Control and operations of tokamaks
Exercise 1 Solutions

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1 Analysis of PF coil controllers for TCV

Recall the coil circuit equation (neglecting the vacuum vessel)

$$\mathbf{M_{aa}}\dot{\mathbf{I}}_\mathbf{a} + \mathbf{R_a}\mathbf{I_a} = \mathbf{V_a} \tag{1}$$

Assume that there is a modeling error such that the true resistivity is $\mathbf{R_a} = \mathbf{R_{a,0}} + \mathbf{\Delta R_a}$, with $\mathbf{\Delta R_a}$ a diagonal matrix.

a) Compute the MIMO transfer function from $\mathbf{V_a}(s)$ to $\mathbf{I_a}(s)$.

In this exercise the Laplace transform of $\mathbf{I_a}(t)$ is written as $\mathcal{L}\{\mathbf{I_a}(t)\} = \mathbf{I_a}(s)$ and the Laplace transform of $\mathbf{V_a}(t)$ is written as $\mathcal{L}\{\mathbf{V_a}(t)\} = \mathbf{V_a}(s)$. For an arbitrary function $f(t)$, for which $\mathcal{L}\{f(t)\} = \mathbf{F}(s)$, the Laplace transform of the derivative is given by $\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = sF(s) - f(0^+)$. Assuming that $\mathbf{I_a}(t=0) = 0$ it follows that $\mathcal{L}\left\{\dot{\mathbf{I}}_\mathbf{a}(t)\right\} = s\mathbf{I_a}(s)$. Recall that $\mathbf{R_a} = \mathbf{R_{a,0}} + \mathbf{\Delta R_a}$. In this way, including the mismatch transfer function matrix $\mathbf{H}(s)$ can now be derived:

$$\begin{array}{rcl}
\mathbf{M_{aa}}\dot{\mathbf{I}}_\mathbf{a}(t) + \mathbf{R_a}\mathbf{I_a}(t) & = & \mathbf{V_a}(t) \\
\Downarrow \mathcal{L} & & \Downarrow \mathcal{L} \\
\mathbf{M_{aa}}\mathbf{s}\mathbf{I_a}(s) + \mathbf{R_a}\mathbf{I_a}(s) & = & \mathbf{V_a}(s) \\
\left(s\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{R_a}\right)\mathbf{I_a}(s) & = & \mathbf{M_{aa}}^{-1}\mathbf{V_a}(s) \\
\mathbf{I_a}(s) & = & \left(s\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{R_a}\right)^{-1}\mathbf{M_{aa}}^{-1}\mathbf{V_a}(s)
\end{array}.$$

Hence

$$\frac{\mathbf{I_a}(s)}{\mathbf{V_a}(s)} = \mathbf{H}(s) = \left(s\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{R_a}\right)^{-1}\mathbf{M_{aa}}^{-1}$$

In the transfer function above $\mathbf{I}$ is the identity matrix of the same size as $\mathbf{M_{aa}}$. Premultiplication by $\mathbf{M_{aa}}^{-1}$ is done for convenience in the next exercises and is allowed because $\mathbf{M_{aa}}$ is be invertible.

b) How could the poles of the nominal transfer function (assuming $\boldsymbol{\Delta}\mathbf{R_a} = 0$) be calculated for given $\mathbf{M_{aa}}$ and $\mathbf{R_{a,0}}$? Is this transfer function always stable?

For the nominal transfer function it is given that $\mathbf{R_a} = \mathbf{R_{a,0}}$. For a SISO system the poles are the values for which the denominator is equal to zero. If one would use apply this same logic to the MIMO system, the poles can be found for values of s for which the denominator $\left(s\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{R_{a,0}}\right)$ is zero. Therefore the poles would follow by solving $\det\left(s\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{R_{a,0}}\right) = 0$. Note that this problem is the same as finding the eigenvalues of $-\mathbf{M_{aa}}^{-1}\mathbf{R_{a,0}}$.

Note: for transfer functions of the shape $\mathbf{G_1}(\mathbf{s}) = (\mathbf{s}I + A)^{-1}B$, with A and B not containing s, the poles are found by solving $\det(\mathbf{s}I + A) = 0$. For a general transfer function $\mathbf{G_2}(\mathbf{s}) = A^{-1}(\mathbf{s})B(\mathbf{s})$, the poles are NOT found by solving $\det(A(\mathbf{s})) = 0$ and one needs to use multivariable theorems (e.g. theorem 4.4 in Skogestad and Postlethwaite 2005) or convert the transfer function to a state space model.

As demonstrated in the slides, inv(Maa)*Raa has only positive eigenvalues, so A has only negative eigenvalues (poles) and the system is stable.

c) Does a voltage applied to coil i also affect currents in other coils? Explain your answer.

The matrix $\mathbf{M_{aa}}$ has no elements equal to zero and therefore $\mathbf{M_{aa}}^{-1}$ also has no elements equal to zero. With $\mathbf{R_a}$ a diagonal full rank matrix, the transfer function it holds that $\mathbf{H}(s = 0) = \mathbf{R_a}^{-1}$, where $\mathbf{R_a}^{-1}$ is a diagonal matrix. Therefore, in steady state ($s = 0$) a voltage on coil i only results in a current in coil i.

However, in the transient response we have to consider also values $s \neq 0$. In this case the transfer function is not diagonal, which shows that a voltage on coil i also affects currents in the other coils.

d) Steady state tracking error with model mismatch

Now, assume (as in the lecture) that the coils are controlled by a controller of the form:

$$\mathbf{V_a}(t) = \mathbf{M_{aa}}k_p\left(\mathbf{I_{ref}}(t) - \mathbf{I_a}(t)\right) + \mathbf{R_{a,0}}\mathbf{I_a}(\mathbf{t}) \tag{2}$$

In this exercise the Laplace transform of $\mathbf{I_{ref}}(t)$ is written as $\mathcal{L}\{\mathbf{I_{ref}}(t)\} = \mathbf{I_{ref}}(s)$.

$$\begin{aligned}
\mathbf{M_{aa}}\dot{\mathbf{I}}_{\mathbf{a}}(t) + \mathbf{R_a}\mathbf{I_a}(t) &= \mathbf{V_a}(t) \\
\mathbf{M_{aa}}\dot{\mathbf{I}}_{\mathbf{a}}(t) + \mathbf{R_a}\mathbf{I_a}(t) &= \mathbf{M_{aa}}k_p\left(\mathbf{I_{ref}}(t) - \mathbf{I_a}(t)\right) + \mathbf{R_{a,0}}\mathbf{I_a}(t) \\
\Downarrow \mathcal{L} & \quad \Downarrow \mathcal{L} \\
\mathbf{M_{aa}}\mathbf{s}\mathbf{I_a}(s) + \mathbf{R_a}\mathbf{I_a}(s) &= \mathbf{M_{aa}}k_p\left(\mathbf{I_{ref}}(s) - \mathbf{I_a}(s)\right) + \mathbf{R_{a,0}}\mathbf{I_a}(s) \\
\left(sI + k_pI + \mathbf{M_{aa}}^{-1}\boldsymbol{\Delta}\mathbf{R_a}\right)\mathbf{I_a}(s) &= k_p\mathbf{I_{ref}}(s) \\
\mathbf{I_a}(s) &= \left((s+k_p)\mathbf{I} + \mathbf{M_{aa}}^{-1}\boldsymbol{\Delta}\mathbf{R_a}\right)^{-1}k_p\mathbf{I_{ref}}(s)
\end{aligned}$$

The following transfer function matrix is found:

$$\mathbf{H_r}(s) = \left((s+k_p)\mathbf{I} + \mathbf{M_{aa}}^{-1}\boldsymbol{\Delta}\mathbf{R_a}\right)^{-1}k_p \tag{3}$$

A step on $\mathbf{I_{ref}}(t)$ means that $\mathbf{I_{ref}}(s) = \frac{1}{s}\mathbf{1}$ (where $\mathbf{1}$ is the vector of ones). The steady-state error $\mathbf{e_{ss}}$ can be found by applying the final value theorem:

$$\begin{aligned}
\mathbf{e_{ss}} &= \lim_{\mathbf{s}\to 0} s\left\{\mathbf{I_{ref}}(s) - \mathbf{H_r}(s)\mathbf{I_{ref}}(s)\right\} \\
&= \lim_{s\to 0} s\left\{\frac{1}{s}\mathbf{I} - \left((s+k_p)\mathbf{I} + \mathbf{M_{aa}}^{-1}\boldsymbol{\Delta}\mathbf{R_a}\right)^{-1}k_p\frac{1}{s}\right\}\mathbf{1}\ . \\
&= \left(\mathbf{I} - \left(k_p\mathbf{I} + \mathbf{M_{aa}}^{-1}\boldsymbol{\Delta}\mathbf{R_a}\right)^{-1}k_p\right)\mathbf{1}
\end{aligned}$$

For $\boldsymbol{\Delta}\mathbf{R_a} = 0$, the error is $\mathbf{e_{ss}} = (1 - \frac{k_p}{k_p})\mathbf{1} = \mathbf{0}$. However, in the case of modelling uncertainty the steady-state tracking error is nonzero.

e) Alternative controller

Consider the alternative controller (in Laplace form)

$$\mathbf{V_a}(s) = (s\mathbf{M_{aa}} + \mathbf{R_{a0}})\frac{k_p}{s}(\mathbf{I_{a,ref}}(s) - \mathbf{I_a}(s))$$

It is possible to apply the same approach as above

$$\begin{aligned}
\mathbf{M_{aa}}\dot{\mathbf{I}}_{\mathbf{a}}(t) + \mathbf{R_a}\mathbf{I_a}(t) &= \mathbf{V_a}(t) \\
\Downarrow \mathcal{L} & \quad \Downarrow \mathcal{L} \\
(s\mathbf{M_{aa}} + \mathbf{R_a})\mathbf{I_a}(s) &= (s\mathbf{M_{aa}} + \mathbf{R_{a0}})\tfrac{k_p}{s}(\mathbf{I_a}(s) - \mathbf{I_{a,ref}}(s)) \\
\left((s\mathbf{M_{aa}} + \mathbf{R_{a0}})(s+k_p) + s\boldsymbol{\Delta}\mathbf{R_a}\right)\mathbf{I_a}(s) &= k_p\left(s\mathbf{M_{aa}} + \mathbf{R_{a0}}\right)\mathbf{I_{a,ref}}(s)
\end{aligned}$$

The transfer function matrix is this time

$$\mathbf{H_r}(s) = \left((s\mathbf{M_{aa}} + \mathbf{R_{a0}})(s+k_p) + s\boldsymbol{\Delta}\mathbf{R_a}\right)^{-1}k_p\left(s\mathbf{M_{aa}} + \mathbf{R_{a0}}\right)\mathbf{I_{a,ref}}(s)$$

The steady-state error $\mathbf{e_{ss}}$ for a step input can be found by applying the final value theorem:

$$\begin{aligned}
\mathbf{e_{ss}} &= \lim_{s\to 0} s\mathbf{I_{a,ref}}\left\{\mathbf{I} - \left((s\mathbf{M_{aa}} + \mathbf{R_{a0}})(s+k_p) + s\boldsymbol{\Delta}\mathbf{R_a}\right)^{-1}k_p\left(s\mathbf{M_{aa}} + \mathbf{R_{a0}}\right)\mathbf{I_{a,ref}}(s)\right\} \qquad (4) \\
&= \lim_{s\to 0} s\mathbf{I_{a,ref}}\left((s\mathbf{M_{aa}} + \mathbf{R_{a0}})(s+k_p) + s\boldsymbol{\Delta}\mathbf{R_a}\right)^{-1}\left(s(s\mathbf{M_{aa}} + \mathbf{R_{a0}} + \boldsymbol{\Delta}\mathbf{R_a})\right) \qquad (5)
\end{aligned}$$

Using $\mathbf{I_{a,ref}} = \frac{1}{s}\mathbf{1}$ we get:

$$\mathbf{e_{ss}} = 0$$

It is therefore possible to track perfectly a step reference in $\mathbf{I_{ref}(s)}$ with this controller also in presence of mismatch $\mathbf{\Delta R_a}$.

f) Ramp tracking

A ramp in $\mathbf{I_{ref}}(t)$ means that $\mathbf{I_{ref}}(s) = \frac{1}{s^2}\mathbf{1}$.

1) Original control law

$$\mathbf{V_a(s)} = \mathbf{M_{aa}}k_p\left(\mathbf{I_{ref}}(s) - \mathbf{I_a}(s)\right) + \mathbf{R_{a,0}I_a(s)} \tag{6}$$

Knowing the transfer function, it is possible to directly estimate the capability to track a ramp evaluating the steady state error

$$\begin{aligned}
\mathbf{e_{ss}} &= \lim_{s\to 0} s\left\{\mathbf{I_{ref}}(s) - \mathbf{H_r}(s)\mathbf{I_{ref}}(s)\right\} \\
&= \lim_{s\to 0} \frac{1}{s}\left\{1 - \left((s+k_p)\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{\Delta R_a}\right)^{-1} k_p\right\}\mathbf{1} \\
&= \lim_{s\to 0} \left\{\frac{1}{s} - \left((s+k_p)\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{\Delta R_a}\right)^{-1} k_p\frac{1}{s}\right\}\mathbf{1}
\end{aligned}.$$

Once more if $\mathbf{\Delta R_a} = 0$, the error is

$$\mathbf{e_{ss}} = \lim_{s\to 0}\left\{\frac{1}{s} - \frac{k_p}{s+k_p}\frac{1}{s}\right\}\mathbf{1} = \lim_{s\to 0}\left\{\frac{s(s+k_p) - sk_p}{s^2(s+k_p)}\right\} = \frac{\mathbf{1}}{k_p}$$

So even in the case without model mismatch, the steady-state error for a ramp reference input is not zero.

Additionally, in the case of model mismatch the steady-state tracking error moves to infinity:

$$\lim_{s\to 0}\left\{\frac{1}{s} - \left((s+k_p)\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{\Delta R_a}\right)^{-1} k_p\frac{1}{s}\right\}\mathbf{1} \tag{7}$$

$$= \lim_{s\to 0}\left\{\frac{1}{s}\left((s+k_p)\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{\Delta R_a}\right)^{-1}\left(\left((s+k_p)\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{\Delta R_a}\right) - k_p\right)\right\}\mathbf{1} \tag{8}$$

$$= \lim_{s\to 0}\left(-\frac{k_p}{s}\right)\mathbf{1} = \infty \tag{9}$$

$$\tag{10}$$

Even though the error will remain limited due to the limited duration of a shot, the tracking error for a ramp reference. In other words, if a ramp signal is set in as a reference, the current in the coils start increasing but with a different slope and a the tracking error grows as time goes on.

2) Alternative control law

$$\mathbf{V_a}(s) = (s\mathbf{M_{aa}} + \mathbf{R_{a0}})\frac{k_p}{s}(\mathbf{I_a(s)} - \mathbf{I_{a,ref}(s)})$$

from (5), filling in this time $\mathbf{I_{a,ref}} = \frac{1}{s^2}\mathbf{1}$ we get:

$$\begin{aligned}
\mathbf{e_{ss}} &= \lim_{s\to 0}\frac{1}{s}\left((s\mathbf{M_{aa}} + \mathbf{R_{a0}})(s+k_p) + s\mathbf{\Delta R_a}\right)^{-1}\left(s(s\mathbf{M_{aa}} + \mathbf{R_{a0}} + \mathbf{\Delta R_a})\right) && (11)\\
&= \lim_{s\to 0}\left((s\mathbf{M_{aa}} + \mathbf{R_{a0}})(s+k_p) + s\mathbf{\Delta R_a}\right)^{-1}\left((s\mathbf{M_{aa}} + \mathbf{R_{a0}} + \mathbf{\Delta R_a})\right) && (12)\\
&= \frac{1}{k_p}\left(\mathbf{R_{a,0}}\right)^{-1}\left(\mathbf{R_{a,0}} + \mathbf{\Delta R_a}\right) = \frac{1}{k_p}\left(\mathbf{I} + \mathbf{R_{a,0}^{-1}}\mathbf{\Delta R_a}\right) && (13)
\end{aligned}$$

Which shows that the error stays bounded also in the case of model mismatch, and close to $1/k_p$ if the mismatch is small.

Note that this controller is equivalent to a PI controller.

g) Upper gain k_p theoretical limit

For the first controller, we derived the closed-loop transfer function (3). From this we see that A sufficient condition for stability of the closed-loop system is that the determinant of

$$\left((s+k_p)\mathbf{I} + \mathbf{M_{aa}}^{-1}\mathbf{\Delta R_a}\right) \qquad (14)$$

does not vanish for any positive value of s. For $\mathbf{\Delta R_a} = 0$, this obviously holds for any $k_p > 0$. For the case $\mathbf{\Delta R_a} \neq 0$, we still see that the first term will dominate for k_p large enough, so a large k_p will also guarantee stability. Hence, in theory, there is no limit to how large k_p can be.

h) Upper gain k_p practical limit

The two main reasons why a large k_p would not be a practical solution are

- an increase in k_p leads to a larger voltage request to the power supply system, possibly leading to saturation of the output, which is undesired.
- real control chains include delays, which are not included in the present model. Increasing k_p would increase the bandwidth of the open loop, leading to phase margin loss when the delay becomes effective at large frequencies, inducing a possible loss of stability or unwanted oscillations.