} = -m_h \xi^r \int d^3v\, \frac{v_\perp^2}{2} \frac{\partial F_h}{\partial r}\,, \quad \delta P_{\parallel h} = -m_h \xi^r \int d^3v\, v_\parallel^2 \frac{\partial F_h}{\partial r}\,,$$

$$\delta W_{hf} = -\frac{1}{2} \int d^3x\, \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa} \xi^r \int d^3v\, m_h \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right) \frac{\partial F_h}{\partial r}\,.$$

For zero and weak hot anisotropy we can take the radial derivative outside the velocity integral. Zero anisotropy means that $F_h = F_h(r, \mathcal{E})$ (i.e. otherwise independent of μ or pitch angle $\mu/\mathcal{E}$), the pressure moments are then independent of poloidal angle:

$$\delta W_{hf}(isotropic) = -\frac{1}{2} \int d^3x\, \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa}\, \xi^r \frac{d}{dr} \left[P_{\perp h} + P_{\parallel h} \right]$$

This hot adiabatic contribution can be compared with the interchange/ballooning term for thermal ions in δW_t, that is,

$$-\frac{1}{2} \int d^3x\, 2\, (\boldsymbol{\xi} \cdot \boldsymbol{\nabla} P_t)\, (\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa}) \equiv -\frac{1}{2} \int d^3x\, 2 \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa}\, \xi^r \frac{dP_t}{dr}$$

For finite anisotropy one cannot take the radial derivative outside the velocity integrals in general. But we can combine some of the hot ion fluid physics conveniently with the thermal contributions, as shown next.

Hot Adiabatic Response

For the adiabatic hot ion case the total (thermal fluid plus hot fluid) δW becomes

$$\delta W_f = \frac{1}{2}\int d^3x\,\left[|\delta B_\perp|^2 + B^2\,|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2 - 2(\boldsymbol{\xi}_\perp\cdot\nabla\overline{P})(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*) - J_\parallel(\boldsymbol{\xi}_\perp^*\times\boldsymbol{b})\cdot\boldsymbol{\delta B}_\perp\right] + \delta W_{fA},$$

$$\delta W_{fA} = -\frac{1}{2}\int d^3x\,\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa}\,\xi^r\left\{\int d^3v\,m_h\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\partial F_h}{\partial r} - \frac{d}{dr}\left(\overline{P_{\perp h}}+\overline{P_{\parallel h}}\right)\right\} \quad (7.8)$$

where

$$\overline{P} = P_c + \left(\frac{\overline{P_{\perp h}}+\overline{P_{\parallel h}}}{2}\right), \quad \text{with} \quad \overline{X} = \frac{\int_{-\pi}^{\pi} d\Theta\,\mathcal{J}X}{\int_{-\pi}^{\pi} d\Theta\,\mathcal{J}}.$$

The term δW_{fA} on the second line of Eq. (7.8) treats specifically the effect of fluid anisotropy, and in particular the effect of the anisotropic pressure corrections becoming functions of Θ. In the isotropic limit the radial derivative acting on F_h can be taken outside the velocity integral, the velocity integral which then defines the pressure moments (these being independent of Θ in the isotropic limit) gives $\delta W_{fA} = 0$.

The effect of anisotropy contained in the brackets $\{\}$ introduces periodic dependence in Θ, as will be seen in the example exercises, the content of the brackets vanishing in the isotropic limit. And for the anisotropic case we have that $\{......\}$ is even in Θ (as it depends on $B \sim 1/R$. In addition we have that $\overline{\{......\}} = 0$. Hence, we need only the leading order component of $\xi^r\,\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa}$, and specifically the even component of it,

$$\xi^r\,\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa} \to |\xi^r|^2\frac{\cos\Theta}{R_0}.$$

Hence, we have that,

$$\delta W_{fA} = \frac{1}{2}\int d^3x\,|\xi^r|^2\frac{\cos\Theta}{R_0}\left\{\int d^3v\,m_h\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\partial F_h}{\partial r} - \frac{d}{dr}\left(\overline{P_{\perp h}}+\overline{P_{\parallel h}}\right)\right\}. \quad (7.9)$$

Notice that this result is not sensitive to the definition of Θ because different choices will cause higher order corrections than the leading order terms which are non-zero providing the distribution function is anisotropic.

Physical fast ion distribution functions

As mentioned earlier, equilibrium fast ion distribution functions should depend on the constants of motion for a single particle, which means that $F_h = F_h(\mathcal{E}, \mu, r)$. In the definition of δW_{fA} the meaning of the partial derivative is

$$\left.\frac{\partial F_h}{\partial r}\right|_{\mathcal{E},\mu}$$

For investigating fast ion anisotropy effects it is convenient to do so via a suitable bi-Maxwellian distribution function, which is an extension of the Maxwellian,

$$\frac{m_h^{3/2} n(r)}{[2\pi T(r)]^{3/2}} \exp\left(-\frac{\mathcal{E}}{T(r)}\right) = \frac{m_h^{3/2} n(r)}{[2\pi T(r)]^{3/2}} \exp\left(-\frac{m v_\parallel^2}{2T(r)} - \frac{m v_\perp^2}{2T(r)}\right).$$

Usually the bi-Maxwellian is written in the form,

$$\frac{m_h^{3/2} n(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{m v_\parallel^2}{2T_\parallel(r)} - \frac{m v_\perp^2}{2T_\perp(r)}\right).$$

But writing the latter in terms of $\mathcal{E}$ and μ we have,

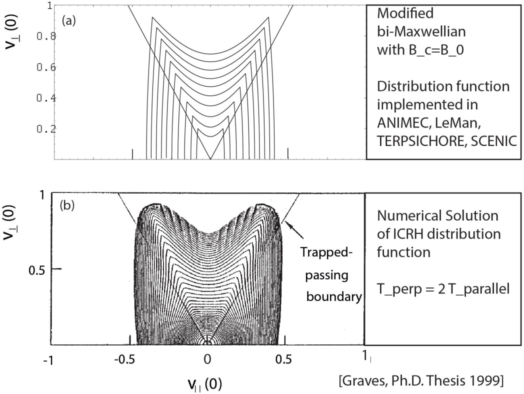
$$\frac{m_h^{3/2} n(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{\mathcal{E} - \mu B(r,\Theta)}{T_\parallel(r)} - \frac{\mu B(r,\Theta)}{T_\perp(r)}\right),$$

the distribution function thus being unphysical because it depends on Θ.

The distribution can be adapted to be physical and experimentally relevant (e.g. for ICRH heating) by replacing $B(r,\Theta)$ with $B_c(r)$, some magnetic field strength that depends only on r. Suitable choices can be $B_c = B_0$, or $B_c = B_{min}(r) \approx B_0(1-\epsilon)$. We also require that the argument of the exponential is negative for all phase space, so we use,

$$F_h(\mathcal{E}, \mu, r) = \frac{m_h^{3/2} n_c(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{|\mathcal{E} - \mu B_c(r)|}{T_\parallel(r)} - \frac{\mu B_c(r)}{T_\perp(r)}\right),$$

Modified bi-Maxwellian

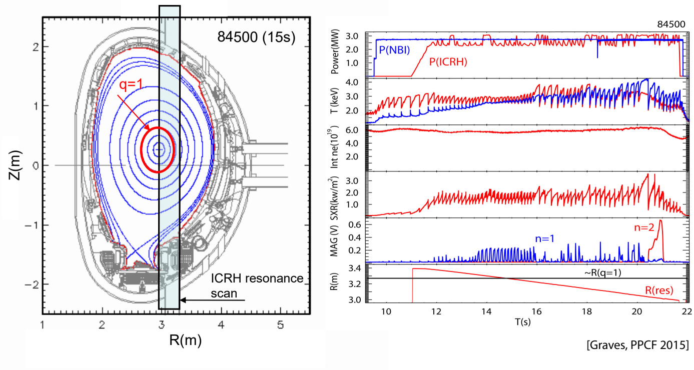


The modified bi-Maxweillian distribution with $B_c(r) = B_0$ models ICRH fast ions with heating applied on-axis. In this course we avoid the inconvenience of the modulus (and hence the sharp corner in the above figure) by setting $B_c = B_{min}(r)$:

$$F_h(\mathcal{E}, \mu, r) = \frac{m_h^{3/2} n_c(r)}{(2\pi)^{3/2} T_\perp(r) T_{||}(r)^{1/2}} \exp\left(-\frac{\mathcal{E} - \mu B_{min}(r)}{T_{||}(r)} - \frac{\mu B_{min}(r)}{T_\perp(r)}\right),$$

which for the ICRH application above is a particularly good model for RF heating where resonance is applied off-axis on the low field side of the device (this is often done experimentally, effective for controlling sawteeth and impurities e.g. in JET). **But, in fact, we may apply the above for NBI and ICRH with any resonance position. The main physics being determined by profiles $n_c(r)$ and $T_\perp(r)$ and $T_{||}(r)$.**

NBI and ICRH JET Sawtooth Control Experiments



[Graves, PPCF 2015]

Internal kink initially stabilised (long sawteeth) by NBI ions, via anisotropic fluid effect and kinetic effects. ICRH can be stabilising or destabilising, depending on deposition or ICRH relative to $q = 1$ surface. For heating outside $q = 1$ the ICRH ion density profile and temperature profiles can become inverted (causing destabilisation).

We may approximately apply the below for NBI and (ICRH with any resonance position):

$$F_h(\mathcal{E}, \mu, r) = \frac{m_h^{3/2} n_c(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{\mathcal{E} - \mu B_{min}(r)}{T_\parallel(r)} - \frac{\mu B_{min}(r)}{T_\perp(r)} \right).$$

The main physics being determined by profiles $n(r)$ and $T_\perp(r)$ and $T_\parallel(r)$.

Modified bi-Maxwellian

The modified bi-Maxwellian can be written in terms of $v_\parallel$ and $v_\perp$ once again. We easily obtain using $\mathcal{E} = m_h v_\perp^2/2 + m_h v_\parallel^2/2$ and $\mu = m_h v_\perp^2/(2B)$,

$$F_h = \frac{m_h^{3/2} n_c(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{m_h v_\parallel^2}{2T_\parallel(r)} - \frac{m_h v_\perp^2}{2\hat{T}_\perp(r,\Theta)}\right), \tag{7.10}$$

$$\frac{1}{\hat{T}_\perp(r,\Theta)} = \frac{1}{T_\perp(r)B(r,\Theta)}\left[B_{min}(r) + \frac{T_\perp(r)}{T_\parallel(r)}\left(B(r,\Theta) - B_{min}(r)\right)\right]. \tag{7.11}$$

Notice that we obtain the standard isotropic Maxwellian for $T_\perp = T_\parallel$ (which would not have occurred in fact if we choose e.g. $B_c = B_0$ for example).

The full velocity integral over all phase space is,

$$\int_{all V} d^3v = 2\pi \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty dv_\perp\, v_\perp$$

For a distribution that is symmetric in $v_\parallel$ (such as the one above) we may use the following for the full velocity integral (note distributions are always symmetric with respect to perpendicular thermal velocity):

$$\int_{all V} d^3v = 2\pi \int_0^\infty dv_\parallel \int_0^\infty dv_\perp^2 \tag{7.12}$$

Then the relevant moments are,

$$n_h = 2\pi \int_0^\infty dv_\parallel \int_0^\infty dv_\perp^2\, F_h, \quad P_{h\perp} = 2\pi \int_0^\infty dv_\parallel \int_0^\infty dv_\perp^2\, m_h F_h \frac{v_\perp^2}{2}, \quad P_{h\parallel} = 2\pi \int_0^\infty dv_\parallel \int_0^\infty dv_\perp^2\, m_h F_h v_\parallel^2$$

Modified bi-Maxwellian - dominant density gradient

In this course we wish to understand the main physics properties, slightly at the expense of more general situations. Let us consider the limit

$$\frac{r}{n_c}\frac{dn_c}{dr} \gg \left(\frac{r}{T_\perp}\frac{dT_\perp}{dr}, \frac{r}{T_\parallel}\frac{dT_\parallel}{dr}, \frac{r}{B_{min}}\frac{dB_{min}}{dr}\right).$$

In which case we have that,

$$\left.\frac{\partial F_h}{\partial r}\right|_{\mathcal{E},\mu} \approx \frac{F_h}{n_c}\frac{dn_c}{dr}$$

From the definition of Eq. (7.9) and the definitions of the parallel and perpendicular pressure we obtain

$$\begin{aligned}\delta W_{fA} &= \frac{1}{2}\int d^3x\,|\xi^r|^2\frac{\cos\Theta}{R_0}\left\{\int d^3v\, m_h\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\partial F_h}{\partial r} - \frac{d}{dr}\left(\overline{P_{\perp h}}+\overline{P_{\parallel h}}\right)\right\}\\ &\approx \frac{1}{2}\int d^3x\,|\xi^r|^2\frac{\cos\Theta}{R_0}\frac{1}{n_c}\frac{dn_c}{dr}\left\{\left(P_{\perp h}+P_{\parallel h}\right) - \left(\overline{P_{\perp h}+P_{\parallel h}}\right)\right\} \qquad (7.13)\end{aligned}$$

where,

$$\begin{aligned}P_{\perp h}+P_{\parallel h} &= 2\pi\int_0^\infty dv_\parallel\int_0^\infty dv_\perp^2\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)m_h F_h\\ &= 2\pi\int_0^\infty dv_\parallel\int_0^\infty dv_\perp^2\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{m_h^{5/2}n_c(r)}{(2\pi)^{3/2}T_\perp(r)T_\parallel(r)^{1/2}}\exp\left(-\frac{m_h v_\parallel^2}{2T_\parallel(r)}-\frac{m_h v_\perp^2}{2\hat{T}_\perp(r,\Theta)}\right)\\ &= n_c(r)T_\perp(r)\left[\left(\frac{\hat{T}_\perp(r,\Theta)}{T_\perp(r)}\right)^2+\left(\frac{\hat{T}_\perp(r,\Theta)}{T_\perp(r)}\right)\frac{1}{A(r)}\right], \quad A(r)=\frac{T_\perp(r)}{T_\parallel(r)}\end{aligned}$$

Modified bi-Maxwellian - large aspect ratio expansion

We now adopt the large aspect ratio expansion $B = B_0(1 - \cos\Theta)$, $B_{min} = B_0(1 - \epsilon)$. Hence,

$$P_{\perp h} + P_{\parallel h} = n_c(r)T_\perp(r)\frac{(1-\epsilon\cos(\Theta))[1-\epsilon+A(1+\epsilon)-2A\epsilon\cos(\Theta)]}{A[1-\epsilon-A\epsilon+A\epsilon\cos(\Theta)]^2}.$$

We note that for a strongly perpendicular distribution function we can have that $\epsilon A \sim 1$ or larger. But, in order to make further analytic simplifications, we assume that $\epsilon A \ll 1$, this still providing essential understanding. Hence we obtain,

$$P_{\perp h} + P_{\parallel h} \approx n_c T_\perp\left[\left(1+\frac{1}{A}\right)+\left(1-\frac{1}{A}\right)(1+2A)\epsilon(\cos\Theta-1)\right]. \tag{7.14}$$

Hence, we obtain,

$$\left\{\left(P_{\perp h}+P_{\parallel h}\right)-\left(\overline{P_{\perp h}+P_{\parallel h}}\right)\right\} = n_c T_\perp\left(1-\frac{1}{A}\right)(1+2A)\epsilon\cos\Theta. \tag{7.15}$$

As expected this expression vanishes for $A = 1$.

- For a deeply passing distribution we have a surplus of pressure on the HFS (due to having diminished trapped fraction)

$$A \ll 1: \quad \delta W_{fA} = \frac{1}{2}\int d^3x\,|\xi^r|^2\frac{1}{R_0^2}\left(-\frac{r}{n_c}\frac{dn_c}{dr}\right)n_c T_\parallel\cos^2\Theta$$

  The pressure weighted average curvature yields improved stability $\delta W_{fA} > 0$ assuming $dn_c/dr < 0$

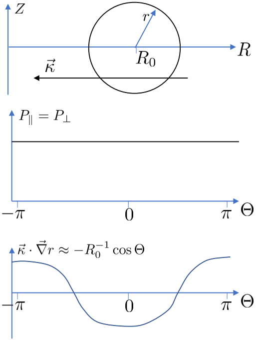
- For a deeply trapped distribution we have a surplus of pressure on the LFS:

$$A \gg 1: \quad \delta W_{fA} = \frac{1}{2}\int d^3x\,|\xi^r|^2\frac{1}{R_0^2}\left(-\frac{1}{n_c}\frac{dn_c}{dr}\right)(-2n_c T_\perp)\cos^2\Theta$$

  The pressure weighted average curvature in yields weakened stability $\delta W_{fA} < 0$ assuming $dn_c/dr < 0$

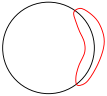
Pressure averaged curvature

Isotropic Plasma

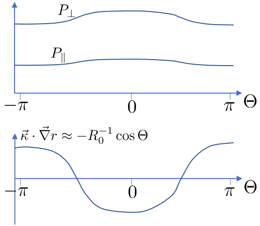


$\delta W_{fA} = 0$

Perp anisotropy $T_\perp > T_\parallel$



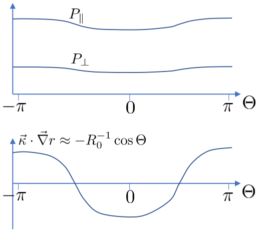
Excess of trapped particles, less passing particles, increases pressure on low field side



$\delta W_{fA} < 0$ i.e. destabilising
Excess pressure in the bad curvature region.
Pressure weighted average curvature unfavourable

Parallel anisotropy $T_\perp < T_\parallel$

Excess of passing particles.
Passing particles spend more time on HFS than LFS.
Excess of pressure on HFS



$\delta W_{fA} > 0$ i.e. stabilising
Excess pressure in the good curvature region.
Pressure weighted average curvature favourable

$\delta W_{fA} \sim \left[-\frac{1}{n_h}\frac{dn_h}{dr}\right]\int_{-\pi}^{\pi} d\Theta\,(\vec{\kappa}\cdot\vec{\nabla}r)[P_\perp + P_\parallel - \overline{P_\perp + P_\parallel}]$ Comments on stabilisation/destabilisation assume core peaking ($dn_h/dr < 0$)

$\delta W_c \sim \int_{-\pi}^{\pi} d\Theta\,(\vec{\kappa}\cdot\vec{\nabla}r)[\overline{P_\perp + P_\parallel}]$ contribution involves ϵ corrections to $\vec{\kappa}$ since $\overline{P_\perp + P_\parallel}$ is independent of Θ.

Passing and trapped fractions

When considering kinetic corrections we will need the velocity integral over passing particles and trapped particles. Particles are passing if $v_\parallel$ does not change sign over an orbit, or if a zero in $v_\parallel^2$ does not exist. Note that,

$$\frac{m_h}{2} v_\parallel^2 = \mathcal{E} - \mu B(r, \Theta)$$

where we consider that μ, $\mathcal{E}$ and r are constant over an orbit (passing or trapped). Hence the following should also **not change sign** over a passing orbit,

$$\frac{m_h}{2\mathcal{E}} v_\parallel^2 = \frac{1}{\mathcal{E}} \left[\mathcal{E} - \mu B(r, \Theta)\right] = 1 - \lambda B(r, \Theta), \quad \lambda = \frac{\mu}{\mathcal{E}}.$$

Passing particles explore all values of B on a given flux surface r, from $B_{min}(r)$ at $\Theta = 0$ to $B_{max}(r)$ at $\Theta = \pi$. Hence, the zero in $v_\parallel^2$ is avoided over the passing particle orbit providing that,

$$0 \leq \lambda < \frac{1}{B_{max}(r)}, \quad i.e. \quad 0 \leq \frac{v_\perp^2}{B(v_\parallel^2 + v_\perp^2)} < \frac{1}{B_{max}}$$

This can be written as a condition on $v_\perp^2$, i.e. $0 \leq v_\perp^2 < (v_\parallel^2 + v_\perp^2)B/B_{max}$ or

$$0 \leq v_\perp^2 < \frac{v_\parallel^2}{B_{max}(r)/B(r, \Theta) - 1}.$$

Hence the passing fraction of velocity space is:

$$\int_{pass} d^3v = 2\pi \int_0^\infty dv_\parallel \int_0^{v_\parallel^2/(B_{max}/B-1)} dv_\perp^2 \tag{7.16}$$

The trapped fraction of velocity space is therefore,

$$\int_{trap} d^3v = \int_{allV} d^3v - 2\pi \int_0^\infty dv_\parallel \int_0^{v_\parallel^2/(B_{max}/B-1)} dv_\perp^2 = 2\pi \int_0^\infty dv_\parallel \int_{v_\parallel^2/(B_{max}/B-1)}^\infty dv_\perp^2 \tag{7.17}$$

Passing and trapped fractions

The trapped fraction of particles for a population F_h can be obtained by evaluating:

$$f_t = \left(\int_{trap} d^3v\, F_h \right) / \int_{allV} d^3v\, F_h$$

The exercises will explore the trapped fraction for an isotropic plasma. The results can be generalised for the modified bi-Maxwellian:

$$f_t = \left[\frac{\frac{\hat{T}_\perp}{T_\parallel}(B_{max} - B)/B}{1 + \frac{\hat{T}_\perp}{T_\parallel}(B_{max} - B)/B} \right]^{1/2} .$$

Using the lowest order expressions for B, B_{max} and $\hat{T}_\perp$ we thus have

$$f_t = \left[\frac{\frac{\hat{T}_\perp}{T_\parallel}\epsilon(1 + \cos\Theta)}{1 + \frac{\hat{T}_\perp}{T_\parallel}\epsilon(1 + \cos\Theta)} \right]^{1/2} [1 + O(\epsilon)] .$$

This expression and the first one at the top of the slide agrees with the trapped fraction of an isotropic species in the limit $T_\perp = T_\parallel$. Letting now $\epsilon T_\perp / T_\parallel \ll 1$ the trapped fraction is

$$f_t = [\epsilon T_\perp / T_\parallel]^{1/2} [1 + \cos\Theta]^{1/2}$$

Notice that the standard textbook definition of the trapped fraction evaluates at $\Theta = 0$ for an isotropic case, giving $f_t = \sqrt{2\epsilon}$. The flux averaged trapped fraction

$$\overline{f_t} \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} f_t = \frac{2}{\pi} \sqrt{2\epsilon T_\perp / T_\parallel}$$

Kinetic fast ion corrections

A more general solution to the perturbed drift kinetic equation is (see e.g. Porcelli PoP 1994, Graves PPCF 2000):

$$\delta F_h = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla} F_h + \begin{cases} q_j \left\langle \dot{\phi} \right\rangle \xi^{\psi} \dfrac{\partial F_h}{\partial \mathcal{E}} \dfrac{(\omega - n\omega_{*h})}{\omega - n \left\langle \dot{\phi} \right\rangle + i\nu_{eff}} & \text{trapped ions} \\ 0 & \text{passing ions.} \end{cases}$$

in the limit where the bounce frequency of fast ions is much larger than the mode frequency ω, the average toroidal precession drift $\left\langle \dot{\phi} \right\rangle$, the collision frequency ν_{eff} and the diamagnetic frequency

$$\omega_{*h} = q_j^{-1} \frac{\partial F_h}{\partial \psi} \bigg/ \frac{\partial F_h}{\partial \mathcal{E}}.$$

Here $\left\langle \dot{\phi} \right\rangle \xi^{\psi}$ represents the orbit averaged Lagrangian which vanishes for passing ions.

- In the highly collisional limit $\nu_{eff} \to \infty$ we obtain the anisotropic (adiabatic) fluid result $\delta F_{hf} = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla} F_h = -\xi^r \partial F_h / \partial r$ of Eq. (7.7) considered until this slide.
- In the limit $\omega \sim n \left\langle \dot{\phi} \right\rangle$ we have possible Landau wave-particle resonant energy transfer. This effect is responsible e.g. for fishbones for the high frequency branch of $n = 1$, $m = 1$ internal kinks (explored if we had more time)
- A low frequency branch $\omega = 0$ which in the collisionless limit is,

$$\delta F_h = \delta F_{hf} + \delta F_{hk}, \quad \text{with} \quad \delta F_{hf} = -\xi^r \frac{\partial F_h}{\partial r}, \quad \delta F_{hk} = \begin{cases} \xi^r \dfrac{\partial F_h}{\partial r} & \text{trapped ions} \\ 0 & \text{passing ions.} \end{cases} \tag{7.18}$$

$$= \begin{cases} 0 & \text{trapped ions} \\ -\xi^r \dfrac{\partial F_h}{\partial r} & \text{passing ions.} \end{cases} \tag{7.19}$$

The high and low frequency branches can co-exist. The kinetic corrections for the low frequency branch are strongly stabilising if $\partial F_h / \partial r < 0$ (core peaked profiles). Note in absence of δF_{hf}, stabilisation would be said to be due to trapped ion physics. Taken all together it might be said to be due passing particles (absence of trapped ions). The literature is confused on this, both interpretations are possible.

Kinetic fast ion corrections

We write the total energy

$$\delta W = \delta W_c + \delta W_{fA} + \delta W_k$$

where from the last slide, Eqs. (7.4) and Eq. (7.8)

$$\delta W_c = \frac{1}{2}\int d^3x \left[|\delta B_\perp|^2 + B^2 \left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right|^2 - 2(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\overline{P})(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*) - J_\parallel(\boldsymbol{\xi}_\perp^*\times\boldsymbol{b})\cdot\boldsymbol{\delta B}_\perp\right]$$

$$\delta W_{fA} = -\frac{1}{2}\int d^3x\, \boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa}\,\xi^r \left\{\int d^3v\, m_h \left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\frac{\partial F_h}{\partial r} - \frac{d}{dr}\left(\overline{P_{\perp h}} + \overline{P_{\parallel h}}\right)\right\}$$

$$\delta W_k = \frac{1}{2}\int d^3x\, \boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa}\,\xi^r \left\{\int_{trap} d^3v\, m_h \left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\frac{\partial F_h}{\partial r}\right\}$$

Again for simplicity we assume gradients in F_h are dominated by gradients in $n_c(r)$. Using the lowest order definition for the curvature, and noting that only the even component of the curvature contributes,

$$\begin{aligned}\delta W_k &= -\frac{1}{2}\int d^3x\, |\xi^r|^2 \frac{\cos\Theta}{R_0}\left\{\int_{trap} d^3v\, m_h\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\frac{\partial F_h}{\partial r}\right\}(1+O(\epsilon)) \\ &= -\frac{1}{2}\int d^3x\, |\xi^r|^2 \frac{\cos\Theta}{R_0}\frac{1}{n_c}\frac{dn_c}{dr}\left\{\Big(P_{\perp h}(trap) + P_{\parallel h}(trap)\Big)\right\}(1+O(\epsilon)) \qquad (7.20)\end{aligned}$$

So that including δW_{fA}, from Eq. (7.13), we may write the total hot response in the convenient form,

$$\delta W_{fA} + \delta W_{kA} = \frac{1}{2}\int d^3x\, |\xi^r|^2 \underbrace{\left(-\frac{\cos\Theta}{R_0}\right)}_{\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}r}\left[-\frac{1}{n_c}\frac{dn_c}{dr}\right]\left\{\Big(P_{\perp h}(pass) + P_{\parallel h}(pass)\Big)\right\}$$

Kinetic fast ion corrections

Simple extension of the exercises show that in the isotropic limit the trapped pressure components are

$$\frac{T_\perp}{T_\parallel} = 1 : \quad P_{\perp h}(trap) + P_{\parallel h}(trap) = n_c T_\perp \frac{(4B_{max} - B)\left[B_{max} - B\right]^{1/2}}{2B_{max}^{3/2}}.$$

Substituting $B = B_0(1 - \epsilon\cos\theta)$ and $B_{max} = B_0(1+\epsilon)$ and expanding in ϵ we obtain,

$$\frac{T_\perp}{T_\parallel} = 1 : \quad P_{\perp h}(trap) + P_{\parallel h}(trap) \approx \frac{3}{2} n_c T_\perp f_t (1 + O(\epsilon)), \quad f_t = \sqrt{\epsilon(1+\cos\Theta)}$$

It is possible to generalise this result for the modified bi-Maxwellian. Assuming again $\epsilon T_\perp / T_\parallel \ll 1$ one obtains

$$P_{\perp h}(trap) + P_{\parallel h}(trap) \approx \frac{3}{2} n_c T_\perp f_t (1 + O(\epsilon)), \quad f_t = \sqrt{\epsilon A(1+\cos\Theta)} \tag{7.21}$$

where $A = T_\perp / T_\parallel$.

A result that is important for the calculation of δW_k is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta \, \cos\Theta f_t = \frac{\overline{f_t}}{3}, \qquad \overline{f_t} = \frac{2}{\pi}\sqrt{2\epsilon A} \tag{7.22}$$

Note that these kinetic corrections are larger than the hot corrections to the fluid δW by a factor $(A\epsilon)^{-1/2}$. But clearly hot fluid contributions will dominate if there are very few trapped particles (i.e. where $A \ll 1$). We now put all the results together.

Total hot contributions to δW

We now consider the sum $\delta W_{fA} + \delta W_k$. Use $\int d^3x = 2\pi R_0 \int_{-\pi}^{\pi} d\Theta \int_0^a dr\, r$. From Eqs. (7.13), (7.15), (7.20), (7.21) and (7.22) we obtain,

$$\delta W_{fA} + \delta W_k = \pi^2 \int_0^a dr\, r|\xi^r|^2 \left(-T_\perp \frac{dn_c}{dr}\right) \left[\overline{f_t} - \epsilon\left(1 - \frac{1}{A}\right)(1 + 2A)\right]$$

we may now look at three limits:

- The strongly parallel anisotropic limit where there are no fast trapped ions $A = 0$:

$$\delta W_{fA} + \delta W_k = \pi^2 \int_0^a dr\, r|\xi^r|^2 \left(-T_\parallel \frac{dn_c}{dr}\right) \epsilon$$

- The isotropic limit where we only have kinetic corrections:

$$\delta W_{fA} + \delta W_k = \pi^2 \int_0^a dr\, r|\xi^r|^2 \left(-T \frac{dn_c}{dr}\right) \overline{f_t}, \qquad \overline{f_t} = \frac{2}{\pi}\sqrt{2\epsilon T_\perp / T_\parallel}$$

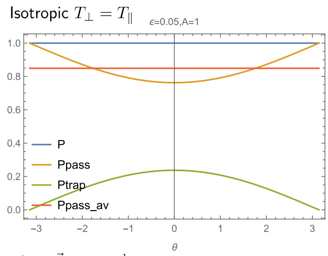
- The strongly perpendicular limit where we have $A \gg 1$:

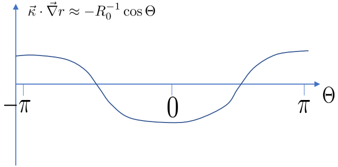
$$\delta W_{fA} + \delta W_k = \pi^2 \int_0^a dr\, r|\xi^r|^2 \left(-T_\perp \frac{dn_c}{dr}\right) \left[\overline{f_t} - 2\epsilon \frac{T_\perp}{T_\parallel}\right]$$

  Note though that this result is valid only for $\epsilon A \ll 1$

Fast particles are stabilising to these terms in all cases if $-dn_c/dr > 0$ (since these δW terms are positive), otherwise destabilising for $-dn_c/dr < 0$. A separate calculation allowing for $\epsilon A \sim 1$ requires numerical calculation, but the opposite conclusion is found. For $\epsilon A \sim 1$ the fraction of passing particles becomes small, so fast ion stabilisation also becomes weak ($\delta F_h \to 0$).

Isotropic Kinetic corrections



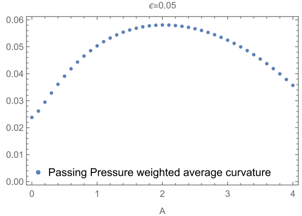


$$\delta W_h = \delta W_{fA} + \delta W_k \sim \left[-\frac{1}{n_h}\frac{dn_h}{dr}\right]\int_{-\pi}^{\pi} d\Theta\,(\vec{\kappa}\cdot\vec{\nabla}r)\left[P_{\perp h}(pass) + P_{\parallel h}(pass)\right]$$
$$\delta W_h > 0$$

$$\delta W_h = \delta W_{fA} + \delta W_k \sim \left[-\frac{1}{n_h}\frac{dn_h}{dr}\right]\int_{-\pi}^{\pi} d\Theta\,(\vec{\kappa}\cdot\vec{\nabla}r)\left[P_{\perp h}(pass) + P_{\parallel h}(pass)\right]$$

- For simplicity we select an isotropic case in the example on the left.
- The total hot contribution involves the average curvature, weighted with the passing pressure.
- The trapped particles are dominantly on the LFS. Since the total pressure is independent of poloidal angle, it means the passing pressure is smaller on the LFS than it is on the HFS.
- The HFS where the passing pressure is largest is the region of good curvature. The weighted average curvature is therefore favourable providing that the hot density is peaked in the core

Total passing pressure, and its poloidal variation, depends on anisotropy A
Effect on δW_h is:

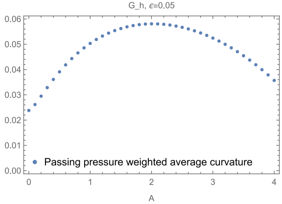


Total fast ion effects: effect of anisotropy

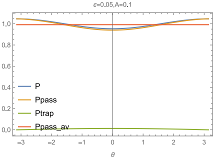
$\overline{P_h} = \frac{1}{2}(\overline{P_{\perp h} + P_{\parallel h}}) = 1$ for all cases
$\overline{P_h}(passing)$ reduces as $A = T_\perp / T_\parallel$ increases

Changing the anisotropy changes variation of the passing pressure with θ

$$\delta W_h = \delta W_{fA} + \delta W_k \sim \left[-\frac{1}{n_h}\frac{dn_h}{dr} \right] \int_{-\pi}^{\pi} d\Theta \, (\vec{\kappa} \cdot \vec{\nabla} r) \left[P_{\perp h}(pass) + P_{\parallel h}(pass) \right]$$
$$\propto \alpha_h G_h$$

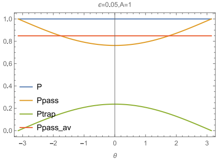


Strong par. anis. $T_\perp = 0.1 T_\parallel$



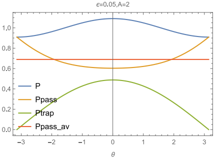
Strong parallel anisotropy.
There are virtually no trapped ions.
Case is the same as the fluid anisotropic case with $A \ll 1$

Isotropic $T_\perp = T_\parallel$



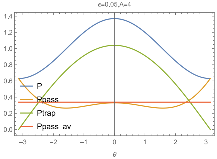
Fluid anisotropic effects are zero. But we have the kinetic trapped effect. Or we can understand it as the passing ion pressure weighted curvature.

Weak perp. anis. $T_\perp = 2T_\parallel$



As the Anisotropy is increased the trapped fraction increases. This causes strong poloidal variation in the passing pressure.
Large passing pressure weighted average curvature is obtained.

Moderate perp. anis. $T_\perp = 4T_\parallel$



As the Anisotropy is increased further, the passing fraction is diminished. As a consequence the weighted avergage curvature is smaller than the isotropic case.

Fast ion corrections to pressure driven instabilities

The fast ion contributions to δW considered to this point are relevant only if they can drive or damp important tokamak instabilities. Note that the drive or damping from the fast ions enters essentially through the fast ion pressure gradient $T_\perp dn_c/dr$.

- Current driven instabilities are not sensitive to these fast ion corrections. Recall that tearing modes, external kink modes etc appear at $\delta W \sim \epsilon^2$. Pressure gradient corrections are not important at that order
- Pressure driven instabilities can be expected to be affected by fast ion populations under empirically relevant conditions. Interchange, infernal and internal kink instabilities are highly relevant to tokamak operations.

The last slides consistently integrates fast ion physics effects into the stability criteria of internal kink modes and interchange modes.

The total energy is $\delta W = \delta W_c + \delta W_{fA} + \delta W_k$. These separate terms require The average pressure,

$$\overline{P} = P_t + \overline{P_h}, \quad \overline{P_h} = \frac{\overline{P_{\perp h}} + \overline{P_{\parallel h}}}{2}$$

For the case where fast ion density gradients are much larger than temperature gradients, we have that

$$\left(-T_\perp \frac{dn_c}{dr}\right) \approx \frac{d}{dr}\left(-2\overline{P_h}\right)\frac{A}{1+A}$$

so that

$$\delta W_c = \frac{1}{2}\int d^3x \left[|\delta B_\perp|^2 + B^2 |\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2 - 2(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\overline{P})(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*) - J_\parallel(\boldsymbol{\xi}_\perp^*\times\boldsymbol{b})\cdot\boldsymbol{\delta B}_\perp\right]$$

$$\delta W_{fA} + \delta W_k = 2\pi^2\int_0^a dr\, r|\xi^r|^2 \frac{d}{dr}\left(-2\overline{P_h}\right) G_h, \quad G_h = \frac{A\overline{f_t} + (1-A)(1+2A)\epsilon}{2(1+A)}, \quad \overline{f_t} = \frac{2}{\pi}\sqrt{2\epsilon A}.$$

Recalling that the grad-Shafranov equilibrium and the definition of $\boldsymbol{\kappa}$ includes $\overline{P_h}$.

Total Energy for toroidal modes

We now refer back to results in week 3. The sum of the field line bending terms of Eq. (3.33) (with boundary terms neglected ($\xi_0^r(a) = 0$)) and the toroidal terms of Eq. (3.35) yield the core contribution δW_c in a torus:

$$\delta W_c = \frac{2\pi^2 B_0^2}{R_0} \int_0^a dr\, r \left\{ \left[\left(r \frac{d\xi_0^r}{dr} \right)^2 + \left(m^2 - 1 \right) (\xi_0^r)^2 \right] \left(\frac{1}{q_s} - \frac{1}{q} \right)^2 + \frac{(\xi^r)^2}{q_s^2} \left[\epsilon\alpha \left(1 - \frac{1}{q_s^2} \right) \right] \right\} + \frac{1}{n^2} \delta W_T(\alpha).$$

The first term involving the ballooning parameter

$$\alpha = -\frac{2q_s^2 R_0}{B_0^2} \frac{dP_c}{dr} + \alpha_h, \quad \alpha_h = -\frac{2q_s^2 R_0}{B_0^2} \frac{d\overline{P_h}}{dr}$$

is $(1 - q_s^2)\delta W_C$ in Eq. (3.35) and subsequent equations. Note α includes the average fast particle pressure. Also $\delta W^T(\alpha)$ is an additional toroidal contribution, e.g. Eq. (3.36) for the case of $m = 1$ assuming the Heaviside solution for $\xi^r(r)$. Adding the hot ion contributions $\delta W_{fA} + \delta W_k$ we obtain

$$\delta W = \frac{2\pi^2 B_0^2}{R_0} \int_0^a dr\, r \left\{ \left[\left(r \frac{d\xi_0^r}{dr} \right)^2 + \left(m^2 - 1 \right) (\xi_0^r)^2 \right] \left(\frac{1}{q_s} - \frac{1}{q} \right)^2 + \frac{(\xi^r)^2}{q_s^2} \left[\alpha\epsilon \left(1 - \frac{1}{q_s^2} \right) + \alpha_h G_h \right] \right\} + \delta W^T(\alpha), \tag{7.23}$$

where

$$G_h = \frac{A\overline{f_t} + (1 - A)(1 + 2A)\epsilon}{2(1 + A)}.$$

Modified Mercier Criterion

In chapter 6 we investigated the Mercier criterion from the infinite ballooning equation. But it can also be obtained from Eq. (7.23). For Mercier instabilities we neglect the extra toroidal term δW^T, which is appropriate for $m \neq 1$ and shear not too small. The corresponding Euler-Lagrange equation is then

$$\frac{1}{r}\frac{d}{dr}\left\{r^3\left(\frac{1}{q_s}-\frac{1}{q}\right)^2\frac{d\xi^r}{dr}\right\} = \frac{1}{q_s^2}\left[\alpha\epsilon\left(1-\frac{1}{q_s^2}\right)+\alpha_h G_h\right]\xi^r$$

Mercier (interchange) instabilities are highly localised. We thus expand using a layer variable (same one as we used for the resistive problems) $x = (r - r_s)/r_s$, where $q(r_s) = q_s$. Noting the definition of the magnetic shear and using $(1/q_s - 1/q)^2 \approx x^2 s^2/q_s^2$, we thus have,

$$\frac{d}{dx}\left(x^2\frac{d\xi^r}{dx}\right)+D_M\xi^r = 0, \quad D_M = \frac{1}{s^2}\left[\epsilon\alpha\left(\frac{1}{q_s^2}-1\right)-\alpha_h G_h\right]$$

where we note the symmetry with Eq. (6.18), obtained from the infinite n ballooning equation. Note the hot particle modified Mercier factor D_M. As explained in chapter 6, Mercier instabilities are unstable for

$$D_M > \frac{1}{4}.$$

Since $G_h > 0$ for all fast ion populations, we find that fast ions are stabilising to interchange modes providing that $\alpha_h > 0$, i.e. centrally peaked profiles. We now see why interchange modes are rarely observed in large hot tokamaks! Notice that the effect of hot particles also enters into the standard Mercier term (since $\alpha = \alpha_c + \alpha_h$) but the contribution involving G_h is usually larger, especially for $|1/q_s^2 - 1| \ll 1$.

Finally, it is possible to obtain an estimate of the growth rate for interchange modes (see Graves *et al* PPCF 2022 for details):

$$\frac{\gamma}{\omega_A} = \frac{16\,s\,\exp\{\pi\,[D_M - 1/4]^{-1/2} - C + \pi/2\}}{q_s\sqrt{1+2q_s}},$$

where $C = 0.577..$ is the Euler-Mascheroni constant.

Modified $m = 1$ internal kink stability

For this problem we simply apply the Heaviside step solution into Eq. (7.23) on setting $m = 1$. Using the normalisation defined after Eq. (3.35), ie. $\delta W = \delta \hat{W}(2\pi^2 R_0 B_0^2 |\xi_0^r|^2)$. We obtain,

$$\delta \hat{W}_4 = \left(1 - q_s^2\right)\delta \hat{W}^C + \frac{1}{n^2}\delta \hat{W}^T + \frac{1}{\epsilon_1^2 q_s^2}\int_0^{r_1} dr\, \frac{r}{r_1^2}\alpha_h G_h, \quad \delta \hat{W}_C = -\frac{1}{\epsilon_s q_s^4}\int_0^{r_s}\frac{dr\, r^2}{r_s^3}\alpha,$$

see the notes page at the end of chapter 3 for more details. Now select the most important case, the $n = m = 1$ internal kink problem. Hence, setting $q_s = 1$ the Mercier term $\left(1 - q_s^2\right)\delta \hat{W}^C$ vanishes, and we are left with,

$$\delta \hat{W} = 3(1 - q_0)\left((\beta_p^c)^2 - \beta_p^2\right) + \frac{1}{\epsilon_1^2}\int_0^{r_1} dr\, \frac{r}{r_1^2}\alpha_h G_h$$

with $q_0 = q(r = 0) < 1$ and $\beta_p^c \approx 0.3$ for a parabolic q-profile and

$$\beta_p = \frac{1}{\epsilon_1}\int_0^{r_1}\frac{dr\, r^2}{r_1^3}\alpha = -\frac{2}{B_0^2\epsilon_1^2}\int_0^{r_1}\frac{dr\, r^2}{r_1^2}\frac{d(P_c + \overline{P_h})}{dr}.$$

Here β_p enters the internal kink problem via the Shafranov shifted equilibrium $\Delta' = \epsilon(l_i/2 + \beta_p)$. Hence we see how the fast particles influence the toroidal term, and indeed why the fast particle distribution cannot be highly anisotropic (the toroidal equilibrium calculation would have to take into account the extra toroidal harmonics in the pressure).

Hence we see that while the fast ions have a destabilising effect in the toroidal term (via the correction to β_p) the dominant effect of fast particles will be the stabilising term associated with $\alpha_h G_h$. It explains why sawteeth are very long in plasmas with energetic particles. The growth rate for the internal kink is (from Eq. (4.18)):

$$\frac{\gamma}{\omega_A} = \epsilon_1^2\frac{\pi}{s_1}\left[3(1 - q_0)\left[\beta_p^2 - (\beta_p^c)^2\right] - \frac{1}{\epsilon_1^2}\int_0^{r_1} dr\, \frac{r}{r_1^2}\alpha_h G_h\right].$$